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Article

# The Solvability of Certain Sixth-Degree Polynomial Equations and the Radical Expressions of Their Roots Under Specific Conditions

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## Abstract

This article deals with sixth-degree polynomial equations and, more specifically, describes the necessary conditions under which the solutions of these polynomials can be expressed in closed-form radicals. For this purpose, eight different cases of sixth-degree polynomial equations will be discussed in detail. It is concluded that if a specific relation exists between the constant term and the remaining coefficients of a sixth-degree polynomial equation, then its solutions can be obtained by formulas expressed as functions of the initial coefficients and ultimately represented in terms of radicals. Each case of the polynomial discussed in this article is presented in the “problem–proof” format.

**Keywords:** sixth polynomial; sextic polynomial; sixth degree polynomial equation; solvability; galois theory; radical solution; radical roots; radical expression

## 1. Introduction

It has been known since the 18th century that mathematicians such as N. Abel, P. Ruffini, and E. Galois contributed to the proof that general polynomial equations of degree five or higher cannot be solved by closed-form radicals. This means that the solutions of polynomial equations must be expressed in terms of relations and functions involving their coefficients, using only the standard operations (addition, multiplication, division, exponentiation, extraction of roots, etc.). For example, the formulas of G. Cardano, S. Ferro, and N. Tartaglia for cubic equations [1], as well as the L. Ferrari's method for quartic equations [2], and the various formulas and methods for quadratic trinomials known to date [3–5], all above formulas produce roots expressible by closed-form solutions. However, certain polynomial equations of degree  $n \geq 5$  are solvable by radicals when specific conditions are satisfied, such as algebraic relations among their coefficients. Moreover, some quintic polynomial equations can be solved by radicals if an appropriate parameterization of their coefficients is found.

For example, the parameterized polynomial of G. P. Young [6], those studied by Spearman and Kenneth [7], as well as other more specialized quintic forms such as the Brioschi equation [8,9] are commonly mentioned. Similar conditions can be introduced for sixth-degree polynomial equations. In the present study, sixth-degree polynomial equations are divided into two categories, based on whether or not they can be solved by radicals. The solvability of an equation in terms of radical expressions depends on the Galois group to which the polynomial belongs. However, determining the Galois group for a sextic polynomial requires further investigation.

Let  $P(x)$  the general sextic polynomial equation:

$$P(x) = x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$$

With  $(a_1, \dots, a_6) \in \mathbb{R}$  or  $\mathbb{C}$  and  $a_6 \neq 0$ . Let  $S(x)$  be a second polynomial in which the coefficients are functions of the coefficients of  $P(x)$ . In addition, set  $P(x) = 0$ . The table below presents the solvable and non-solvable by radicals, sixth-degree polynomials  $S(x)$ .

**Table 1.** Sixth-degree polynomial equations that are solvable by radicals and the methodology for solving them.

Coefficients of P(x)	Polynomial S(x)	Solution methodology (S(x) = 0)
$(a_1, a_3, a_5) = 0$	$x^6 + a_2x^4 + a_4x^2 + a_6$	S(x) can be transformed into a cubic equation by applying the substitution, $x^2 = y$ . The resulting cubic polynomial can be solved using Cardano formula.
$(a_1, a_2, a_4, a_5) = 0$	$x^6 + a_3x^3 + a_6$	1. S(x) can be transformed into a quadratic trinomial by applying the substitution, $x^3 = y$ . 2. Alternatively, S(x) can be transformed into a biquadratic (fourth-degree) trinomial using the transformation $x^3 = y^2$ . The resulting equation can then be solved either by applying the further transformation $y^2 = z$ or by using Ferrari's formula, although the latter is a more complex methodology.
All coefficients are equal to zero except $a_6$ .	$x^6 + a_6$	$x^6 = (-a_6)^{\frac{1}{6}}$
$(a_1, a_2, a_3) = 0$ $a_4 = 1$	$x^6 + x^2 + a_5x + a_6$	If S(x) can be transformed into the form $x^6 + (x+p)(x+q)$ , then it can be solved in terms of the hypergeometric function ${}_2F_1$ [10,11].
$(a_1, a_2, a_3, a_4) = 0$ $a_5 = -1$	$x^6 - x - a_6$	The S(x) can be solved in terms of the H Fox hypergeometric function [12,13].

**Table 2.** Sixth-degree polynomial equations that are solvable by radicals, only when certain conditions are satisfied for the constant term  $a_6$ .

Coefficients of P(x)	Polynomial S(x)	Conditions for S(x)
-	$x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$	$a_6 = f(a_1 \dots a_5)$
$a_1 = 0$	$x^6 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$	$a_6 = f(a_2 \dots a_5)$
$(a_2, a_3, a_4, a_5) = 0$	$x^6 + a_1x^5 + a_6$	$a_6 = f(a_1)$
$(a_1, a_2, a_3, a_4) = 0$	$x^6 + a_5x + a_6$	$a_6 = f(a_5)$

The necessary conditions under which the polynomial S(x) is able to produce radical expressions for its roots are discussed in the following sections (Sections 2-4). For completeness and clarity of presentation, formulas that calculate the roots of cubic and quartic polynomial equations are also included.

### 1.1. The Roots of Third Degree Polynomial Equations

Let the general cubic equation  $H(x) = x^3 + ax^2 + bx + c$  with  $(a, b, c) \in \mathbb{R}$  or  $\mathbb{C}$  and  $c \neq 0$ . If  $H(x) = 0$  then the roots of H(x) can be calculated using Cardano's formula as follows:

$$x_1 = -\frac{a}{3} + \left( -\frac{B}{2} + \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2} \right)^{\frac{1}{3}} + \left( -\frac{B}{2} - \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2} \right)^{\frac{1}{3}}$$

$$x_2 = -\frac{a}{3} + \omega_1 \left( -\frac{B}{2} + \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2} \right)^{\frac{1}{3}} + \omega_2 \left( -\frac{B}{2} - \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2} \right)^{\frac{1}{3}}$$

$$x_3 = -\frac{a}{3} + \omega_2 \left( -\frac{B}{2} + \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2} \right)^{\frac{1}{3}} + \omega_1 \left( -\frac{B}{2} - \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2} \right)^{\frac{1}{3}}$$

where A and B are coefficients expressed as functions of the original coefficients a, b, c of H(x), and  $\omega_1, \omega_2$  are the cubic roots of unity.

$$A = b - \frac{a^2}{3}, \quad B = c - \frac{ab}{3} + \frac{2a^3}{27}, \quad \omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \omega_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

### 1.2. The Roots of Fourth Degree Polynomial Equations

Let the general quartic equation  $H(x) = x^4 + ax^3 + bx^2 + cx + d$  with  $(a, b, c, d) \in \mathbb{R}$  or  $\mathbb{C}$  and  $d \neq 0$ . If  $H(x) = 0$  then the roots of H(x) can be calculated using Ferrari's formula as follows:

$$x_{1,2} = -\frac{a}{4} + \frac{1}{2} \left( -\sqrt{2y - A} \pm \sqrt{-2y - A + \frac{2B}{\sqrt{2y - A}}} \right)$$

$$x_{3,4} = -\frac{a}{4} + \frac{1}{2} \left( \sqrt{2y - A} \pm \sqrt{-2y - A + \frac{2B}{\sqrt{2y - A}}} \right)$$

where y is the root of the resolvent of H(x), which is a cubic equation.

$$y = \frac{A}{6} + \left( -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{\frac{1}{3}} - \frac{p}{3 \left( -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{\frac{1}{3}}}$$

where p and q are the coefficients expressed as functions of the resolvent coefficients.

$$p = -\frac{A^2}{12} - C, \quad q = -\frac{A^3}{108} + \frac{AC}{3} - \frac{B^2}{8}$$

where A, B and C are coefficients expressed as a function of the initial coefficients a, b, c, and d of H(x).

$$A = -\frac{3a^2}{8} + b, \quad B = \frac{a^3}{8} - \frac{ab}{2} + c, \quad C = -\frac{3a^4}{256} + \frac{a^2b}{16} - \frac{ac}{4} + d$$

## 2. First Case – The General Sextic Polynomial Equation

**Problem 1.** Let the polynomial  $P(x) = x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$  with  $a_6 \neq 0$  and  $(a_1 \dots a_6) \in \mathbb{R}$  or  $\mathbb{C}$ . Prove that the roots  $x_1 \dots x_6$  of  $P(x)$  can be expressed by radicals; that is,  $(x_1 \dots x_6) = f(a_1 \dots a_6)$ , if the constant term of  $P(x)$  satisfies  $a_6 = f(a_1 \dots a_5)$ ,

**Proof.** The polynomial  $P(x)$  can be transformed into a depressed sextic polynomial  $S(y)$  without the quintic term (i.e., with  $a_1 = 0$ ) using the Tschirnhaus transformation  $x = y - a_1/6$ .

$$\left(y - \frac{a_1}{6}\right)^6 + a_1\left(y - \frac{a_1}{6}\right)^5 + a_2\left(y - \frac{a_1}{6}\right)^4 + a_3\left(y - \frac{a_1}{6}\right)^3 + a_4\left(y - \frac{a_1}{6}\right)^2 + a_5\left(y - \frac{a_1}{6}\right) + a_6$$

$$\iff S(y) = y^6 + py^4 + qy^3 + ry^2 + sy + t, \quad t \neq 0$$

where p, q, r, s, t are real or complex coefficients expressed as functions of the original coefficients of  $P(x)$ ; that is,  $(p, q, r, s, t) = f(a_1 \dots a_6)$ . More explicitly:

$$p = -\frac{5}{12}a_1^2 + a_2$$

$$q = \frac{5}{27}a_1^3 - \frac{2}{3}a_1a_2 + a_3$$

$$r = -\frac{5}{144}\alpha_1^4 + \frac{1}{6}\alpha_1^2\alpha_2 - \frac{1}{2}\alpha_1\alpha_3 + \alpha_4$$

$$s = \frac{1}{324}\alpha_1^5 - \frac{1}{54}\alpha_1^3\alpha_2 + \frac{1}{12}\alpha_1^2\alpha_3 - \frac{1}{3}\alpha_1\alpha_4$$

$$t = -\frac{5}{46656}\alpha_1^6 + \frac{1}{1296}\alpha_1^4\alpha_2 - \frac{1}{216}\alpha_1^3\alpha_3 + \frac{1}{36}\alpha_1^2\alpha_4 + \frac{1}{6}\alpha_1\alpha_5 + \alpha_6$$

In the next step, consider  $G(y)$  as the product of two cubic trinomials without quadratic term:

$$G(y) = (y^3 + my + n)(y^3 + uy + v) = 0$$

where  $(m, n, u, v) \in \mathbb{R}$  or  $\mathbb{C}$ . Both trinomials as well as  $G(y)$  can be solved by Cardano formula. The result of the product  $G(y)$  is as follows:

$$G(y) = y^6 + (m + u)y^4 + (n + v)y^3 + muy^2 + (mv + nu)y + nv = 0$$

□

It can be observed that  $G(y) \equiv S(y)$ ; that is, the two equations coincide and the corresponding terms of  $G(y)$  and  $S(y)$  are the same. Consequently, we obtain the following system of equations, consisting of five known and four unknown variables:

$$m + u = p$$

$$n + v = q$$

$$mu = r$$

$$mv + nu = s$$

$$nv = t$$

The above system provides four solutions for  $m, n, u, v$  and one equation. The solutions are as follows:

$$m = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - r}, \quad n = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} - t}$$

$$u = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - r}, \quad v = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} - t}$$

By substituting  $m, n, u, v$  into  $G(y)$ , we obtain the two following cubic trinomials.

$$y^3 + \left(\frac{p}{2} + \sqrt{\frac{p^2}{4} - r}\right)y + \left(\frac{q}{2} + \sqrt{\frac{q^2}{4} - t}\right)$$

$$y^3 + \left(\frac{p}{2} - \sqrt{\frac{p^2}{4} - r}\right)y + \left(\frac{q}{2} - \sqrt{\frac{q^2}{4} - t}\right)$$

The solutions of the cubic trinomials are also roots of  $S(y)$ . The last term  $t$  of  $S(y)$  is calculated using the following equation:

$$t = nv = \frac{q^2}{4} - \frac{(pq - 2s)^2}{4(p^2 - 4r)}, \quad p^2 - 4r \neq 0$$

It can be seen that the last term  $t$  is expressed as a function of the coefficients of  $S(y)$ ; that is,  $t = f(p, q, r, s)$ . Furthermore, it has already been shown that  $(p, q, r, s, t) = f(a_1 \dots a_6)$  and hence  $t = f(a_1 \dots a_6)$ . For the last term  $a_6$  of  $P(x)$  we have:

$$a_6 = \frac{5}{46656}a_1^6 - \frac{1}{1296}a_1^4a_2 + \frac{1}{216}a_1^3a_3 - \frac{1}{36}a_1^2a_4 - \frac{1}{6}a_1a_5 + t$$

If we set  $f_1$  and  $f_2$  as follows:

$$f_1(a_1 \dots a_5) = \frac{5}{46656}a_1^6 - \frac{1}{1296}a_1^4a_2 + \frac{1}{216}a_1^3a_3 - \frac{1}{36}a_1^2a_4 - \frac{1}{6}a_1a_5$$

$$f_2(a_1 \dots a_6) = t$$

Finally we have:

$$a_6 = f_1(a_1 \dots a_5) + f_2(a_1 \dots a_6) \iff$$

$$a_6 = f(a_1 \dots a_6)$$

Finally, we conclude that the general sextic polynomial  $P(x)$  has six roots that can be expressed by radicals, provided that the last term of  $P(x)$  can be expressed as a function of its coefficients. The roots of  $P(x)$  are given as follows:

$$\begin{aligned} x_1 &= -\frac{a_1}{6} + \left(-\frac{q}{4} - \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_3}\right)^{\frac{1}{3}} + \left(-\frac{q}{4} - \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_3}\right)^{\frac{1}{3}} \\ x_2 &= -\frac{a_1}{6} + \omega_1 \left(-\frac{q}{4} - \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_3}\right)^{\frac{1}{3}} + \omega_2 \left(-\frac{q}{4} - \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_3}\right)^{\frac{1}{3}} \\ x_3 &= -\frac{a_1}{6} + \omega_2 \left(-\frac{q}{4} - \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_3}\right)^{\frac{1}{3}} + \omega_1 \left(-\frac{q}{4} - \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_3}\right)^{\frac{1}{3}} \\ x_4 &= -\frac{a_1}{6} + \left(-\frac{q}{4} + \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_4}\right)^{\frac{1}{3}} + \left(-\frac{q}{4} + \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_4}\right)^{\frac{1}{3}} \\ x_5 &= -\frac{a_1}{6} + \omega_1 \left(-\frac{q}{4} + \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_4}\right)^{\frac{1}{3}} + \omega_2 \left(-\frac{q}{4} + \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_4}\right)^{\frac{1}{3}} \\ x_6 &= -\frac{a_1}{6} + \omega_2 \left(-\frac{q}{4} + \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_4}\right)^{\frac{1}{3}} + \omega_1 \left(-\frac{q}{4} + \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_4}\right)^{\frac{1}{3}} \end{aligned}$$

where  $\Delta_1 \dots \Delta_4$  are the discriminants of  $P(x)$  and  $\Omega_1, \Omega_2$  are the cubic roots of unity. It is observed that  $(\Delta_3, \Delta_4) = f(\Delta_1, \Delta_2)$ .

$$\Delta_1 = \frac{q^2}{4} - t, \quad \Delta_2 = \frac{p^2}{4} - r$$

$$\Delta_3 = \left(\frac{q}{4} + \frac{\sqrt{\Delta_1}}{2}\right)^2 + \left(\frac{p}{6} + \frac{\sqrt{\Delta_2}}{3}\right)^3, \quad \Delta_4 = \left(\frac{q}{4} - \frac{\sqrt{\Delta_1}}{2}\right)^2 + \left(\frac{p}{6} - \frac{\sqrt{\Delta_2}}{3}\right)^3$$

where  $(p, q, r, s, t)$  are coefficients calculated by  $a_1 \dots a_6$ . The basic conditions that must be satisfied to express the roots of both  $S(y)$  and  $P(x)$  polynomials as functions of their initial coefficients are as follows [14]:

$$t = \frac{q^2}{4} - \frac{(pq - 2s)^2}{4(p^2 - 4r)} \quad \text{and} \quad p^2 - 4r \neq 0$$

### 3. Second Case – The Sextic Equation $x^6 + a_1x^5 + a_6$

**Problem 2.** Let the polynomial  $P(x) = x^6 + a_1x^5 + a_6$  where  $a_1a_6 \neq 0$  and  $(a_1, a_6) \in \mathbb{R}^* \text{ or } \mathbb{C}$ . Prove that the roots  $x_1 \dots x_6$  of  $P(x)$  can be expressed by radicals, hence  $(x_1 \dots x_6) = f(a_1, a_6)$ , provided that the constant term of  $P(x)$  satisfies  $a_6 = f(a_1)$ ,

**Proof.** We follow a technique similar to that used in the previous case. Using the Tschirnhaus transformation  $x = y - a_1/6$ , we construct a new polynomial  $S(y)$ . The above substitution eliminates the quintic term of  $P(x)$ , and therefore the transformed polynomial  $S(y)$  takes the following form:

$$P\left(y - \frac{a_1}{6}\right) = \left(y - \frac{a_1}{6}\right)^6 + a_1\left(y - \frac{a_1}{6}\right)^5 + a_6 \iff$$

$$S(y) = y^6 - \left(\frac{5}{12}a_1^2\right)y^4 + \left(\frac{5}{27}a_1^3\right)y^3 - \left(\frac{5}{144}a_1^4\right)y^2 + \left(\frac{1}{324}a_1^5\right)y - \frac{5}{46656}a_1^6 + a_6 \iff$$

$$S(y) = y^6 + py^4 + qy^3 + ry^2 + sy + t, \quad t \neq 0$$

where  $p, q, r, s, t$  are real or complex coefficients expressed as functions of the initial coefficients of  $P(x)$ . Hence  $(p, q, r, s) = f(a_1)$  and  $t = f(a_1, a_6)$ . The roots of  $S(y)$  can be calculated and expressed by radicals using a procedure similar to that of the previous case. It has already been proven that  $t = f(a_1, a_6)$ . Therefore, the polynomial  $P(x)$  has the following roots:

$$x_1 = -\frac{a_1}{6} + \left(-\frac{5}{108}a_1^3 - \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_3}\right)^{\frac{1}{3}} + \left(-\frac{5}{108}a_1^3 - \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_3}\right)^{\frac{1}{3}}$$

$$x_2 = -\frac{a_1}{6} + \omega_1\left(-\frac{5}{108}a_1^3 - \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_3}\right)^{\frac{1}{3}} + \omega_2\left(-\frac{5}{108}a_1^3 - \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_3}\right)^{\frac{1}{3}}$$

$$x_3 = -\frac{a_1}{6} + \omega_2\left(-\frac{5}{108}a_1^3 - \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_3}\right)^{\frac{1}{3}} + \omega_1\left(-\frac{5}{108}a_1^3 - \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_3}\right)^{\frac{1}{3}}$$

$$x_4 = -\frac{a_1}{6} + \left(-\frac{5}{108}a_1^3 + \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_4}\right)^{\frac{1}{3}} + \left(-\frac{5}{108}a_1^3 + \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_4}\right)^{\frac{1}{3}}$$

$$x_5 = -\frac{a_1}{6} + \omega_1\left(-\frac{5}{108}a_1^3 + \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_4}\right)^{\frac{1}{3}} + \omega_2\left(-\frac{5}{108}a_1^3 + \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_4}\right)^{\frac{1}{3}}$$

$$x_6 = -\frac{a_1}{6} + \omega_2\left(-\frac{5}{108}a_1^3 + \frac{\sqrt{\Delta_1}}{2} + \sqrt{\Delta_4}\right)^{\frac{1}{3}} + \omega_1\left(-\frac{5}{108}a_1^3 + \frac{\sqrt{\Delta_1}}{2} - \sqrt{\Delta_4}\right)^{\frac{1}{3}}$$

where  $\Delta_1 \dots \Delta_4$  are the discriminants of the  $P(x)$  and  $\Omega_1, \Omega_2$  are the cubic roots of unity. It is observed that  $(\Delta_3, \Delta_4) = f(\Delta_1, \Delta_2)$ .

$$\Delta_1 = \frac{5}{576}a_1^6 - a_6, \quad \Delta_2 = \frac{5}{64}a_1^4$$

$$\Delta_3 = \left(\frac{5}{108}a_1^3 + \frac{\sqrt{\Delta_1}}{2}\right)^2 + \left(-\frac{5}{72}a_1^2 + \frac{\sqrt{\Delta_2}}{3}\right)^3$$

$$\Delta_4 = \left(\frac{5}{108}a_1^3 - \frac{\sqrt{\Delta_1}}{2}\right)^2 + \left(-\frac{5}{72}a_1^2 - \frac{\sqrt{\Delta_2}}{3}\right)^3$$

The conditions that must be satisfied in order to express the roots of both of both  $S(y)$  and  $P(x)$  polynomials by radicals are the same as in the previous case:

$$t = \frac{q^2}{4} - \frac{(pq - 2s)^2}{4(p^2 - 4r)} \quad \text{and} \quad p^2 - 4r \neq 0$$

More specifically, after substituting the coefficients p, q, r, s, t, we obtain:

$$a_6 = -\frac{6439}{233280}a_1^6 \text{ and } \left(\frac{5}{12}a_1^2\right)^2 + 4\left(\frac{5}{144}a_1^4\right) \neq 0 \iff a_1 \neq 0$$

□

#### 4. Third Case – The Bring–Jerrard Sextic Equation $x^6 + a_5x + a_6$

**4.1. Problem 3, subcase 1:** Let the polynomial  $P(x) = x^6 + a_5x + a_6$  where  $a_5a_6 \neq 0$  and  $(a_5, a_6) \in \mathbb{R}^*$  or  $\mathbb{C}$ . Prove that the roots  $x_1 \dots x_6$  of  $P(x)$  can be expressed by radicals, that is,  $(x_1 \dots x_6) = f(a_5)$ , provided that the constant term of  $P(x)$  satisfies  $a_6 = f(a_5)$ .

**Proof.** The methodology used in the two previous cases cannot be applied in this particular case. If we consider  $a_1 = a_2 = a_3 = a_4 = 0$  in the general sextic polynomial  $P(x)$ , then we obtain the sextic polynomial Bring–Jerrard  $S(y) = y^6 + py + q$ . However,  $S(y)$  cannot be solved using the previous formulas, since in this case the following limitations:

$$p^2 - 4r = 0, pqrs = 0, t = a_6.$$

Therefore, an alternative technique is required to solve this particular polynomial. If we apply the substitution  $x = y + k$  with  $k \in \mathbb{R}^*$  to the Bring–Jerrard form of  $P(x)$ , we obtain the following expression:

$$P(y+k) = (y+k)^6 + a_5(y+k) + a_6 \iff$$

$$S(y) = y^6 + 6ky^5 + 15k^2y^4 + 20k^3y^3 + 15k^4y^2 + (6k^5 + a_5)y + k^6 + ka_5 + a_6$$

The k parameter is chosen so that the term  $6k^5 + a_5$  is eliminated. Hence:

$$6k^5 + a_5 = 0 \iff k = \left(-\frac{a_5}{6}\right)^{\frac{1}{5}}$$

Equation  $S(y)$  obtains the following form without primary term:

$$S(y) = y^6 + 6ky^5 + 15k^2y^4 + 20k^3y^3 + 15k^4y^2 + k^6 + ka_5 + a_6$$

We set  $a_6 = -(k^6 + ka_5)$  to make  $S(y)$  solvable by radicals. This requirement eliminates the constant coefficient of  $S(y)$ . Consequently,  $S(y)$  obtains the following form:

$$S(y) = y^6 + 6ky^5 + 15k^2y^4 + 20k^3y^3 + 15k^4y^2$$

The above form of  $S(y)$  can be solved by radicals as follows:

$$S(y) = y^2 \left( y^4 + 6ky^3 + 15k^2y^2 + 20k^3y + 15k^4 \right), y_{1,2} = 0 \text{ or}$$

$$y^4 + 6ky^3 + 15k^2y^2 + 20k^3y + 15k^4 = 0 \iff$$

$$y^4 + 6\left(-\frac{a_5}{6}\right)^{\frac{1}{5}}y^3 + 15\left(\frac{a_5}{6}\right)^{\frac{2}{5}}y^2 + 20\left(-\frac{a_5}{6}\right)^{\frac{3}{5}}y + 15\left(\frac{a_5}{6}\right)^{\frac{4}{5}} = 0$$

The remaining four roots of  $S(y)$  can be determined using Ferrari's formula. The Bring–Jerrard sextic polynomial  $P(x)$  ultimately has one double root and four additional roots, which are as follows.

$$x_{1,2} = \left(-\frac{a_5}{6}\right)^{\frac{1}{5}}$$

$$x_{3,4} = -\frac{2}{3}\left(-\frac{a_5}{6}\right)^{\frac{1}{5}} + \frac{1}{2}\left(-\sqrt{2y-A} \pm \sqrt{-2y-A + \frac{2B}{\sqrt{2y-A}}}\right)$$

$$x_{5,6} = -\frac{2}{3}\left(-\frac{a_5}{6}\right)^{\frac{1}{5}} + \frac{1}{2}\left(\sqrt{2y-A} \pm \sqrt{-2y-A + \frac{2B}{\sqrt{2y-A}}}\right)$$

where,  $y$  is the root of the resolvent of  $S(y)$ .

$$y = \frac{A}{6} + \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} - \frac{p}{3\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}}$$

Where  $p$  and  $q$  are:

$$p = -\frac{15}{4}\left(\frac{a_5}{6}\right)^{\frac{4}{5}}, \quad q = \frac{4279}{3424}\left(\frac{a_5}{6}\right)^{\frac{6}{5}}$$

where  $A$ ,  $B$ , and  $C$  are coefficients expressed as functions of the initial coefficients of  $S(y)$ .

$$A = \frac{3}{2}\left(\frac{a_5}{6}\right)^{\frac{2}{5}}, \quad B = -2\left(\frac{a_5}{6}\right)^{\frac{3}{5}},$$

The condition that must be satisfied for the constant term in order to express the roots of both  $S(y)$  and  $P(x)$  polynomials by radicals is as follows:

$$a_6 = -(k^6 + ka_5) \iff$$

$$a_6 = -\left[\left(-\frac{a_5}{6}\right)^{\frac{6}{5}} + a_5\left(-\frac{a_5}{6}\right)^{\frac{1}{5}}\right] \iff$$

$$a_6 = -a_5^{\frac{6}{5}}\left(\frac{1}{6^{\frac{6}{5}}} - \frac{1}{6^{\frac{1}{5}}}\right) \iff$$

$$a_6 = 5\left(\frac{a_5}{6}\right)^{\frac{6}{5}}$$

□

**4.2. Problem 3, subcase 2:** Let the polynomial  $P(x) = x^6 + a_5x + a_6$  where  $a_5a_6 \neq 0$  and  $(a_5, a_6) \in \mathbb{R}^*$  or  $\mathbb{C}$ . Prove that the roots  $x_1 \dots x_6$  of  $P(x)$  can be expressed by radicals, hence  $(x_1 \dots x_6) = f(a_5)$ , provided that the constant term of  $P(x)$  satisfies  $a_6 = f(a_5, y)$ .

**Proof.** The same methodology as in subcase 1 is followed. The difference in this subcase focused on the condition on the constant term of  $S(y)$ . More specifically, the  $S(y)$  polynomial is given by:

$$S(y) = y^6 + 6ky^5 + 15k^2y^4 + 20k^3y^3 + 15k^4y^2 + k^6 + ka_5 + a_6 \quad \text{with } k = \left(-\frac{a_5}{6}\right)^{\frac{1}{5}}$$

We set  $a_6 = -(6ky^5 + 20k^3y^3)$  to make  $S(y)$  solvable by radicals. This requirement eliminates both quintic and cubic term of  $S(y)$ . Consequently,  $S(y)$  obtains the following form:

$$S(y) = y^6 + 15k^2y^4 + 15k^4y^2 + k^6 + ka_5 \iff$$

$$S(y) = y^6 + 15\left(-\frac{a_5}{6}\right)^{\frac{2}{5}}y^4 + 15\left(-\frac{a_5}{6}\right)^{\frac{4}{5}}y^2 + \left(-\frac{a_5}{6}\right)^{\frac{6}{5}} + \left(-\frac{a_5}{6}\right)^{\frac{1}{5}}a_5 \iff$$

$$S(y) = y^6 + 15\left(-\frac{a_5}{6}\right)^{\frac{2}{5}}y^4 + 15\left(-\frac{a_5}{6}\right)^{\frac{4}{5}}y^2 + 5\left(-\frac{a_5}{6}\right)^{\frac{6}{5}}$$

The above sixth-degree equation can be transformed into a cubic equation using the transformation  $y^2 = z$  which can then be solved by Cardano's formula. Finally, in this particular subcase, the Bring-Jerrard polynomial  $P(x)$  has the following six roots, all of which can be expressed by radicals:

$$x_{1,2} = \left(-\frac{a_5}{6}\right)^{\frac{1}{5}} \pm \sqrt{-5\left(\frac{a_5}{6}\right)^{\frac{2}{5}} + \left(-\frac{B}{2} + \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2}\right)^{\frac{1}{3}} + \left(-\frac{B}{2} - \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2}\right)^{\frac{1}{3}}}$$

$$x_{3,4} = \left(-\frac{a_5}{6}\right)^{\frac{1}{5}} \pm \sqrt{-5\left(\frac{a_5}{6}\right)^{\frac{2}{5}} + \omega_1\left(-\frac{B}{2} + \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2}\right)^{\frac{1}{3}} + \omega_2\left(-\frac{B}{2} - \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2}\right)^{\frac{1}{3}}}$$

$$x_{5,6} = \left(-\frac{a_5}{6}\right)^{\frac{1}{5}} \pm \sqrt{-5\left(\frac{a_5}{6}\right)^{\frac{2}{5}} + \omega_2\left(-\frac{B}{2} + \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2}\right)^{\frac{1}{3}} + \omega_1\left(-\frac{B}{2} - \sqrt{\left(\frac{A}{3}\right)^3 + \left(\frac{B}{2}\right)^2}\right)^{\frac{1}{3}}}$$

where  $A$  and  $B$  are coefficients expressed as functions of initial coefficients of  $S(y)$ .

$$A = -60\left(\frac{a_5}{6}\right)^{\frac{4}{5}}, \quad B = 180\left(\frac{a_5}{6}\right)^{\frac{6}{5}}$$

The condition that must be satisfied for the constant term in order to express the roots of both  $S(y)$  and  $P(x)$  polynomials by radicals is as follows:

$$a_6 = -(6ky^5 + 20k^3y^3) \iff$$

$$a_6 = 6\left(\frac{a_5}{6}\right)^{\frac{1}{5}}y^5 + 20\left(\frac{a_5}{6}\right)^{\frac{3}{5}}y^3 \iff$$

$$a_6 = 50\sqrt{125}\left(\frac{a_5}{6}\right)^{\frac{6}{5}}$$

where  $y$  is one of the six roots of  $S(y)$ :

$$y = \sqrt{5}\left(\frac{a_5}{6}\right)^{\frac{4}{5}}$$

□

**4.3. Problem 3, subcase 3:** Let the polynomial  $P(x) = x^6 + a_5x + a_6$  with  $a_5a_6 \neq 0$  and  $(a_5, a_6) \in \mathbb{R}^*$  or  $\mathbb{C}$ . Prove that the roots  $x_1 \dots x_6$  of  $P(x)$  can be expressed by radicals—hence  $(x_1 \dots x_6) = f(a_5)$ —if for the constant term of  $P(x)$  satisfies  $a_6 = f(a_5, y)$ .

**Proof.** The same methodology as in subcase 1 is followed. If we apply the substitution  $x = y + k$  with  $k \in \mathbb{R}^*$  to the Bring-Jerrard polynomial  $P(x)$ , then we obtain the following expression:

$$P(y + k) = (y + k)^6 + a_5(y + k) + a_6 \iff$$

$$S(y) = y^6 + 6ky^5 + 15k^2y^4 + 20k^3y^3 + 15k^4y^2 + (6k^5 + a_5)y + k^6 + ka_5 + a_6$$

The  $k$  parameter is chosen so that the term  $k^6 + ka_5$  is eliminated. Hence:

$$k^6 + ka_5 = 0 \iff k = 0 \text{ or } k = (-a_5)^{\frac{1}{5}}$$

With replacement of the k parameter, the equation S(y) obtains the following form:

$$S(y) = y^6 + 6(-a_5)^{\frac{1}{5}}y^5 + 15(-a_5)^{\frac{2}{5}}y^4 + 20(-a_5)^{\frac{3}{5}}y^3 + 15(-a_5)^{\frac{4}{5}}y^2 - 5a_5y + a_6$$

Both S(y) and the initial polynomial P(x) can be solved by radicals depending on the relationship between the constant term  $a_6$ , the coefficient  $a_5$ , and the root of S(y). The various relations of  $a_6$  as a function of  $a_5$  are described in the following subcases.  $\square$

#### Subcase 4.3.1

I

$$S(y) = y^6 + 6(-a_5)^{\frac{1}{5}}y^5 + 15(-a_5)^{\frac{2}{5}}y^4 + 20(-a_5)^{\frac{3}{5}}y^3 + 15(-a_5)^{\frac{4}{5}}y^2 \iff$$

$$y = 0 \text{ or } y^4 - 6(a_5)^{\frac{1}{5}}y^3 + 15(a_5)^{\frac{2}{5}}y^2 - 20(a_5)^{\frac{3}{5}}y + 15(a_5)^{\frac{4}{5}} = 0$$

The roots of S(y) can be calculated using Ferrari's formula. The same method as in problem 3 – subcase 1 as well as problem 1, is followed. After numerous calculations, the roots y of S(y) are as follows:

$$x_{1,2} = (a_5)^{\frac{1}{5}} \left[ -\frac{3}{2} + \sqrt{-\frac{z^2}{2} - \frac{z}{8} - 1} \pm \sqrt{-\frac{z^2}{2} - \frac{z}{8} - 1 - \frac{1}{\sqrt{2z^2 + \frac{z}{2} + 4}}} \right]$$

$$x_{3,4} = (a_5)^{\frac{1}{5}} \left[ -\frac{3}{2} - \sqrt{-\frac{z^2}{2} - \frac{z}{8} - 1} \pm \sqrt{-\frac{z^2}{2} - \frac{z}{8} - 1 - \frac{4}{\sqrt{2z^2 + \frac{z}{2} + 4}}} \right]$$

where:

$$z = \left( -\frac{5}{8}a_5^{\frac{6}{5}} + \frac{5}{4}i \right)^{\frac{1}{3}}$$

#### Subcase 4.3.2

If we set the constant term:

$$a_6 = 5a_5y - 15(-a_5)^{\frac{4}{5}}y^2$$

Then we obtain the polynomial:

$$S(y) = y^6 + 6(-a_5)^{\frac{1}{5}}y^5 + 15(-a_5)^{\frac{2}{5}}y^4 + 20(-a_5)^{\frac{3}{5}}y^3 \iff$$

$$y = 0 \text{ or } y^3 - 6(a_5)^{\frac{1}{5}}y^2 + 15(a_5)^{\frac{2}{5}}y - 20(a_5)^{\frac{3}{5}} = 0$$

The roots of S(y) can be calculated using Cardano's formula, following the same method as in problem 3 – subcase 2 and problem 2.

$$y_1 = 0 \text{ and } y_{2,3} = (a_5)^{\frac{1}{5}} \left( 1 \pm 2i\sqrt{3} \right)$$

The constant term  $a_6$  has three different values:

$$a_{6,1} = 0 \text{ or } a_{6,(2,3)} = 10(a_5)^{\frac{6}{5}} \left( -16 \pm 7i\sqrt{3} \right)$$

The roots of P(x) in this case are:

$$x_{1,2,3,4} = (-a_5)^{\frac{1}{5}} \text{ and } x_{5,6} = \pm 2i\sqrt{3} (a_5)^{\frac{1}{5}}$$

#### Subcase 4.3.3

If we set the constant term:

$$a_6 = 5a_5y - 15(-a_5)^{\frac{4}{5}}y^2 - 20(-a_5)^{\frac{3}{5}}y^3$$

Then we obtain the polynomial:

$$S(y) = y^6 + 6(-a_5)^{\frac{1}{5}}y^5 + 15(-a_5)^{\frac{2}{5}}y^4 \iff$$

$$y = 0 \text{ or } y^2 - 6(a_5)^{\frac{1}{5}}y + 15(a_5)^{\frac{2}{5}} = 0$$

The  $S(y)$  is a quadratic trinomial. The roots of  $P(x)$  are:

$$x_{1,2,3,4} = (-a_5)^{\frac{1}{5}}$$

$$x_{5,6} = 2(a_5)^{\frac{1}{5}} \pm \sqrt{9(a_5)^{\frac{1}{5}} - 15(a_5)^{\frac{2}{5}}}$$

The following is the relationship of the constant term  $a_6$  based on the calculated root of  $S(y)$ :

$$a_6 = (a_5)^{\frac{4}{5}} \left[ \left( 180 - 295a_5^{\frac{1}{5}} \right) \sqrt{9a_5^{\frac{1}{5}} - 15a_5^{\frac{2}{5}}} - 135a_5^{\frac{1}{5}} + 225a_5^{\frac{2}{5}} \right]$$

#### Subcase 4.3.4

If we set the constant term:

$$a_6 = 5a_5y - 15(-a_5)^{\frac{4}{5}}y^2 - 20(-a_5)^{\frac{3}{5}}y^3 - 15(-a_5)^{\frac{2}{5}}y^4$$

Then we obtain the polynomial:

$$S(y) = y^6 + 6(-a_5)^{\frac{1}{5}}y^5 \iff$$

$$y = 0 \text{ or } y - 6(a_5)^{\frac{1}{5}} = 0 \iff y = 6(a_5)^{\frac{1}{5}}$$

The roots of  $P(x)$  are as follows:

$$x_{1,2,3,4,5} = (-a_5)^{\frac{1}{5}}$$

$$x_6 = 2(a_5)^{\frac{1}{5}} + (-a_5)^{\frac{1}{5}} = a_5^{\frac{1}{5}}$$

The following is the simplified relationship of the constant term  $a_6$  based on the calculated root of  $S(y)$ :

$$a_6 = 5a_5y - 15(-a_5)^{\frac{4}{5}}y^2 - 20(-a_5)^{\frac{3}{5}}y^3 - 15(-a_5)^{\frac{2}{5}}y^4 \iff$$

$$a_6 = 30(a_5)^{\frac{6}{5}} - 540(-a_5)^{\frac{6}{5}} - 4320(-a_5)^{\frac{6}{5}} - 19440(-a_5)^{\frac{6}{5}} \iff$$

$$a_6 = -24270(a_5)^{\frac{6}{5}}$$

## 5. Resume

Contents	Polynomial P(x)	Relation among coefficients in the last term $a_6$
First case	$x^6 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$	$a_6 = \frac{a_2^2}{4} - \frac{(a_2a_3 - 2a_5)^2}{4(a_2^2 - 4a_4)}$
Second case	$x^6 + a_1x^5 + a_6$	$a_6 = -\frac{6439}{233280}a_1^6$
Third case Problem 3 – Subcase 1		$a_6 = 5\left(\frac{a_5}{6}\right)^{\frac{6}{5}}$
Third case Problem 3 – Subcase 2		$a_6 = 6\left(\frac{a_5}{6}\right)^{\frac{1}{5}}y^5 + 20\left(\frac{a_5}{6}\right)^{\frac{3}{5}}y^3 \Leftrightarrow a_6 = 50 \times 5^{\frac{3}{5}}\left(\frac{a_5}{6}\right)^{\frac{6}{5}}$
Third case Problem 3 – Subcase 3.1	$x^6 + a_5x + a_6$ <b>With:</b>	$a_6 = 5a_5y = 5a_5\left(-\frac{6}{4}\left(-\frac{a_5}{6}\right)^{\frac{1}{5}} + \frac{1}{2}\left(-\sqrt{2y-A} \pm \sqrt{-2y-A + \frac{2B}{\sqrt{2y-A}}}\right)\right)$ <p style="text-align: center;">where:</p> $y = \frac{a_5^{\frac{2}{3}}}{4}\left(-\frac{5}{8}a_5^{\frac{6}{3}} + \frac{5}{4}i\right) + \left(-\frac{5}{8}a_5^{\frac{6}{3}} + \frac{5}{4}i\right)^{\frac{2}{3}}a_5^{\frac{2}{3}} + \frac{5}{4}a_5^{\frac{2}{3}}$
Third case Problem 3 – Subcase 3.2		$a_6 = 0 \text{ or } a_6 = 10(a_5)^{\frac{6}{5}}(-16 \pm 7i\sqrt{3})$
Third case Problem 3 – Subcase 3.3		$a_6 = (a_5)^{\frac{4}{5}}\left[180 - 295a_5^{\frac{1}{5}}\right]\sqrt[4]{9a_5^{\frac{1}{5}} - 15a_5^{\frac{2}{5}} - 135a_5^{\frac{3}{5}} + 225a_5^{\frac{4}{5}}}$
Third case Problem 3 – Subcase 3.4		$a_6 = -24270(a_5)^{\frac{6}{5}}$

The previous table presents all cases of sixth-degree polynomial equations that can be solved in radicals under certain conditions. Specifically, each constant term must be a function of the other coefficients of the polynomial. In summary, eight different cases are finally mentioned.

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