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Article

A Study of the Jacobi Stability of the Rosenzweig–MacArthur Predator–prey System by the KCC Geometric Theory

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Abstract: In this paper, we will consider an autonomous two-dimensional ODE Kolmogorov type system with three parameters, which is a particular system of the general predator–prey systems with a Holling type II. By reformulating this system as a set of two second order differential equations, we will investigate the nonlinear dynamics of the system from the Jacobi stability point of view, using the Kosambi-Cartan-Chern (KCC) geometric theory. We will determine the nonlinear connection, the Berwald connection and the five KCC-invariants which express the intrinsic geometric properties of the system, including the deviation curvature tensor. Furthermore, we will obtain necessary and sufficient conditions on the parameters of the system in order to have the Jacobi stability near the equilibrium points and we will point out these on a few examples.

Keywords: predator–prey systems; Kolmogorov systems; KCC-theory; the deviation curvature tensor; Jacobi stability

1. Introduction

M. Rosenzweig and R. MacArthur introduced in [1] this predator-prey system in order to understanding the relationship between two populations which involves to destruction by members of one population of members of the other for the purpose of obtaining food. This system is a particular case of the predator-prey system of Holling's type II [2], [3]. The original Rosenzweig–MacArthur predator–prey system has the following form [1]:

$$\begin{cases} \dot{x} &= rx\left(1-\frac{x}{K}\right) - y\frac{mx}{b+x} \\ \dot{y} &= y\left(-\delta + c\frac{mx}{b+x}\right) \end{cases} \quad (1)$$

where the dot denotes the derivative with respect to the time t, $x \geq 0$ denotes the prey density (#/unit of area) and $y \geq 0$ denotes the predator density (#/unit of area), the parameter $\delta > 0$ is the death rate of the predator, the function $x \mapsto mx/(b+x)$ is the prey caught by predator per unit time, the function $x \mapsto rx/(1-x/K)$ is the growth of the prey in the absence of predator (by the logistic growth model), and $c > 0$ is the rate of the conversion of prey to predator.

The general case of the predator-prey systems with a Holling type response function P is given by

$$\begin{cases} \dot{x} &= rx\left(1-\frac{x}{K}\right) - yP(x) \\ \dot{y} &= y(-\delta + cP(x)) \end{cases} \quad (2)$$

with the same conditions on x, y and the parameters.

Depending of the expression of the function P, there are four types of predator-prey Holling's type systems, as follow [4]. If $P = mx$, then (2) is a predator-prey system of Holling's type I. If $P = \frac{mx}{b+x}$, then (2) is a predator-prey system of Holling's type II. In this case the function $P = \frac{mx}{b+x}$ is an increasing function and tends to $m > 0$ when $x \to \infty$, and P is often called a Michaelis-Menten function or a response function of Holling's type II.

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If $P = \frac{mx^2}{a+x^2}$ or $P = \frac{mx^2}{a+bx+x^2}$, where a, b, m strictly positive, then the function P is called a response function of Holling's type III. If $P = \frac{mx}{a+x^2}$ or $P = \frac{mx}{a+bx+x^2}$, then the function P is called a response function of Holling's type IV or a Monod-Haldane function. For more details about the predator-prey models (2) with Hollding type of functional responses, see the papers [4–9].

The predator–prey systems with response functions of Holling type represent the mathematical model for slow-fast dynamics in biology, more exactly the models where both the death rate and the conversion rate of prey to predator are kept very small ([10]).

In [11] R. Huzak reduced the study of the Rosenzweig and MacArthur system to study a polynomial differential system. In order to do that the first step is to do the rescaling $(\bar{x}, \bar{y}, \bar{b}, \bar{c}, \bar{\delta}) = \left(\frac{x}{K}, \frac{m}{rK}y, \frac{b}{K}, \frac{cm}{r}, \frac{\delta}{r}\right)$. After denoting again $(\bar{x}, \bar{y}, \bar{b}, \bar{c}, \bar{\delta})$ by (x, y, b, c, δ) and doing a time rescaling multiplying by $b + x$, the obtained polynomial differential system of degree three is

$$\begin{cases} \dot{x} &= x(-x^2 - (1-b)x - y + b) \\ \dot{y} &= y((c-\delta)x - \delta b) \end{cases} \tag{3}$$

where b, c, and δ are positive parameters.

Of course, the study of this system will be done only in the positive quadrant of the plane where it has an ecological meaning [11], [12].

The system (3) is a particular case of a Kolmogorov type system. These systems were proposed by Kolmogorov in the year 1936, in [13], as an extension of the Lotka–Volterra systems to arbitrary dimension and arbitrary degree.

The classical (linear or Lyapunov) stability Rosenzweig–MacArthur predator–prey system was completely studied in [11] and [12]. In the present paper we will study for the first time another kind of stability for this system, namely *Jacobi stability*. The Jacobi stability is a natural generalization of the stability of the geodesic flow on a differentiable manifold equipped with a Riemannian or Finslerian metrics to a manifold without a metric [14–19]. The Jacobi stability examines the robustness of a dynamical system defined by a system of second-order differential equations (SODEs), where the robustness is a measure of insensitivity and adaptation to change of the system internal parameters and the environment. Jacobi stability analysis of dynamical systems has been recently studied by several authors in [15], [16], [20–27], using the Kosambi-Cartan-Chern (KCC) theory [28–30]. More exactly, the dynamics of the system is studied with the help of the geometric objects associated to the system of second order differential equations obtained from the initial first order differential system.

The KCC theory deals with the study of the deviation of neighboring trajectories, which allow us to to estimate the perturbation permitted around the equlibrium points of the second-order differential system. Initially, this approach was linked with the study of the variation equations (or Jacobi field equations) associated to the geometry on the differentiable manifold. More exactly, P. L. Antonelli, R. Ingarden and M. Matsumoto started the study of the Jacobi stability for the geodesics corresponding to a Riemannian or Finslerian metric, by deviating the geodesics and using the KCC-covariant derivative for the differential system in variations [14–16]. So, it appeared the second KCC-invariant, called *the deviation curvature tensor*, which is essential for establish the Jacobi stability for geodesics and, generally, for the trajectories associated to a system of second order differential equations. In differential geometry's theory, a system of second-order differential equations (or, shortly SODE) is called semispray. Using a semispray, we can define a nonlinear connection on the manifold and conversely, using a nonlinear connection we can define a semispray. Consequently, any SODEs can define a geometry on the manifold by the associated geometric objects and conversely [17], [31–33]. Of course, these geometric objects are tensors which can check properties of symmetry or not, depending on the particularity of the SODE.

The KCC theory originated from the works of D. D. Kosambi [28], E. Cartan [29] and S. S. Chern [30], and hence the abbreviation KCC (Kosambi-Cartan-Chern). This geometric

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theory has a lot of applications in engineering, physics, chemistry and biology [20], [23,25–27], [34]. Also, new developments and applications of KCC theory in gravitation and cosmology was done in [35], [36]. Moreover, in [22] C.G. Boehmer, T. Harko and S.V. Sabau have analyzed Jacobi stability and its relations with the linear Liapunov stability analysis of dynamical systems, and presented a comparative study of these methods in the fields of gravitation and astrophysics.

In the second section a short presentation of the Rosenzweig–MacArthur predator–prey system will be done and we will point out the main results about the local stability of this system. Next, in section 3, we review the main notions and tools of the KCC theory in order to analyze the Jacobi stability of the system. We presented the five invariants of the KCC theory and definition of the Jacobi stability. In section 4, a reformulation of the Rosenzweig–MacArthur predator–prey system (3) as a system of second order differential equations is obtained and the five geometrical invariants are computed. The obtained results about the Jacobi stability of this predator-prey system near the equilibria are presented in section 5. More exactly, we will find necessary and sufficient conditions in order to have the Jacobi stability of the system near the equilibrium points. Consequently, for this values of the parameters, it is not possible to have a chaotic behaviour for Rosenzweig–MacArthur predator–prey system. Finally, a lot of examples will be presented in the sixth section and the conclusions in the seventh section. As usually, in differential geometry, the sum over crossed repeated indices is understood.

2. The Rosenzweig–MacArthur predator–prey system

Next, we will study the Rosenzweig and MacArthur system having the following form:

$$\begin{cases} \dot{x} &= x\left(-x^2+(1-b)x-y+b\right) \\ \dot{y} &= y((c-\delta)x-\delta b) \end{cases} \tag{4}$$

where $b, c, \delta > 0$ and $x, y \geq 0$.

In order to find the equilibrium points of this system and following [1] and [11], by analyzing the system,

$$\begin{cases} x\left(-x^2+(1-b)x-y+b\right) &= 0 \\ y((c-\delta)x-\delta b) &= 0 \end{cases}$$

we have at most three equlibria:

- $E_0(0,0)$ with eigenvalues $\lambda_1 = b$, $\lambda_2 = -\delta b$
- $E_1(1,0)$ with eigenvalues $\lambda_1 = -b-1$, $\lambda_2 = -(b\delta+\delta-c)$
- $E_2\left(\frac{b\delta}{c-\delta}, \frac{-bc(b\delta+\delta-c)}{(c-\delta)^2}\right)$ with eigenvalues $\lambda_{1,2} = \frac{b}{2(c-\delta)^2}\left(A \pm \sqrt{\delta B}\right)$, where

$$A = -\delta(b\delta+\delta-c+bc)$$

and

$$B = \delta(b\delta+\delta-c+bc)^2 + 4c(c-\delta)^2(b\delta+\delta-c)\,.$$

Let us remark that the third equilibria E_2 **exists only if** $c \neq \delta$ and $0 < b\delta < c-\delta$. In this case the corresponding eigenvalues satisfies

$$\lambda_1+\lambda_2 = \frac{bA}{(c-\delta)^2} \text{ and } \lambda_1\lambda_2 = \frac{b^2}{4(c-\delta)^4}\left(A^2-\delta B\right).$$

Also, $A^2-\delta B = -4c\delta(c-\delta)^2(b\delta+\delta-c) > 0$ whenever E_2 exists.

Note that if $b\delta = c-\delta$, then $E_2 = E_1$.

In [12] was obtained the following table which describe the type of the equilibria according to the values of the parameters b, c and δ.

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Case	Conditions	Equilibrium points type
1	$b\delta > c - \delta$	E_0 saddle, E_1 stable node
2	$b\delta = c - \delta$	E_0 saddle, E_1 saddle-node
3	$0 < b\delta < c - \delta$, $B \geq 0$, $A > 0$	E_0 saddle, E_1 saddle point, E_2 unstable node
4	$0 < b\delta < c - \delta$, $B \geq 0$, $A < 0$	E_0 saddle, E_1 saddle point, E_2 stable node
5	$0 < b\delta < c - \delta$, $B < 0$, $A > 0$	E_0 saddle, E_1 saddle point, E_2 unstable focus
6	$0 < b\delta < c - \delta$, $B < 0$, $A < 0$	E_0 saddle, E_1 saddle point, E_2 stable focus
7	$0 < b\delta < c - \delta$, $B < 0$, $A = 0$	E_0 saddle, E_1 saddle point, E_2 weak stable focus (center)

Table 1. The equilibrium points in the closed positive quadrant.

Let us remark that if $A = 0$, i.e. $b\delta + \delta - c = -bc$, then we have $B = -4bc^2(c - \delta)^2 < 0$. Also, let us remark that in the last case can occurs a Hopf bifurcation at the equilibria E_2 because $B < 0$ and $A = 0$.

Even though the system has only two equations and three parameters, let us remark that it is not so easy to obtain the behaviour of the system near to the equilibrium points because the system has no symmetry properties and computations can be very difficult. However, a deeply study of the linear stability around equilibrium points was done in 2022 by E. Diz-Pita, J. Llibre and M. V. Otero-Espinar in the paper [12], where was obtained the following results about the existence of limit cycles and a Hopf bifurcation at E_2:

Theorem 2.1. *a) If $0 < b\delta < c - \delta$ and $A > 0$, then there exists at least one limit cycle surrounding equilibrium point E_2.*

b) The equilibrium E_2 of the Rosenzweig–MacArthur predator–prey system (4) undergoes a supercritical Hopf bifurcation at $b_0 = \frac{c-\delta}{c+\delta}$ (i.e. $A = 0$).
For $b > b_0$ the system (4) has a unique stable limit cycle bifurcating from the equilibrium point E_2.

c) If $0 < b\delta < c - \delta$ and $A > 0$, the limit cycle surrounding the equilibrium point E_2 is unique.

Consequently, from this result, it was obtained that the unique limit cycle of system (4) appears from the equilibrium point E_2 in a Hopf bifurcation.

Moreover, from the proof of this theorem (see [12]), it results that the equilibrium point E_2 is a weak stable focus when $B < 0$ and $A = 0$.

However, for the cases 4, 6 and 7 from the Table 1 was not proved that there are or not limit cycles. Using the Bendixson–Dulac Theorem ([12]), only for some subcases of this cases was established that there are not limit cycles.

Proposition 2.2. *If $0 < b\delta < c - \delta$, $A < 0$ and $1 + c < b + \delta + b\delta$, then the Rosenzweig–MacArthur system (4) does not have periodic orbits in the set $\{(x, y) \in \mathbf{R}^2 | x, y \geq 0\}$.*

Finally, for this cases 4, 6 and 7 from the Table 1, in [12] was enounced the following conjecture about which only some numerical evidences was claimed.

Conjecture 2.3. *If $0 < b\delta < c - \delta$, $A < 0$ and $1 + c > b + \delta + b\delta$, then there are not limit cycles for the Rosenzweig–MacArthur predator–prey system (4).*

In the next sections, we will focused only to the study of the Jacobi stability in order to clarify the behaviour of the system and to confirm or not this conjecture.

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3. Kosambi-Cartan-Chern (KCC) geometric theory and Jacobi stability

According to [26] and [27], in this section we will made a brief presentation of the basic notions and main results from Kosambi-Cartan-Chern (KCC) theory [15], [16], [20], [21], [28–30]. The Kosambi-Cartan-Chern (KCC) theory is a modern and nice geometric approach of the dynamical systems which associates a semispray, a nonlinear connection and a Berwald connection to the SODE corresponding to the dynamical system. More exactly, for every SODE, we can associate the five geometrical invariants which determine the dynamics of the system, ε^i - the external force, P^i_j - the deviation curvature tensor, P^i_{jk} - the torsion tensor, P^i_{jkl} - the Riemann–Christoffel curvature tensor and D^i_{jkl} - the Douglas curvature tensor. But, the Jacobi stability of a dynamical system is determined only by the second invariant, the deviation curvature tensor.

We now introduce the main ideas of KCC theory from [14–16], [20], [21]. Let M be a real, smooth n-dimensional manifold and let TM be its tangent bundle. Usually, $M = \mathbf{R}^n$ or M is an open subset of $\mathbf{R}^n$. Let $u = (x, y)$ be a point in TM, where $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$, which means $y^i = \frac{dx^i}{dt}$, $i = 1, \ldots, n$. Consider the following system of second order differential equations in normalized form [14]

$$\left\{ \frac{d^2x^i}{dt^2} + 2G^i(x, y) = 0, \; i = 1, \ldots, n. \right. \tag{5}$$

where $G^i(x, y)$ are smooth functions defined in a local system of coordinates on TM, usually an open neighborhood of some initial conditions (x_0, y_0). In fact the system (5) is motivated by Euler-Lagrange equations of classical dynamics [14], [31]

$$\left\{ \begin{array}{rcl} \frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} & = & F^i \\ y^i & = & \frac{dx^i}{dt} \end{array} \right. , \; i = 1, \ldots, n. \tag{6}$$

where $L(x, y)$ is a regular Lagrangian of TM and F^i are the external forces.

The system (5) has a geometrical meaning if and only if "the accelerations" $\frac{d^2x^i}{dt^2}$ and "the forces" $G^i(x^j, y^j)$ are $(0, 1)$-type tensors under the local coordinates transformation

$$\left\{ \begin{array}{rcl} \tilde{x}^i & = & \tilde{x}^i(x^1, \ldots, x^n) \\ \tilde{y}^i & = & \frac{\partial \tilde{x}^i}{\partial x^j} y^j \end{array} \right. , \; i = 1, \ldots, n. \tag{7}$$

More precisely, the system (5) has a geometrical meaning (and it is called *a semispray*) if and only if the functions $G^i(x^j, y^j)$ are changing under the local coordinates transformation (7) following the rules [14], [31]

$$2\tilde{G}^i = 2G^j \frac{\partial \tilde{x}^i}{\partial x^j} - \frac{\partial \tilde{y}^i}{\partial x^j} y^j \, . \tag{8}$$

The main idea of KCC theory is to change the second order differential equations (5) into an equivalent system (i.e. with the same solutions), but with geometrical meaning, and then to show that it defines five tensor fields, also called *geometric invariants* of the KCC theory [15], [16]. In order to determine the five KCC geometrical invariants of the system (5), under the local change coordinates (7), we will introduce the KCC-covariant differential of a vector field $\xi = \xi^i \frac{\partial}{\partial x^i}$ defined in a open set of TM (usually $TM = \mathbf{R}^n \times \mathbf{R}^n$) [15], [28–30]

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N^i_j \xi^j \, , \tag{9}$$

where $N^i_j = \frac{\partial G^i}{\partial y^j}$ are the coefficients of *a nonlinear connection* N on the tangent bundle TM associated to the semispray (5).

For $\xi^i = y^i$ it obtains

$$\frac{Dy^i}{dt} = -2G^i + N^i_j y^j = -\varepsilon^i\,. \tag{10}$$

The contravariant vector field ε^i is called *the first invariant* of KCC theory. This invariant plays the role of an external force and the terms ε^i has a geometrical character since with respect to coordinates transformation (7), we have [15]

$$\tilde{\varepsilon}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\varepsilon^j\,.$$

If the functions G^i are 2-homogeneous with respect to y^i, i.e. $\frac{\partial G^i}{\partial y^j}y^j = 2G^i$, for all $i = 1, \ldots, n$, then $\varepsilon^i = 0$, for all $i = 1, \ldots, n$. Therefore, the first invariant of the KCC theory vanishes if and only if the semispray is a spray. This is always available for the geodesic spray corresponding to a Riemannian or Finsler metric [14], [31].

The main goal of Kosambi-Cartan-Chern theory is to study the trajectories which are slightly deviated upon a certain trajectory of (5). Practically, the dynamics of the system in variations will be studied and then, it will be varying the trajectories $x^i(t)$ of (5) into nearby ones described by

$$\tilde{x}^i(t) = x^i(t) + \eta\xi^i(t) \tag{11}$$

where $|\eta|$ is a small parameter and $\xi^i(t)$ are the components of a contravariant vector field defined along the trajectories $x^i(t)$. So, after substituting (11) into (5) and doing the limit $\eta \to 0$, it will be obtain the following variational equations [14–16]:

$$\frac{d^2\xi^i}{dt^2} + 2N^i_j\frac{d\xi^j}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0 \tag{12}$$

By using the KCC-covariant derivative from (9), the equations (12) can be written in the following covariant form [14–16]:

$$\frac{D^2\xi^i}{dt^2} = P^i_j\xi^j \tag{13}$$

where in the right side we have the $(1,1)$-type tensor P^i_j with components

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l\frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j\,. \tag{14}$$

The folowing coefficients

$$G^i_{jl} = \frac{\partial N^i_j}{\partial y^l} \tag{15}$$

represent the *Berwald connection* associated to the nonlinear connection N, according to [14], [31]. If all coefficients of nonlinear connection and Berwald connection are identically zero, then the deviation curvature tensor from (14) becomes $P^i_j = -2\frac{\partial G^i}{\partial x^j}$.

Then, according to [34], we can introduce the so-called *zero-connection curvature tensor* Z given by

$$Z^i_j = 2\frac{\partial G^i}{\partial x^j}\,. \tag{16}$$

For two-dimensional systems, the zero-connection curvature Z corresponds to the Gaussian curvature K of the potential surface $V(x^i) = 0$, where $\dot{x}^i = f^i(x^j) = -\frac{\partial V}{\partial x^i}(x^j)$. When the potential surface is minimal, then we have $P = -K$.

The coefficients P^i_j represent the so called *deviation curvature tensor* and is *the second invariant* of Kosambi-Cartan-Chern theory. The equations (12) are called *the deviation equations* (or *Jacobi equations*) and the invariant equations (13) are called also the *Jacobi*

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equation. In Riemannian or Finslerian geometry, when the second order system of equations represents the geodesic motion, then the equations (12) (or even (13)) are exactly the Jacobi field equations corresponding to the given geometry.

Finally, we can introduce the *third*, the *fourth* and the *fifth invariants* of the Kosambi-Cartan-Chern (KCC) theory for the second order system of equations (5). These invariants are defined by

$$P^i_{jk} = \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right),$$
$$P^i_{jkl} = \frac{\partial P^i_{jk}}{\partial y^l},\ D^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}. \tag{17}$$

From geometrical point of view the third KCC invariant P^i_{jk} can be interpreted as a *torsion tensor*. The fourth and the fifth KCC invariants P^i_{jkl} and D^i_{jkl} represent the *Riemann-Christoffel curvature tensor*, and the *Douglas tensor*, respectively.

It is important to point out that these tensors always exist [14–16], [21], [31].

According to [14], [29], [31], these *five invariants* are the basic mathematical quantities which describe the geometrical properties of the system and give us the geometrical interpretation for an arbitrary system of second-order differential equations.

Next, we present a basic result of KCC theory, obtained by P.L. Antonelli in [15]:

Theorem 3.1. *Two second order differential systems of the type of (5)*

$$\frac{d^2x^i}{dt^2} + 2G^i(x^j, y^j) = 0,\ y^j = \frac{dx^j}{dt}$$

and

$$\frac{d^2\tilde{x}^i}{dt^2} + 2\tilde{G}^i(\tilde{x}^j, \tilde{y}^j) = 0,\ \tilde{y}^j = \frac{d\tilde{x}^j}{dt}$$

can be locally transformed one into another via changing coordinates transformation (7) if and only if the five invariants ε^i, P^i_j, P^i_{jk}, P^i_{jkl} and D^i_{jkl} are equivalent tensors with $\tilde{\varepsilon}^i$, $\tilde{P}^i_j$, $\tilde{P}^i_{jk}$, $\tilde{P}^i_{jkl}$ and respectively $\tilde{D}^i_{jkl}$.

In particular, there are local coordinates $(x^1, \ldots, x^n)$ on the base manifold M, for which $G^i = 0$, for all i, if and only if all five invariant tensors vanish. In this case, the trajectories of the dynamical systems are straight lines.

The term "Jacobi stability" in the Kosambi-Cartan-Chern theory comes from the fact that, when (5) represents the second order differential equations for the geodesic equations in Riemannian or Finslerian geometry, then (13) is the Jacobi field equations for the geodesic deviation. Generally, the Jacobi equations (13) of the Finslerian manifold (M, F) can be written in the scalar form [18]:

$$\frac{d^2v}{ds^2} + K \cdot v = 0 \tag{18}$$

where $\xi^i = v(s)\eta^i$ is the Jacobi field along the geodesic $\gamma : x^i = x^i(s)$, η^i is the unit normal vector field along the geodesic γ and K is the flag curvature of Finslerian space (M, F).

The sign of the flag curvature K affects the geodesic rays as follows: if $K > 0$, then the geodesics bunch together (or are Jacobi stable), and if $K < 0$, then the geodesics disperse (or are Jacobi unstable). Therefore, if we take into account the equivalence between (13) and (18), then we obtain that a positive (or negative) flag curvature is equivalent to negative (or positive) eigenvalues of the curvature deviation tensor P^i_j.

We present below a well-known result from the Kosambi-Cartan-Chern theory [22]:

Theorem 3.2. *The trajectories of the system (5) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor P^i_j are strictly negative everywhere, and Jacobi unstable, otherwise.*

Now, we can write a rigorous definition of the Jacobi stability for a geodesic on a manifold endowed with an Euclidean, Riemannian or Finslerian metric or, even for a trajectory $x^i = x^i(s)$ of the dynamical system corresponding to (5) [19–22]:

Definition 3.3. *A trajectory $x^i = x^i(s)$ of (5) is said to be Jacobi stable if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|\tilde{x}^i(s) - x^i(s)\| < \varepsilon$ holds for all $s \geq s_0$ and for all trajectories $\tilde{x}^i = \tilde{x}^i(s)$ for which $\|\tilde{x}^i(s_0) - x^i(s_0)\| < \delta(\varepsilon)$ and $\|\frac{d\tilde{x}^i}{ds}(s_0) - \frac{dx^i}{ds}(s_0)\| < \delta(\varepsilon)$.*

According with [19–22], we take the trajectories of (5) as curves in a Euclidean space $\mathbf{R}^n$, where the norm $\|\cdot\|$ is the induced norm by the canonical inner product $< \cdot, \cdot >$ on $\mathbf{R}^n$. More that, we will assume that the deviation vector ξ from (13) verify the initial conditions $\xi(s_0) = O$ and $\dot{\xi}(s_0) = W \neq O$, where O is the null vector from $\mathbf{R}^n$. Additionally, if we assume that $s_0 = 0$ and $\|W\| = 1$, then, for $s \searrow 0$, the trajectories of (5) merge together if and only if the real parts of all eigenvalues of $P^i_j(0)$ are strictly negative or the trajectories of (5) disperse if and only if at least one of the real parts of the eigenvalues of $P^i_j(0)$ are positive.

This type of stability is about the focusing tendency in a small neighborhood of $s_0 = 0$ of the trajectories of (5) with respect to the variation of the trajectories (11) that satisfy the conditions $\|\tilde{x}^i(0) - x^i(0)\| = 0$ and $\|\frac{d\tilde{x}^i}{ds}(0) - \frac{dx^i}{ds}(0)\| \neq 0$.

Let us remark that the system of second order differential equations (SODE) (5) or the semispray (5) is Jacobi stable if and only if the system in variations (12) (or, in the covariant form (13)) is Lyapunov (or linear) stable. Consequently, the Jacobi stability analysis is based on the study of Lyapunov stability of all trajectories in a region without considering the velocity. Therefore, even when is reduced at an equilibrium point, this theory yield information about the behaviour of the trajectories in an open region which contain this equilibrium point.

4. SODE formulation of the Rosenzweig–MacArthur predator–prey system

We consider the Rosenzweig–MacArthur predator–prey system (4). By taking the derivative with respect to time t in both equations of this system, we obtain

$$\begin{cases} \ddot{x} + 3x^2\dot{x} - 2(1-b)x\dot{x} - (b-y)\dot{x} + x\dot{y} & = & 0 \\ \ddot{y} - ((c-\delta)x - \delta b)\dot{y} - y(c-\delta)\dot{x} & = & 0 \end{cases}$$

If we change the notations of variables as follows

$$x = x^1, \dot{x} = y^1, y = x^2, \dot{y} = y^2$$

then this system of second order differential equations (SODEs) becomes

$$\begin{cases} \ddot{x}^1 + 3(x^1)^2y^1 + 2(b-1)x^1y^1 + (x^2-b)y^1 + x^1y^2 & = & 0 \\ \ddot{x}^2 + (\delta b - (c-\delta)x^1)y^2 - (c-\delta)x^2y^1 & = & 0 \end{cases} \quad (19)$$

or, equivalently,

$$\begin{cases} \frac{d^2x^1}{dt^2} + 3(x^1)^2y^1 + 2(b-1)x^1y^1 + (x^2-b)y^1 + x^1y^2 & = & 0 \\ \frac{d^2x^2}{dt^2} + (\delta b - (c-\delta)x^1)y^2 - (c-\delta)x^2y^1 & = & 0 \end{cases} \quad (20)$$

where $\frac{dx^i}{dt} = y^i, i = 1, 2$.

This system can be write like a SODEs from KCC-theory

$$\begin{cases} \frac{d^2x^1}{dt^2} + 2G^1(x^1, x^2, y^1, y^2) = 0 \\ \frac{d^2x^2}{dt^2} + 2G^2(x^1, x^2, y^1, y^2) = 0 \end{cases} \tag{21}$$

where $\frac{dx^i}{dt} = y^i$, $i = 1, 2$ and

$$\begin{aligned} G^1(x^i, y^i) &= \tfrac{1}{2}\left[3(x^1)^2y^1 + 2(b-1)x^1y^1 + \left(x^2 - b\right)y^1 + x^1y^2\right] \\ G^2(x^i, y^i) &= \tfrac{1}{2}\left[(\delta - c)\left(x^1y^2 + x^2y^1\right) + \delta by^2\right] \end{aligned} \tag{22}$$

The zero-connection curvature $Z^i_j = 2\frac{\partial G^i}{\partial x^j}$ has the coefficients 212

$$\begin{aligned} Z^1_1 &= 6x^1y^1 + 2(b-1)y^1 + y^2 \\ Z^1_2 &= y^1 \\ Z^2_1 &= (\delta - c)y^2 \\ Z^2_2 &= (\delta - c)y^1 \end{aligned}$$

Since $N^i_j = \frac{\partial G^i}{\partial y^j}$, the nonlinear connection N is given by the following coefficients:

$$\begin{cases} N^1_1 = \frac{\partial G^1}{\partial y^1} = \frac{1}{2}\left[3(x^1)^2 + 2(b-1)x^1 + x^2 - b\right] \\ N^1_2 = \frac{\partial G^1}{\partial y^2} = \frac{1}{2}x^1 \\ N^2_1 = \frac{\partial G^2}{\partial y^1} = \frac{1}{2}(\delta - c)x^2 \\ N^2_2 = \frac{\partial G^2}{\partial y^2} = \frac{1}{2}\left[(\delta - c)x^1 + \delta b\right] \end{cases} \tag{23}$$

Then, it results that all coefficients of the Berwald connection $G^i_{jk} = \frac{\partial N^i_j}{\partial y^k}$ are null. 213

The first invariant of KCC theory $\varepsilon^i = -\left(N^i_jy^j - 2G^i\right)$ has the components

$$\begin{cases} \varepsilon^1 = \frac{3}{2}(x^1)^2y^1 + (b-1)x^1y^1 + \frac{1}{2}\left(x^2 - b\right)y^1 + \frac{1}{2}x^1y^2 \\ \varepsilon^2 = \frac{1}{2}(\delta - c)\left(x^1y^2 + x^2y^1\right) + \frac{1}{2}\delta by^2 \end{cases} \tag{24}$$

Let us remark that $\varepsilon^i = G^i$, for $i = 1, 2$, that means $\frac{\partial G^i}{\partial y^j}y^j = 1 \cdot G^i$, for $i = 1, 2$, or equivalently 214
the functions G^i are 1-homogeneous with respect to y^i. 215

Next, taking into account of (14), we obtain the components of the second invariant of the KCC theory, that means the deviation curvature tensor of the Rosenzweig–MacArthur system (4):

$$\begin{aligned} P^1_1 &= -3x^1y^1 - (b-1)y^1 - \tfrac{1}{2}y^2 + \tfrac{1}{4}\left[3(x^1)^2 + 2(b-1)x^1 + x^2 - b\right]^2 \\ &+ \tfrac{1}{4}(\delta - c)x^1x^2 \\ P^1_2 &= -\tfrac{1}{2}y^1 + \tfrac{1}{4}\left[3(x^1)^3 + 2(b-1)(x^1)^2 + (x^2 - b)x^1 + (\delta - c)(x^1)^2 + \delta bx^1\right] \\ P^2_1 &= -\tfrac{1}{2}(\delta - c)y^2 + \tfrac{1}{4}(\delta - c)x^2 \cdot \left[3(x^1)^2 + 2(b-1)x^1 + x^2 - b\right. \\ &+ \left.(\delta - c)x^1 + \delta b\right] \\ P^2_2 &= -\tfrac{1}{2}(\delta - c)y^1 + \tfrac{1}{4}(\delta - c)x^1x^2 + \tfrac{1}{4}\left[(\delta - c)x^1 + \delta b\right]^2 \end{aligned} \tag{25}$$

Then the trace and the determinant of the deviation curvature matrix

$$P = \begin{pmatrix} P^1_1 & P^1_2 \\ P^2_1 & P^2_2 \end{pmatrix}$$

are $\text{trace}(P) = P^1_1 + P^2_2$ and $\det(P) = P^1_1P^2_2 - P^2_1P^1_2$. 216

Therefore, following the results from the previous section, we have: 217

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Theorem 4.1. *All roots of the characteristic polynomial of P are negative or have negative real parts (i.e. Jacobi stability) if and only if*

$$P_1^1 + P_2^2 < 0 \text{ and } P_1^1 P_2^2 - P_1^2 P_2^1 > 0 .$$

Taking into account that $P_{jk}^i = \frac{1}{3}\left(\frac{\partial P_j^i}{\partial y^k} - \frac{\partial P_k^i}{\partial y^j}\right)$, $P_{jkl}^i = \frac{\partial P_{jk}^i}{\partial y^l}$, $D_{jkl}^i = \frac{\partial G_{jk}^i}{\partial y^l}$, we obtained the third, fourth and fifth invariants of the Rosenzweig–MacArthur predator–prey system as follows:

Theorem 4.2. *The third KCC invariant P_{jk}^i, called the torsion tensor, has all eight components null, i.e.*

$$P_{jk}^i = 0 \text{ for all } i, j, k. \tag{26}$$

The fourth KCC invariant P_{jkl}^i, called the Riemann-Christoffel curvature tensor, has all sixteen components null, i.e.

$$P_{jkl}^i = 0 \text{ for all } i, j, k, l. \tag{27}$$

The fifth KCC invariant D_{jkl}^i, called the Douglas tensor, has all sixteen components null, that means

$$D_{jkl}^i = 0 \text{ for all } i, j, k, l. \tag{28}$$

5. Jacobi stability analysis of the Rosenzweig–MacArthur predator–prey system

In this section, we will determine the first two invariants at the equilibrium points of the Rosenzweig–MacArthur predator–prey system (4) and we will analyze the Jacobi stability of the system near to each equilibrium points.

Further, for the equilibrium points $E_0(0,0)$, $E_1(1,0)$ and $E_2\left(\frac{b\delta}{c-\delta}, \frac{-bc(b\delta+\delta-c)}{(c-\delta)^2}\right)$ of the initial the Rosenzweig–MacArthur system (4) we have the corresponding equilibrium points $E_0(0,0,0,0)$, $E_1(1,0,0,0)$ and $E_2\left(\frac{b\delta}{c-\delta}, \frac{-bc(b\delta+\delta-c)}{(c-\delta)^2}, 0, 0\right)$ for SODE (20).

For $E_0(0,0,0,0)$, the first invariant of KCC theory ε^i has the components $\varepsilon^1 = \varepsilon^2 = 0$ and the matrix with the components of the second KCC-invariant is

$$P = \begin{pmatrix} \frac{1}{4}b^2 & 0 \\ 0 & \frac{1}{4}\delta^2 b^2 \end{pmatrix}.$$

Since $tr\, P = P_1^1 + P_2^2 = \frac{1}{4}b^2(1+\delta^2) > 0$ and $\det\, P = P_1^1 P_2^2 - P_1^2 P_2^1 = \frac{1}{16}\delta^2 b^4 > 0$, using Theorem 4.1 we obtain:

Theorem 5.1. *The trivial equilibrium point E_0 is always Jacobi unstable.*

For $E_1(1,0,0,0)$ the first invariant of KCC theory ε^i has the components $\varepsilon^1 = 0, \varepsilon^2 = 0$ and the matrix with the components of the second KCC-invariant is

$$P = \begin{pmatrix} \frac{1}{4}(b+1)^2 & \frac{1}{4}[1+b+(b\delta+\delta-c)] \\ 0 & \frac{1}{4}(b\delta+\delta-c)^2 \end{pmatrix}.$$

Since $tr\, P = \frac{1}{4}(b+1)^2 + \frac{1}{4}(\delta - c + \delta b)^2 > 0$ and $\det\, P = \frac{1}{16}(b+1)^2 \cdot (\delta - c + \delta b)^2 > 0$, using Theorem 4.1 we obtain:

Theorem 5.2. *The equilibrium point E_1 is always Jacobi unstable.*

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If $c \neq \delta$ and $0 < b\delta < c - \delta$, then the third equilibria E_2 exists and the first invariant of KCC theory ε^i at E_2 has all components null $\varepsilon^i = 0$. For the second invariant (i.e. the curvature deviation tensor) we obtain the following components P^i_j at E_2:

$$\begin{cases} P^1_1 &= \frac{1}{4}\frac{b^2\delta^2}{(c-\delta)^4}[(b\delta+\delta-c)+bc]^2 + \frac{1}{4}\frac{b^2\delta c(b\delta+\delta-c)}{(c-\delta)^2} \\ P^1_2 &= \frac{1}{4}\frac{b^2\delta^2}{(c-\delta)^3}[(b\delta+\delta-c)+bc] \\ P^2_1 &= \frac{1}{4}\frac{b^2\delta c}{(c-\delta)^3}(b\delta+\delta-c)[(b\delta+\delta-c)+bc] \\ P^2_2 &= \frac{1}{4}\frac{b^2\delta c}{(c-\delta)^2}(b\delta+\delta-c) \end{cases}$$

Taking into account that $tr\, P = P^1_1 + P^2_2 = \frac{1}{4}\frac{b^2\delta}{(c-\delta)^4}E(b,c,\delta)$, where

$$E(b,c,\delta) = \delta[(b\delta+\delta-c)+bc]^2 + 2c(b\delta+\delta-c)(c-\delta)^2$$

and $\det P = P^1_1P^2_2 - P^2_1P^1_2 = \frac{1}{16}\frac{b^4\delta^2c^2}{(c-\delta)^4}(b\delta+\delta-c)^2 > 0$, we obtain the result: 234

Theorem 5.3. *The equilibrium point E_2 is Jacobi stable if and only if $E(b,c,\delta) < 0$, or equivalently,*

$$\frac{1}{2c\delta}\cdot\left(\frac{A}{c-\delta}\right)^2 < -(b\delta+\delta-c)$$

where $A = -\delta[(b\delta+\delta-c)+bc]$. 235

Tacking into account that $B = \delta(b\delta+\delta-c+bc)^2 + 4c(b\delta+\delta-c)(c-\delta)^2$ and then 236
$B = E + 2c(b\delta+\delta-c)(c-\delta)^2$ or $E = B - 2c(b\delta+\delta-c)(c-\delta)^2$, it follows the next result: 237

Theorem 5.4. *If the third equilibria E_2 exists and it is Jacobi stable, then E_2 is a stable focus or* 238
unstable focus ($B < 0$). 239

The converse is not always true! It is possible to have $B < 0$, but E can be positive, 240
because $-(b\delta+\delta-c) > 0$. However, if $A = 0$, then $B < 0$ and also $E < 0$, because 241
$E = -2bc^2(c-\delta)^2$. 242

Remark 5.5. *Whenever E_2 exists and is Jacobi stable, a chaotic behaviour in a enough small* 243
neighborhood of this point is not possible. 244

6. Examples 245

Let be the Rosenzweig–MacArthur predator–prey system (4):

$$\begin{cases} \dot{x} &= x(-x^2+(1-b)x-y+b) \\ \dot{y} &= y((c-\delta)x-\delta b) \end{cases},$$

where $b, c, \delta > 0$ and $x_1, x_2 \geq 0$. 246

Example 6.1. *If $b = 1.5$, $c = 3.5$, $\delta = 1$, then we have the equilibria E_2 with coordinates $x = 0.6$,* 247
$y = 0.84$ and $A = -4.25$, $B = -69.438$, $E = -25.688$. Then E_2 is Jacobi stable. 248

Example 6.2. *If $b = 1.5$, $c = 2.5$, $\delta = 1$, then $x = 1$, $y = 0$ and $E_2 = E_1$. We have only E_0 saddle* 249
point and E_1 saddle-node. 250

Example 6.3. *If $b = 2$, $c = 1.5$, $\delta = 1$, then $x = 4$, $y = -18$ and E_2 is a virtual equilibrium. We* 251
have E_0 saddle point and E_1 stable node. 252

Example 6.4. *For* $b = 1.2$, $c = 2.5$, $\delta = 1$, *we have* $E_2(0.8, 0.4)$, *with* $A = -2.7$, $B = 0.54$, $E = 3.915$. *Then* E_2 *is Jacobi unstable with* $A < 0$, $B > 0$ *and* $E > 0$.

Example 6.5. *For* $b = 0.2$, $c = 0.8$, $\delta = 0.5$, *we have* $E_2(0.33, 0.35)$, *with* $A = 0.02$, $B = -0.0568$, $E = -0.028$. *Then* E_2 *is Jacobi stable with* $A > 0$.

Example 6.6. *For* $b = 0.1$, $c = 0.6$, $\delta = 0.5$, *we have* $E_2(0.5, 0.3)$, *with* $A = -0.005$, $B = -0.00115$, $E = -0.00055$. *Then* E_2 *is Jacobi stable with* $A < 0$ *and* $b\delta + b + \delta - 1 - c = -0.95 < 0$, *like in the Conjecture 2.3 from [12].*

Example 6.7. *For* $b = 1.5$, $c = 2.9$, $\delta = 1$, *we have* $E_2(0.78947, 0.48199)$, *with* $A = -3.95$, $B = -1.1479$ *and* $E = 7.2273$. *Then* E_2 *is Jacobi unstable with* $B < 0$, *but* $b\delta + b + \delta - 1 - c = 0.1 > 0$ *like in Theorem 2.1 from [12].*

Example 6.8. *For* $b = 1.5$, $c = 3.1$, $\delta = 1$, *we have* $E_2(0.71429, 0.63265)$, *with* $A = -4.05$, $B = -16.408$ *and* $E = -0.0027$. *Then* E_2 *is Jacobi stable with* $B < 0$, *but* $b\delta + b + \delta - 1 - c = -0.1$ *like in the Conjecture 2.3 from [12].*

Example 6.9. *For* $b = 1.5$, $c = 3$, $\delta = 1$, *we have* $E_2(0.75, 0.5625)$, *with* $A = -4$, $B = -8$ *and* $E = 4$. *Then* E_2 *is Jacobi unstable with* $B < 0$ *and* $b\delta + b + \delta - 1 - c = 0$.

Example 6.10. *For* $b = 0.1$, $c = 0.58$, $\delta = 0.5$ *we have* E_2 *with coordinates* $x = 0.625$, $y = 0.2718$ *and* $A = -0.014$, $B = -5.344 \times 10^{-5}$, $E = 1.6928 \times 10^{-4}$. *Then* E_2 *is Jacobi unstable with* $B < 0$.

Example 6.11. *For* $b = 0.5$, $c = 2.5$, $\delta = 0.5$ *we have* $E_2(0.125, 0.5468)$, *and* $A = 0.25$, $B = -69.875$, $E = -34.875$. *Then* E_2 *is Jacobi stable with* $A > 0$.

7. Conclusions

In this work we have done a study of the Jacobi stability of Rosenzweig–MacArthur predator–prey system using the geometric tools of the KCC theory. We reformulated the first order nonlinear differential system into a system of second-order differential equations, in order to determine the five geometrical invariants of the KCC theory. We calculated the first and the second invariant of the Kosambi-Cartan-Chern theory and we obtained that the third, fourth and fifth invariants are with all components null, also like the Berwald connection. Moreover, we have determined the nonlinear connection associated to the semispray (SODE) and we have computed the deviation curvature tensor at each equilibrium point in order to determine the Jacobi stability conditions. A future approach can be done by a computational study of the time variation of the deviation vector and of his curvature in order to illustrate the behaviour of Rosenzweig–MacArthur predator–prey system near to the equilibrium points.

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