Aerodynamic Shape and Drag Scaling Laws of Flexible Fibre in Flowing Medium

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The study of a flexible body immersed in a flowing medium is one of best way to find its aerodynamic shape. This Letter revisited the problem first studied by Alben, Shelley and Zhang (Nature 420, 479-481, 2002). The aerodynamic shape of the fibre is found by simpler approach and universal drag scaling laws of the flexible fibre in flowing medium are proposed by using dimensional analysis. The Alben scaling laws is being generalized and confirmed to be universal. Our study show that the Alben number is a measurement of maximum curvature of the fibre forced by dynamic pressure. A complete Maple code is provided for finding aerodynamic shape of the fibre in the flowing medium.

Keywords: flexible fibre; low medium; aerodynamic shape; drag; scaling laws

I. INTRODUCTION

Trees bend in a strong wind in order to absorb deformation energy and reduce the drag force of the wind on the trees as shown in Fig.1. This phenomenon of nature can inspire us to use deformable body to design streamlined shape and obtain drag reduction, which are the two major and intertwined issues of aerodynamics.

FIG. 1: Trees bend and survive in the wind.

Regarding streamlined shape finding, the study of a flexible body immersed in a flowing medium is one of best way to find its aerodynamic shape, since the immersed body will adjust its shape to counteract flow resistance to minimizes drag by reducing, or delaying, the turbulence in boundary layer of the flow nearest the moving body. Such a shape is the streamlined shape we want.

Other than that, drag reduction reduce in flowing medium is another key issue. The current literature mainly studies the resistance of bodies with fixed shape. But what happens if the body is flexible and bends in response to the flow and just how much can a flexible object reduce drag by changing its shape?

Alben, Shelley and Zhang [1] presented results from their experimental and theoretical studies of drag reduction by shape reconfiguration for a flexible body immersed in a flowing medium. They made breakthrough regarding the scaling law of drag of flexible fibre in flowing medium, Alben, Shelley and Zhang [1] reported a remarkable result on the scaling laws: For $\eta \leq 1$, the fibre is nearly straight, the viscous drag $D \sim \eta^2$, and for $\eta \geq 1$, the viscous drag $D \sim \eta^{4/3}$, where η is a non-dimensional parameter. For presentation purpose, these scaling laws will be called Alben scaling laws.

However, Alben, Shelley and Zhang [1] also left an open and intriguing question, whether the scaling laws are universal. It is clear that this problem has no exact solution due to turbulence nature of wake, hence numerical solutions and experimental methods have to be anticipated. Unfortunately, the numerical and experimental results can only be obtained for a specific case, it would be very difficulty to predict the general scaling trends on the problem by using limited numerical results. To find the general scaling law, it would be a natural attempts to take an alternative way – dimensional analysis [3, 4, 6].

From de Gennes [5] and Sun [6–12], the dimensional analysis is an universal method, which can, of course, be used to study the drag scaling laws of flexible fibre in flowing medium. Why we use the dimensional analysis, it is because that there is consensus the universality can be verified by the dimensional analysis, which means the relation is universal if it could be formulated and passed by the dimensional methods.

To interpret the experiment, Alben, Shelley and Zhang [1] abstracted its crucial features and build a mathematical model coupling hydrodynamics to body flexibility. The fluid pressure jump was solved by Alben, Shelley and Zhang [2] through constructing steady solutions to the inviscid, incompressible Euler equations, using free-streamline theory, however, from mathematical point of view, their solving process is quite complicated and may not be applicable in practices. It is necessary to have a simpler strategy, which assumes the fluid pressure jump as $[p] \approx \frac{1}{2}\rho U^2$ and ignores flow wake pressure.

In this Letter, for the flexible fibre acted fluid pressure jump $[p] \approx \frac{1}{2}\rho U^2$, we propose a simpler solution to find the aerodynamic shape and universal scaling laws for the

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drag.

II. AERODYNAMIC SHAPE OF FLEXIBLE FIBRE IN FLOWING MEDIUM

Consider a flexible fibre in the flow medium as shown in Fig.2. The fibre is modelled as a thin, inextensible elastic beam loaded by the difference in fluid pressure p between its upstream and downstream sides. The flow characteristic velocity is U and no wake is considered.

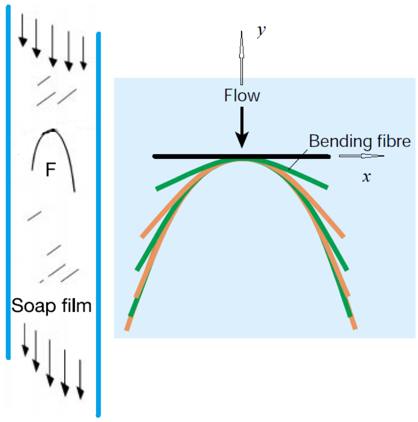

FIG. 2: A glass fibre (F) is inserted into a flowing soap-film tunnel (partly shown). The fibre is supported by a thin stainless-steel rod, which is clamped at one end. Fluid drag force acting on the fibre deflects this support slightly downwards. Driven by gravity, soapy water (1.5% Dawn dish detergent; density $\rho = 1$ gm cm^{-3}) leaves an elevated reservoir and spreads into a vertical soap film (thickness $f = 2\,\mu$m) descending between two straight nylon lines (tunnel width, 9.0 cm). Adjusting reservoir efflux rate adjusts flow velocity U through the range 0.53.0ms^{-1}; breakage occurs at velocities outside this range. Half-way down the tunnel, a thin, flexible glass fibre (length $L = 15$ cm; diameter, 34 mm; rigidity $E = 2.8$ erg cm), glued at its midpoint to a thin rod, is inserted transverse to the flow. The drag force on the fibre is measured, record its shape, and visualize the flow structures using interferometry, all as a function of flow speed. For comparison, a much more rigid fibre is also used ($L = 2.0$ cm; $E = 2,000$ erg cm). On the basis of fibre lengths, flow velocities, and soap film viscosity ν, the Reynolds number $Re = LU/\nu$ is typically large, in the range $2,000 - 40,000$.

The centerline of the beam is represented by an inextensible curve $\boldsymbol{x}(s)$ with arch length s and curvature $\kappa(s)$, Assume $\boldsymbol{x}(s)$ as a reference (middle) centerline and $\boldsymbol{n}(s)$ as the unit normal vector to the centerline of inextensible planar curve. The unit tangent of the centerline is given by $\boldsymbol{t} = \frac{d\boldsymbol{x}}{ds}$, $|\boldsymbol{t}| = 1$, which is orthogonal to the normal, ie., $\boldsymbol{t} \cdot \boldsymbol{n} = 0$. The curvature of the reference (middle) inextensible curve is $\kappa(s) = \frac{d\theta}{ds} = |\frac{d^2\boldsymbol{x}}{ds^2}| = |\frac{d\boldsymbol{t}}{ds}|$, where θ is denoted as the angle between $\boldsymbol{t}$ and horizontal axis x. The shape of the centerline can be reconstructed by relations: $\frac{dx}{ds} = \cos\theta(s)$ and $\frac{dy}{ds} = -\sin\theta(s)$.

The Euler-Bernoulli beam under fluid dynamic pressure was formulated by Alben, Shelley and Zhang [1] as follows:

$$-\frac{d}{ds}(T\boldsymbol{t}) + \frac{d}{ds}(E\frac{d\kappa}{ds}\boldsymbol{n}) = f[p]\boldsymbol{n}, \tag{1}$$

where f is the soap film thickness, T is the line tension, $E = YI$ is the bending rigidity, the Young modulus is Y, area moment of inertia is I, $[p]$ is the fluid pressure jump across the fibre.

At the fibre ends, there are no bending moment, transverse shear force and extensional force, so that the boundary conditions are:

$$T = \kappa = \frac{d\kappa}{ds} = 0. \tag{2}$$

Using planar Frenet's frame formula, namely $\frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{n}$ and $\frac{d\boldsymbol{n}}{ds} = -\kappa\boldsymbol{t}$, Eq.1 can be decomposed into tangential and normal components:

$$\boldsymbol{t} : -\frac{dT}{ds} - \frac{1}{2}E\kappa\frac{d\kappa}{ds} = 0, \tag{5}$$

$$\boldsymbol{n} : -T\kappa + \frac{d^2\kappa}{ds^2} = \frac{1}{2}\rho U^2 f. \tag{6}$$

Integrating Eq.5 with respect to arc length and applying the boundary condition Eq.2 we have $T = -\frac{1}{2}E\kappa^2$. Inserting $T = -\frac{1}{2}E\kappa^2$ into Eq.6 leads to a single ordinary differential equation (ODE):

$$E\frac{d^2\kappa}{ds^2} + \frac{1}{2}E\kappa^3 = \frac{1}{2}\rho U^2 f. \tag{3}$$

If introducing $\bar{s} = s/L$ and $\bar{\kappa} = d\theta/d\bar{s} = L\kappa$, The ODE and corresponding boundary conditions can be noncommissioned as to the form:

$$\frac{d^2\bar{\kappa}}{d\bar{s}^2} + \frac{1}{2}\bar{\kappa}^3 = \eta^2, \quad \bar{\kappa}_{\bar{s}=\frac{1}{2}} = \left(\frac{d\bar{\kappa}}{d\bar{s}}\right)_{\bar{s}=\frac{1}{2}} = 0, \tag{4}$$

in which, the Alben number was introduced by Alben, Shelley and Zhang [1] as follows

$$\eta = \left[\frac{\frac{1}{2}\rho U^2 f L^2}{(E/L)}\right]^{1/2} = (\frac{L}{L_0})^{3/2}, \tag{5}$$

where $E = YI$ and $L_0 = \left(\frac{2E}{\rho U^2 f}\right)^{1/3}$.

The exact solution of Eq.5 can be obtained

$$\bar{\kappa}(\bar{s}) = c_2 \sin\left(\text{JacobiAM}\left[(\frac{1}{2}\bar{s} + c_1)c_2, \sqrt{-1}\right]\right) + \eta^{2/3}, \tag{6}$$

where JacobiAM(ϕ, k) is the inverse of the normal trigonometric form of the incomplete elliptic integral of the first kind, and c_1, c_2 are constants can be determined by the boundary conditions. However, the constants c_1, c_2 are difficulty to be obtained analytically due to the nature of the incomplete elliptic integral. We can

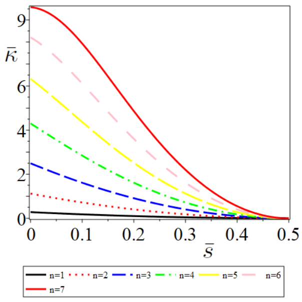

FIG. 3: The curvature $\bar{\kappa}$ vs. $\bar{s}$ for different parameter $\eta = 3n/2$, $n = 1, 2, 3..., 7$, the aerodynamic shape curvature is symmetric to the y axis.

compute it numerically. The curvature $\bar{\kappa}$ vs. $\bar{s}$ for different parameter $\eta = 3n/2$, $n = 1, 2, 3..., 7$ is shown in Fig.3.

The aerodynamic shape of the fibre can be reconstructed by relations: $\frac{d\bar{x}}{d\bar{s}} = \cos\theta(s)$ and $\frac{d\bar{y}}{d\bar{s}} = -\sin\theta(s)$, where $\bar{x} = x/L$, $\bar{y} = y/L$, and shown in Fig.4. Multiplying both

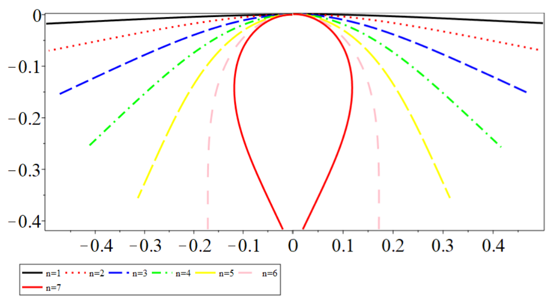

FIG. 4: The aerodynamics shape $(\bar{x}, \bar{y})$ for different parameter $\eta = 3n/2$, $n = 1, 2, 3..., 7$, the aerodynamic shape is symmetric to the y axis.

sides of Eq.5 and integrating respect to $\bar{s}$, after applying boundary conditions, we have its the 1st integral

$$\left(\frac{d\bar{\kappa}}{d\bar{s}}\right)^2 = \frac{1}{4}\bar{\kappa}(8\eta^2 - \bar{\kappa}) \geq 0. \tag{7}$$

$\frac{d\bar{\kappa}}{d\bar{s}} = 0$ results to the maximum curvature $\bar{\kappa}_{max} = 8\eta^2 = \frac{4\rho U^2 f L^3}{E}$, or equivalently

$$\kappa_{max} = \frac{8\eta^2}{L} = \frac{4\rho U^2 f L^2}{E}. \tag{8}$$

These expressions are profoundly revealed that the Alben number η is a measurement of maximum curvature of the fibre forced by dynamic pressure.

III. GENERALIZATION OF ALBEN SCALING LAWS OF DRAG

Regarding the scaling law of drag of flexible fibre in flowing medium, by some numerical and experiments, Alben, Shelley and Zhang [1] reported a remarkable result on the scaling laws: For $\eta \leq 1$, the fibre is nearly straight, the viscous drag $D \sim \eta^2$, and for $\eta \geq 1$, the viscous drag $D \sim \eta^{4/3}$. As we pointed out, are the Ablen scaling laws are universal ? To answer this question, we are going to examine this problem by only using dimensional method without carrying out any particular analysis. We wish to see what form will the most general scaling laws take and generalize the results obtained by Alben, Shelley and Zhang [1].

For the problem, there are 6 quantities, namely D, ρ, E, ν, U, A, where ν is kinematic viscosity and $A = fL$. The quantity dimensions are listed in the following table I:

TABLE I: Dimensions of physical quantities

Variables	Notation	Dimensions
Drag	D	MLT^{-2}
Mass density	ρ	ML^{-3}
Rigidity	E	ML^3T^{-2}
Kinematic viscosity	ν	L^2t^{-1}
Flow velocity	U	LT^{-1}
Area($= fL$)	A	L^2

The dimensional basis used is length (L), mass (M) and time (T).

The drag D can be expressed as the function of quantities ρ, E, , ν, U, A, namely,

$$D = F(\rho,\ E, ,\nu,\ U,\ A). \tag{9}$$

In the above relation, there are 6 quantities, two of them are dimensionless. Since only 3 dimensional basis L,M,T are used, so according to dimensional analysis Bridgman [4], Sun [7], Eq. 9 produces 3 dimensionless quantities Π. The first one is: $\Pi_D = D\rho^a U^b A^c = L^0 M^0 T^0$,, we have

$$\Pi_D = \frac{D}{\frac{1}{2}\rho U^2 A}, \tag{10}$$

Replacing D by E, we have

$$\Pi_E = \frac{E}{\frac{1}{2}\rho U^2 A^2}, \tag{11}$$

and third one is

$$\Pi_\nu = \frac{\nu}{\rho U^2 \sqrt{A}} = \frac{1}{Re}, \tag{12}$$

From Bumingham Π theorem, the Eq.9 can be equivalency expressed as $\Pi_D = F(\Pi_E, \Pi_\nu)$, namely

$$D = \frac{1}{2}\rho U^2 A F\left(\frac{E}{\frac{1}{2}\rho U^2 A^2}, Re\right). \tag{13}$$

For infinite Reynolds number, we can propose an approximate $F(\frac{E}{\frac{1}{2}\rho U^2 A^2}, Re) \approx C\left(\frac{E}{\frac{1}{2}\rho U^2 A^2}\right)^\alpha$, therefore

$$D \approx C\frac{1}{2}\rho U^2 A\left(\frac{E}{\frac{1}{2}\rho U^2 A^2}\right)^\alpha, \tag{14}$$

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where C is a constant and α is an exponent to be determined by experiments.

To determined the exponent α, by using the Alben number η and $A = fL$, the above relation in Eq.14 can be also expressed as follows

$$D \approx CEf^{-\alpha}L^{-2-\alpha}\eta^{2(1-\alpha)}, \tag{15}$$

From experiments, Alben, Shelley and Zhang [1] found that, for $\eta \leq 1$, the fibre is nearly straight, the viscous drag $D \sim \eta^2$, and for $\eta \geq 1$, the viscous drag $D \sim \eta^{4/3}$.

Applying these experimental results, for $\eta \leq 1$, we have $\alpha = 0$, for $\eta \geq 1$, we have $\alpha = \frac{1}{3}$, therefore

$$D = \begin{cases} C_1 \frac{1}{2}\rho f L U^2, & (\eta \leq 1), \\ C_2 \left[(\frac{1}{2}\rho)^2 E f L\right]^{1/3} U^{4/3}, & (\eta \geq 1). \end{cases} \tag{16}$$

By data fitting, Alben, Shelley and Zhang [2] obtained, for $\eta \leq 1$, $D = \frac{2\pi}{\pi+4}\eta^2$ for fibre with $L = 2$ cm and $E = 2000$ erg cm; for $\eta \geq 1$, $D = 1.87\eta^{4/3}$. After checking, we found that the fitting coefficients in the Alben scaling laws have big error and must be corrected. If we set film thickness $f = 2\mu$m $= 0.0002$cm, by data fitting, we can get the constants

$$C_1 = 8.094175640, \ C_2 = 0.01393099429. \tag{17}$$

The scaling laws in Eq.16 are depicted in Fig.5. The comparison shows that the scaling laws are very close to the experimental data.

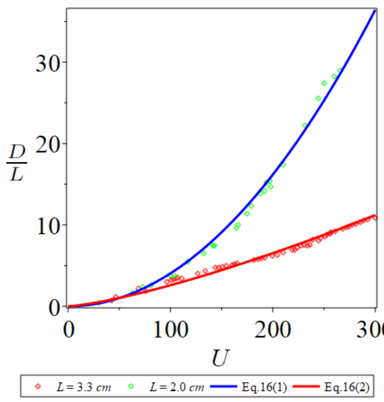

FIG. 5: Comparison of drag data from experiment and Eq.16. Drag per unit fibre length $\frac{D}{L}$ versus flow velocity U for a flexible fibre ($L = 3.3$ cm; red circles for experiment and red solid line from the 2nd of Eq.16) and a rigid fibre($L = 2.0$ cm; green squares experiment and blue solid line from the 1st of Eq.16).

IV. DISCUSSIONS AND PERSPECTIVES

The Alben scaling laws has been successfully generalized in terms of flow velocity and confirmed to be universal by using dimensional analysis.

For $\eta \leq 1$, the $D = C_1\frac{1}{2}\rho f L U^2$, indicates that drag is proportional to U^2 and noting to do with the rigidity E, which means that the fibre is considered as a rigid body. For $\eta \geq 1$, the $D = C_2\left[(\frac{1}{2}\rho)^2 E f L\right]^{1/3} U^{4/3}$, reveals that the drag is proportional to $U^{4/3}$, $E^{1/3}$ and $(fL)^{1/3}$.

If we set the film thickness f equal to the diameter of the fibre d, namely $f = d$, then we have draw scaling law of a flexible fibre in flowing medium as follows

$$D = C_2\left[(\frac{1}{2}\rho U^2)^2 E d L\right]^{1/3}, \quad (U \geq \sqrt{\frac{2E}{\rho d L^3}}). \tag{18}$$

This drag scaling way reveals that the softer fibre the smaller the drag, which is the secret of reducing drag by fine hairs [13].

V. CONCLUSIONS

The aerodynamic shape of the fibre has obtained by simpler approach and universal drag scaling laws of the flexible fibre in flowing medium has been formulated by using dimensional analysis. The Alben scaling laws has been generalized and confirmed to be universal. Our study shown that the Alben number is a measurement of maximum curvature of the fibre forced by dynamic pressure. A complete Maple code is provided for finding aerodynamic shape of the fibre in the flowing medium. The research and understanding here is of great significance for streamlined design and drag reduction.

Acknowledgements

This work was supported by Xi'an University of Architecture and Technology (Grant No. 002/2040221134). The authors wish to express their gratitude for the constructive discussions with Prof. Jun Zhang from New York University.

Appendix: Complete Maple Code of the Problem

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restart:with(plots): for n to 10 do eta := (3*n)/2; eq1 := diff(theta(s), s) = k(s); eq2 := diff(k(s), s, s) + 1/2*k(s)*k(s)*k(s) = eta*eta; eq3 := diff(phi(s), s) = cos(theta(s)); eq4 := diff(psi(s), s) = -sin(theta(s)); equs := eq1, eq2, eq3, eq4; bc := phi(0) = 0, psi(0) = 0, theta(0) = 0, k(1/2) = 0, D(k)(1/2) = 0; sys := [bc, equs]; vars := [k(s), phi(s), psi(s), theta(s)]; sol := dsolve(sys, vars, numeric, output = listprocedure); kn[n] := rhs(sol[2]); xn[n] := rhs(sol[4]); yn[n] := rhs(sol[5]); thetan[n] := rhs(sol[6]); print(n); end do:
odeplot(sol, [seq([xn[n](s), yn[n](s)], n = 1 .. 7)], s = 0 .. 1/2, legend = ["n=1", "n=2", "n=3", "n=4", "n=5", "n=6", "n=7"], linestyle = [1, 2, 3, 4, 5, 6, 1], thickness = 3, color = ["black", "red", blue, green, yellow, pink, red, purple], axes = boxed)
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