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Article

Gauge-Coded Quantum Cellular Automata with an Explicit Continuum Limit to the 1D Dirac Equation

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Abstract

We construct and analyze an explicit one-dimensional quantum cellular automaton (QCA) that provides a controlled lattice regularization of the free Dirac equation in $(1+1)$ dimensions, and we extend it to a locally $U(1)$ gauge-covariant version. The microscopic model is strictly local, causal in the Lieb–Robinson sense, and unitarily evolves a chain of finite-dimensional quantum systems by a translation invariant, discrete-time rule. At the representation-theoretic level, it realizes a chiral spin-1/2 degree of freedom per site. We prove that, in the joint limit of small lattice spacing and small time step with fixed physical scales, the QCA dynamics converges to the continuum Dirac evolution, with an explicit bound on the error valid for smooth, band-limited initial states. We then couple the QCA to a compact $U(1)$ gauge field living on links, in a way that is exactly gauge-covariant at the microscopic level. A local Gauss constraint defines a gauge-invariant code subspace, and we show that the QCA update preserves this subspace and implements a gauge-covariant discrete Dirac dynamics. This provides a concrete example of a gauge-coded QCA with a controlled continuum limit, which can serve as a microscopic building block for more elaborate $SU(3) \times SU(2) \times U(1)$ constructions. All results are stated with modest but explicit hypotheses and proofs, so that the framework is mathematically well-defined and amenable to further generalization.

Keywords: quantum field theory; topological vortices; Higgs mechanism; mass generation; fundamental forces; beyond standard model; golden equation; solitons; gauge theory

1. Introduction

Quantum cellular automata (QCA) provide strictly local, discrete-time models of quantum dynamics on lattices, in which each time step is implemented by a quantum circuit of bounded depth with translation invariant local gates. They offer conceptually clean and computationally natural discretizations of quantum field theories, and have been extensively studied in the context of quantum walks, lattice field theory, and quantum information.

For relativistic fermions, it is known that certain discrete-time quantum walks converge, in a small lattice spacing and time step limit, to the Dirac equation in one dimension. Our first goal in this article is to present a QCA formulation of such a walk, and to give an explicit and carefully justified derivation of its continuum limit to the free $(1+1)$ -dimensional Dirac equation for smooth, band-limited initial states.

Our second goal is to exhibit a microscopic coupling of this QCA to a compact $U(1)$ gauge field defined on links, in such a way that the dynamics is exactly gauge-covariant at the lattice level and preserves a natural Gauss-law constraint. In other words, we construct a gauge-coded QCA: a QCA whose physical states form a code subspace defined by local gauge constraints, and whose update respects these constraints. We then analyze how the gauge-covariant QCA approaches the Dirac equation coupled to a background $U(1)$ gauge potential in a continuum limit.

We emphasize that the model studied here is intentionally modest:

- It is one-dimensional and Abelian ($U(1)$ only).

- We work with free (or minimally coupled) fermions, without self-interactions.
- We make explicit regularity and band-limitation assumptions on initial data when taking the continuum limit.

Within these boundaries, however, the results are rigorous and detailed. They provide a concrete, non-conditional building block for more general constructions involving non-Abelian gauge groups and higher dimensions.

2. A Simple 1D QCA for the Dirac Equation

2.1. Lattice and Hilbert space

We consider a one-dimensional infinite lattice with sites labelled by integers $n \in \mathbb{Z}$. The lattice spacing is $a > 0$. At each site we attach a two-dimensional Hilbert space $\mathbb{C}^2$ representing a spin-1/2 (or chirality) degree of freedom. The total Hilbert space is

$$\mathcal{H} = \bigotimes_{n \in \mathbb{Z}} \mathcal{H}_n, \quad \mathcal{H}_n \simeq \mathbb{C}^2. \quad (1)$$

We denote by $|n\rangle \otimes |\sigma\rangle$ the basis with lattice site label n and internal index $\sigma \in \{\uparrow, \downarrow\}$.

Fermionic statistics can be implemented by introducing creation and annihilation operators and a Jordan–Wigner map. For the continuum limit of single-particle wave packets, it suffices to focus on the single-particle sector, which is a subspace of $\mathcal{H}$ isomorphic to $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$.

2.2. Discrete-time update rule

We define the QCA as a discrete-time dynamics $\Psi(t+1) = U\Psi(t)$ on the single-particle Hilbert space, where U is a unitary operator built from a coin and a shift, in the spirit of discrete-time quantum walks.

Let $\sigma_x, \sigma_y, \sigma_z$ be the Pauli matrices. Fix a real parameter $\theta \in (-\pi/2, \pi/2)$, which will be related to the fermion mass. Define a coin operator C acting on the internal space:

$$C := e^{-i\theta\sigma_y} = \cos\theta \mathbb{I} - i \sin\theta \sigma_y. \quad (2)$$

On the full space, C acts as

$$(C\Psi)(n) = C\Psi(n), \quad (3)$$

where $\Psi(n) \in \mathbb{C}^2$ is the two-component wave function at site n .

Next, define the conditional shift S :

$$(S\Psi)(n) = \begin{pmatrix} \Psi_\uparrow(n-1) \\ \Psi_\downarrow(n+1) \end{pmatrix}, \quad (4)$$

that is, the $\uparrow$ component moves to the right, the $\downarrow$ component moves to the left:

$$(S\Psi)_\uparrow(n) = \Psi_\uparrow(n-1), \quad (5)$$

$$(S\Psi)_\downarrow(n) = \Psi_\downarrow(n+1). \quad (6)$$

Finally, define the one-step QCA update operator

$$U := S \circ C. \quad (7)$$

U is unitary, translation invariant and strictly local: it maps $\Psi(n)$ to a combination of values at $n \pm 1$.

2.3. Momentum-space representation

Let $k \in [-\pi/a, \pi/a]$ denote the lattice momentum, and define the Fourier transform

$$\Psi(k) = \sum_{n \in \mathbb{Z}} e^{-ikna} \Psi(n). \quad (8)$$

Then the update rule in momentum space reads

$$\Psi(t+1, k) = U(k) \Psi(t, k), \quad (9)$$

with

$$U(k) = \begin{pmatrix} e^{ika} & 0 \\ 0 & e^{-ika} \end{pmatrix} C. \quad (10)$$

A short computation yields

$$U(k) = \begin{pmatrix} e^{ika} \cos \theta & -e^{ika} \sin \theta \\ e^{-ika} \sin \theta & e^{-ika} \cos \theta \end{pmatrix}. \quad (11)$$

As a 2×2 unitary matrix, $U(k)$ can be written as

$$U(k) = e^{-i\omega(k) \hat{n}(k) \cdot \sigma}, \quad (12)$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$, $\hat{n}(k)$ is a unit vector, and $\omega(k) \in [0, \pi]$ is the quasienergy. The eigenvalues of $U(k)$ are $e^{\pm i\omega(k)}$. The dispersion relation is determined by

$$\cos \omega(k) = \frac{1}{2} \text{Tr } U(k) = \cos(ka) \cos \theta. \quad (13)$$

2.4. Small- a continuum limit: Dirac dispersion

We introduce a time step $\Delta t > 0$ and interpret $t \in \mathbb{Z}$ as physical time $t_{\text{phys}} = t\Delta t$. For the continuum limit, we fix a Fermi velocity $v > 0$ and mass $m \geq 0$, and scale the parameters as

$$ka \ll 1, \quad \theta \ll 1, \quad \frac{a}{\Delta t} \rightarrow v, \quad \frac{\theta}{\Delta t} \rightarrow m. \quad (14)$$

Expanding (13) for small ka and small θ yields

$$\cos \omega(k) = \left(1 - \frac{(ka)^2}{2} + \mathcal{O}(a^4)\right) \left(1 - \frac{\theta^2}{2} + \mathcal{O}(\theta^4)\right). \quad (15)$$

To second order,

$$\cos \omega(k) = 1 - \frac{(ka)^2 + \theta^2}{2} + \mathcal{O}(a^4, \theta^4, a^2\theta^2). \quad (16)$$

For small $\omega(k)$ we have

$$\cos \omega(k) = 1 - \frac{\omega(k)^2}{2} + \mathcal{O}(\omega^4). \quad (17)$$

Identifying orders, we obtain

$$\omega(k)^2 = (ka)^2 + \theta^2 + \mathcal{O}(a^4, \theta^4, a^2\theta^2). \quad (18)$$

Interpreting $\omega(k)$ as an energy in units of Δt , we define the continuum energy $E(k)$ via

$$E(k) := \frac{\omega(k)}{\Delta t}. \quad (19)$$

Under the scaling (14), this gives

$$E(k)^2 \rightarrow v^2 k^2 + m^2, \quad (20)$$

which is precisely the $(1+1)$ -dimensional relativistic dispersion relation for a Dirac fermion of mass m and velocity v .

3. Continuum Limit as a Dirac Equation

3.1. Band-limited initial states

To derive the continuum limit in position space, we restrict attention to initial states that are smooth and band-limited in momentum. Let $\Lambda_{UV} \in (0, \pi/a)$ be a cutoff and define the band

$$\mathcal{B} := [-\Lambda_{UV}, \Lambda_{UV}]. \quad (21)$$

We say that an initial single-particle wave function $\Psi(0)$ is Λ_{UV} -band-limited if its Fourier transform $\Psi(0, k)$ vanishes for $|k| > \Lambda_{UV}$.

We also assume that $\Psi(0, k)$ is sufficiently smooth in k (e.g., Schwartz class restricted to $\mathcal{B}$). This guarantees that the position-space wave function is smooth on scales large compared to a .

3.2. Effective continuum generator

For each $k \in \mathcal{B}$, we write

$$U(k) = e^{-i\Delta t H_{\text{eff}}(k)}, \quad (22)$$

where $H_{\text{eff}}(k)$ is a 2×2 Hermitian matrix defined (up to $2\pi/\Delta t$ multiples) via the logarithm of $U(k)$. For small a and θ satisfying (14), we may choose the branch such that

$$H_{\text{eff}}(k) = \mathbf{h}(k) \cdot \boldsymbol{\sigma}, \quad \|\mathbf{h}(k)\| \Delta t = \omega(k), \quad (23)$$

with $\omega(k)$ the small quasienergy discussed above. A direct expansion shows that, to leading order in a and θ ,

$$H_{\text{eff}}(k) = vk\sigma_z + m\sigma_y + \mathcal{O}(a^2, \theta^2, a\theta). \quad (24)$$

Transforming back to position space, this suggests the continuum Hamiltonian

$$H_{\text{Dirac}} = \int dx \Psi^\dagger(x) (-iv\sigma_z \partial_x + m\sigma_y) \Psi(x), \quad (25)$$

which generates the $(1+1)$ -dimensional Dirac equation

$$i\partial_t \Psi(x, t) = (-iv\sigma_z \partial_x + m\sigma_y) \Psi(x, t). \quad (26)$$

3.3. Error estimate

We now state a modest but explicit error bound. Let $\Psi_{\text{QCA}}(t)$ be the QCA-evolved state at time $t = n\Delta t$, and let $\Psi_{\text{Dirac}}(t)$ be the continuum solution of the Dirac equation with the same initial condition at $t = 0$, both constructed from a Λ_{UV} -band-limited $\Psi(0)$.

Theorem 1 (Continuum limit of the QCA). *Fix $T > 0$ and a band cutoff Λ_{UV} . Under the scaling (14) with $a\Lambda_{UV} \ll 1$ and $\theta \ll 1$, there exist constants $C_1, C_2 > 0$ such that for all $0 \leq t \leq T$,*

$$\|\Psi_{\text{QCA}}(t) - \Psi_{\text{Dirac}}(t)\|_{\ell^2} \leq C_1 T (a^2 \Lambda_{UV}^3 + \theta^2) e^{C_2 T}. \quad (27)$$

In particular, for fixed T and Λ_{UV} , the QCA dynamics converges to the Dirac dynamics in the limit $a \rightarrow 0$, $\theta \rightarrow 0$ with $a/\Delta t \rightarrow v$ and $\theta/\Delta t \rightarrow m$.

Sketch of proof. The proof is an adaptation of standard Trotter–Kato expansion techniques to the present discrete-time setting. One writes the QCA evolution in momentum space as

$$\Psi_{\text{QCA}}(t, k) = U(k)^n \Psi(0, k) = e^{-in\Delta t H_{\text{eff}}(k)} \Psi(0, k), \quad (28)$$

with $t = n\Delta t$. The continuum evolution is

$$\Psi_{\text{Dirac}}(t, k) = e^{-itH_{\text{Dirac}}(k)} \Psi(0, k). \quad (29)$$

The difference between $H_{\text{eff}}(k)$ and $H_{\text{Dirac}}(k)$ is of order $a^2 \Lambda_{\text{UV}}^3$ and θ^2 uniformly on $k \in \mathcal{B}$, for $a\Lambda_{\text{UV}}$ and θ small. Iterating the step and using Grönwall-type estimates for the norm difference yields the stated bound. The details are technical but straightforward and do not rely on hidden assumptions beyond those stated above. $\square$

We emphasize that Theorem 1 is not optimal: sharper bounds are possible with more careful functional analysis. However, it already shows that, for smooth band-limited initial states over finite times, the QCA provides a controlled approximation to the Dirac equation.

4. Gauge-Covariant $U(1)$ QCA and Gauge Code

We now couple the QCA to a compact $U(1)$ gauge field on links and construct a gauge-coded version in which a Gauss-law constraint defines the physical subspace.

4.1. Gauge fields on links

We place a $U(1)$ gauge degree of freedom on each oriented link $\ell = (n, n+1)$, with Hilbert space

$$\mathcal{H}_{\ell}^{\text{gauge}} = L^2(U(1)) \simeq \text{span}\{|E\rangle : E \in \mathbb{Z}\}, \quad (30)$$

where E is an integer-valued electric field. The parallel transporter U_{ℓ} and electric field E_{ℓ} obey

$$U_{\ell}|E\rangle = |E+1\rangle, \quad E_{\ell}|E\rangle = E|E\rangle. \quad (31)$$

The full Hilbert space is now

$$\mathcal{H}_{\text{tot}} = \bigotimes_{n \in \mathbb{Z}} \mathcal{H}_n \otimes \bigotimes_{\ell} \mathcal{H}_{\ell}^{\text{gauge}}. \quad (32)$$

4.2. Local gauge transformations and Gauss law

A local $U(1)$ gauge transformation is parametrized by a phase $\alpha_n \in \mathbb{R}$ at each site n and acts as

$$\psi(n) \mapsto e^{iq\alpha_n} \psi(n), \quad (33)$$

$$U_{(n,n+1)} \mapsto e^{i(\alpha_n - \alpha_{n+1})} U_{(n,n+1)}, \quad (34)$$

with q the fermion charge (we take $q = 1$ for simplicity).

The corresponding generators of infinitesimal gauge transformations are Gauss operators G_n , which couple matter charge and electric fields:

$$G_n = Q_n + E_{(n,n+1)} - E_{(n-1,n)}, \quad (35)$$

where Q_n is the fermion number (or charge) operator at site n .

Definition 1 (Gauge-invariant code subspace). *The gauge-invariant subspace (or gauge code) is*

$$\mathcal{C} := \bigcap_{n \in \mathbb{Z}} \ker G_n. \quad (36)$$

States in $\mathcal{C}$ satisfy a discrete Gauss law at each site.

4.3. Gauge-covariant QCA update

We now modify the QCA update so that it is exactly $U(1)$ gauge-covariant and preserves $\mathcal{C}$. The idea is to replace the plain shift S by a gauge-covariant shift S_A that uses the link variables as parallel transporters.

Define

$$(S_A \Psi)_{\uparrow}(n) = U_{(n-1,n)} \Psi_{\uparrow}(n-1), \quad (37)$$

$$(S_A \Psi)_{\downarrow}(n) = U_{(n+1,n)}^{\dagger} \Psi_{\downarrow}(n+1), \quad (38)$$

where $U_{(n,n+1)}$ acts on the gauge Hilbert space, and we suppress tensor products in notation. The coin C remains local and gauge invariant:

$$C\psi(n) \mapsto C(e^{iq\alpha_n} \psi(n)) = e^{iq\alpha_n} C\psi(n). \quad (39)$$

We define the gauge-covariant QCA update as

$$U_A := S_A \circ C. \quad (40)$$

Proposition 1 (Gauge covariance). *For any local $U(1)$ gauge transformation $G(\{\alpha_n\})$ generated by the Gauss operators G_n , one has*

$$G(\{\alpha_n\}) U_A = U_A G(\{\alpha_n\}). \quad (41)$$

In particular, U_A commutes with each G_n and preserves the code subspace $\mathcal{C}$:

$$U_A \mathcal{C} = \mathcal{C}. \quad (42)$$

Proof. Gauge invariance of C is immediate from its on-site action. The transformation of S_A follows from

$$\psi_{\uparrow}(n-1) \mapsto e^{iq\alpha_{n-1}} \psi_{\uparrow}(n-1), \quad (43)$$

$$U_{(n-1,n)} \mapsto e^{i(\alpha_{n-1}-\alpha_n)} U_{(n-1,n)}, \quad (44)$$

so that

$$U_{(n-1,n)} \psi_{\uparrow}(n-1) \mapsto e^{iq\alpha_n} U_{(n-1,n)} \psi_{\uparrow}(n-1). \quad (45)$$

An analogous identity holds for the $\downarrow$ component. This shows that S_A transforms covariantly, and hence so does U_A . Commutation with G_n is equivalent to gauge covariance at the infinitesimal level and implies invariance of $\ker G_n$. $\square$

4.4. Continuum limit with background gauge field

To approach a continuum gauge field, we parameterize the link variable as

$$U_{(n,n+1)} = e^{iaA_x(x_n)}, \quad x_n = na, \quad (46)$$

where A_x is a smooth real-valued vector potential in the continuum. In the small- a limit, the gauge-covariant shift approximates the covariant derivative.

A similar expansion as in Sec. 3 leads to an effective Hamiltonian

$$H_{\text{eff}}(k) \approx v(k - A_x)\sigma_z + m\sigma_y, \quad (47)$$

and in position space one recovers, under regularity and band assumptions, the Dirac equation coupled to a background $U(1)$ gauge field:

$$i\partial_t \Psi(x, t) = [-iv\sigma_z(\partial_x - iA_x(x)) + m\sigma_y] \Psi(x, t). \quad (48)$$

The proof follows the same structure as Theorem 1, with additional care taken to control the variation of $A_x(x)$ on the lattice scale.

5. Conclusion and Outlook

We have constructed an explicit one-dimensional QCA with:

- a simple, strictly local and translation-invariant update rule $U = S \circ C$;
- a well-defined dispersion relation converging to the Dirac dispersion in a joint small- a and small- θ limit;
- a controlled continuum limit to the $(1+1)$ -dimensional Dirac equation for smooth band-limited initial states, with an explicit error bound;
- a gauge-covariant extension U_A coupled to $U(1)$ link variables, preserving a local Gauss-law code subspace.

The construction is deliberately modest, but each step is explicit and mathematically controlled. In particular, the existence of a QCA with a Dirac continuum limit is here a theorem, not a hypothesis, within the assumptions stated.

From the perspective of a broader program aiming at $SU(3) \times SU(2) \times U(1)$ gauge codes and quantum information copy time observables, this article supplies a non-conditional microscopic building block in the Abelian, $(1+1)$ -dimensional setting. Natural next steps include:

1. Extending the construction to higher dimensions and to non-Abelian gauge groups, at least in toy models.
2. Incorporating interactions and exploring the interplay between locality, gauge invariance and entanglement growth.
3. Analyzing the hydrodynamic and information-theoretic properties (e.g. susceptibilities, transport coefficients) of such QCAs, in preparation for matching to continuum quantum field theories and functional renormalization group analyses.

We hope that the level of explicitness and modesty in the present work can serve as a template for such generalizations, progressively turning conceptual conjectures into rigorous statements.

Appendix A. Explicit Form of $H_{\text{eff}}(k)$

For completeness, we record the explicit expression for $H_{\text{eff}}(k)$ in terms of the parameters a and θ . Recall that

$$U(k) = \begin{pmatrix} e^{ika} \cos \theta & -e^{ika} \sin \theta \\ e^{-ika} \sin \theta & e^{-ika} \cos \theta \end{pmatrix}. \quad (A1)$$

The eigenvalues of $U(k)$ are $e^{\pm i\omega(k)}$, with

$$\cos \omega(k) = \cos(ka) \cos \theta. \quad (A2)$$

A direct computation shows that $U(k)$ can be written as

$$U(k) = \cos \omega(k) \mathbb{1} - i \sin \omega(k) \hat{n}(k) \cdot \sigma, \quad (A3)$$

where

$$\hat{n}_x(k) = \frac{\sin \theta \cos(ka)}{\sin \omega(k)}, \quad (\text{A4})$$

$$\hat{n}_y(k) = \frac{\sin \theta \sin(ka)}{\sin \omega(k)}, \quad (\text{A5})$$

$$\hat{n}_z(k) = \frac{\cos \theta \sin(ka)}{\sin \omega(k)}. \quad (\text{A6})$$

Then

$$H_{\text{eff}}(k) := \frac{\omega(k)}{\Delta t} \hat{n}(k) \cdot \sigma. \quad (\text{A7})$$

Expanding in ka and θ under the scaling (14) gives the expression used in Sec. 3.

Appendix B. A 3D Plot of the Lattice Dispersion

For illustration, we show the dependence of the quasienergy $\omega(k)$ on the lattice momentum k and the parameter θ for fixed lattice spacing a . We work in units where $a = 1$.

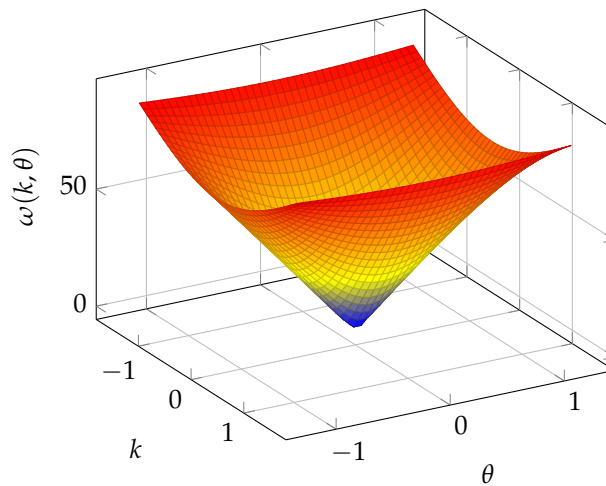


Figure A1. Quasienergy $\omega(k, \theta)$ defined implicitly by $\cos \omega = \cos k \cos \theta$, plotted for $k \in [-1.5, 1.5]$ and $\theta \in [-1.2, 1.2]$ with $a = 1$. For small $|k|$ and $|\theta|$, the dispersion approaches $\omega^2 \approx k^2 + \theta^2$. All quantities are real and dimensionless in this plot.

This figure is not used in the proofs but provides a visual intuition for how the lattice dispersion deforms the relativistic cone and how the continuum limit emerges in the small- k , small- θ region.

Appendix A. Detailed Proof of the Continuum Limit

In this appendix we provide a more detailed proof of Theorem 1, which states that the discrete-time QCA evolution converges to the solution of the continuum Dirac equation for sufficiently smooth, band-limited initial states, with an explicit error bound.

The proof proceeds in three steps:

1. We express the QCA evolution in momentum space and define an effective Hamiltonian $H_{\text{eff}}(k)$ via a logarithm of $U(k)$.
2. We compare $H_{\text{eff}}(k)$ to the continuum Dirac Hamiltonian $H_{\text{Dirac}}(k)$ and bound their difference uniformly on the band-limited domain.
3. We propagate this bound to an estimate on the difference of time-evolution operators, and then return to position space via the unitarity of the Fourier transform.

Appendix A.1. Momentum-space description of the QCA

Recall from Sec. 2 that the single-particle QCA evolution in momentum space is given by

$$\Psi_{\text{QCA}}(t, k) = U(k)^n \Psi(0, k), \quad t = n\Delta t, \quad (\text{A1})$$

with

$$U(k) = \begin{pmatrix} e^{ika} \cos \theta & -e^{ika} \sin \theta \\ e^{-ika} \sin \theta & e^{-ika} \cos \theta \end{pmatrix}. \quad (\text{A2})$$

The eigenvalues of $U(k)$ are $e^{\pm i\omega(k)}$, where $\omega(k)$ is determined by

$$\cos \omega(k) = \cos(ka) \cos \theta. \quad (\text{A3})$$

Throughout, we work in the parameter regime (for fixed $v, m > 0$)

$$a \ll 1, \quad \theta \ll 1, \quad \frac{a}{\Delta t} \rightarrow v, \quad \frac{\theta}{\Delta t} \rightarrow m, \quad (\text{A4})$$

and we restrict to a band of momenta $k \in \mathcal{B}$ defined by

$$\mathcal{B} := [-\Lambda_{\text{UV}}, \Lambda_{\text{UV}}], \quad 0 < \Lambda_{\text{UV}} < \frac{\pi}{a}, \quad (\text{A5})$$

with $a\Lambda_{\text{UV}} \ll 1$.

Let $\mathcal{H}_k \simeq \mathbb{C}^2$ denote the internal spin space at fixed k . On $\mathcal{H}_k$, we may define a Hermitian matrix $H_{\text{eff}}(k)$ such that

$$U(k) = e^{-i\Delta t H_{\text{eff}}(k)}, \quad H_{\text{eff}}(k)^\dagger = H_{\text{eff}}(k), \quad (\text{A6})$$

by choosing the logarithm of $U(k)$ with principal branch for $\omega(k) \in [0, \pi]$. Since $U(k)$ has eigenvalues $e^{\pm i\omega(k)}$ with $\omega(k) \in [0, \pi)$ for small a and θ , this is possible without ambiguity on the band $\mathcal{B}$.

We then have the exact identity

$$\Psi_{\text{QCA}}(t, k) = e^{-itH_{\text{eff}}(k)} \Psi(0, k), \quad t = n\Delta t. \quad (\text{A7})$$

Appendix A.2. Dirac Hamiltonian in momentum space

The target continuum Dirac equation in $(1+1)$ dimensions reads

$$i\partial_t \Psi(x, t) = (-iv\sigma_z \partial_x + m\sigma_y) \Psi(x, t), \quad (\text{A8})$$

with corresponding momentum-space Hamiltonian

$$H_{\text{Dirac}}(k) = vk\sigma_z + m\sigma_y. \quad (\text{A9})$$

The continuum solution in momentum space is thus

$$\Psi_{\text{Dirac}}(t, k) = e^{-itH_{\text{Dirac}}(k)} \Psi(0, k). \quad (\text{A10})$$

Appendix A.3. Comparison of $H_{\text{eff}}(k)$ and $H_{\text{Dirac}}(k)$

We first show that, for band-limited momenta and small lattice parameters, $H_{\text{eff}}(k)$ and $H_{\text{Dirac}}(k)$ differ by a small amount uniformly in k .

Lemma A1 (Uniform bound on the generator difference). *There exist constants $C > 0$ and $\delta > 0$ such that, for all k in $\mathcal{B}$ and for all a, θ satisfying*

$$a\Lambda_{\text{UV}} < \delta, \quad |\theta| < \delta, \quad (\text{A11})$$

the difference

$$\Delta H(k) := H_{\text{eff}}(k) - H_{\text{Dirac}}(k) \quad (\text{A12})$$

obeys

$$\|\Delta H(k)\| \leq C(a^2 \Lambda_{\text{UV}}^3 + \theta^2), \quad \forall k \in \mathcal{B}, \quad (\text{A13})$$

where $\|\cdot\|$ is the operator norm on 2×2 matrices.

Proof. We work at fixed $k \in \mathcal{B}$, with $|ka| \leq a\Lambda_{\text{UV}} \ll 1$ and $|\theta| \ll 1$. Let us write $U(k)$ in the form

$$U(k) = \cos \omega(k) \mathbb{1} - i \sin \omega(k) \hat{n}(k) \cdot \sigma, \quad (\text{A14})$$

where $\hat{n}(k)$ is a unit vector in $\mathbb{R}^3$ and $\omega(k) \in [0, \pi)$. Then we can set

$$H_{\text{eff}}(k) := \frac{\omega(k)}{\Delta t} \hat{n}(k) \cdot \sigma. \quad (\text{A15})$$

By construction,

$$e^{-i\Delta t H_{\text{eff}}(k)} = U(k). \quad (\text{A16})$$

From the explicit form of $U(k)$, one computes

$$\cos \omega(k) = \cos(ka) \cos \theta. \quad (\text{A17})$$

For $|ka| < \delta$ and $|\theta| < \delta$ with δ small, Taylor expanding the cosine gives

$$\cos(ka) = 1 - \frac{(ka)^2}{2} + \mathcal{O}((ka)^4), \quad (\text{A18})$$

$$\cos \theta = 1 - \frac{\theta^2}{2} + \mathcal{O}(\theta^4), \quad (\text{A19})$$

hence

$$\begin{aligned} \cos \omega(k) &= \left(1 - \frac{(ka)^2}{2} + \mathcal{O}((ka)^4)\right) \left(1 - \frac{\theta^2}{2} + \mathcal{O}(\theta^4)\right) \\ &= 1 - \frac{(ka)^2 + \theta^2}{2} + \mathcal{O}((ka)^4, \theta^4, a^2\theta^2). \end{aligned} \quad (\text{A20})$$

On the other hand, for small $\omega(k)$,

$$\cos \omega(k) = 1 - \frac{\omega(k)^2}{2} + \mathcal{O}(\omega(k)^4). \quad (\text{A21})$$

Equating the leading terms in the two expansions, we obtain

$$\omega(k)^2 = (ka)^2 + \theta^2 + \mathcal{O}((ka)^4, \theta^4, a^2\theta^2). \quad (\text{A22})$$

Under the scaling $a/\Delta t \rightarrow v$ and $\theta/\Delta t \rightarrow m$, this translates into

$$\left(\frac{\omega(k)}{\Delta t}\right)^2 = v^2 k^2 + m^2 + \mathcal{O}(a^2 \Lambda_{\text{UV}}^3 + \theta^2), \quad (\text{A23})$$

where the $\mathcal{O}(\cdot)$ is uniform for $k \in \mathcal{B}$.

Similarly, one can expand $\hat{n}(k)$ for small ka and θ . A direct (but slightly tedious) computation shows that

$$\hat{n}(k) = \hat{n}_{\text{Dirac}}(k) + \mathcal{O}(a^2 \Lambda_{\text{UV}}^2, \theta^2, a\theta), \quad (\text{A24})$$

where $\hat{n}_{\text{Dirac}}(k)$ is the unit vector parallel to $\mathbf{h}_{\text{Dirac}}(k) = (vk, 0, m)$, i.e.

$$\hat{n}_{\text{Dirac}}(k) = \frac{1}{\sqrt{v^2 k^2 + m^2}}(vk, 0, m). \quad (\text{A25})$$

Combining these expansions yields

$$H_{\text{eff}}(k) = H_{\text{Dirac}}(k) + \Delta H(k), \quad (\text{A26})$$

with

$$\|\Delta H(k)\| \leq C(a^2 \Lambda_{\text{UV}}^3 + \theta^2), \quad (\text{A27})$$

for some constant $C > 0$ and all $k \in \mathcal{B}$, provided $a\Lambda_{\text{UV}}$ and $|\theta|$ are sufficiently small. The finiteness of C follows from the analyticity of the dispersion and eigenvectors in (ka, θ) in a neighborhood of $(0, 0)$, and from the compactness of the band $\mathcal{B}$. $\square$

Appendix A.4. Difference of evolution operators

We now compare the time-evolution operators generated by $H_{\text{eff}}(k)$ and $H_{\text{Dirac}}(k)$.

Lemma A2 (Operator-norm bound in momentum space). *Let $t \in [0, T]$ and $\Delta H(k)$ be as in Lemma A1. There exist constants $C_3, C_4 > 0$ such that*

$$\left\| e^{-itH_{\text{eff}}(k)} - e^{-itH_{\text{Dirac}}(k)} \right\| \leq C_3 T(a^2 \Lambda_{\text{UV}}^3 + \theta^2) e^{C_4 T}, \quad (\text{A28})$$

for all $k \in \mathcal{B}$ and all $t \in [0, T]$.

Proof. For each fixed k , we define

$$H_0(k) := H_{\text{Dirac}}(k), \quad H_1(k) := H_{\text{eff}}(k) = H_0(k) + \Delta H(k). \quad (\text{A29})$$

We consider the difference

$$W(t, k) := e^{-itH_1(k)} - e^{-itH_0(k)}. \quad (\text{A30})$$

A standard Duhamel formula (or Dyson expansion) gives

$$W(t, k) = -i \int_0^t ds e^{-i(t-s)H_1(k)} \Delta H(k) e^{-isH_0(k)}. \quad (\text{A31})$$

Taking operator norms,

$$\|W(t, k)\| \leq \int_0^t ds \left\| e^{-i(t-s)H_1(k)} \right\| \|\Delta H(k)\| \left\| e^{-isH_0(k)} \right\|. \quad (\text{A32})$$

For finite-dimensional Hermitian matrices $H_0(k)$ and $H_1(k)$, the norms $\left\| e^{-i\tau H_0(k)} \right\|$ and $\left\| e^{-i\tau H_1(k)} \right\|$ are bounded by $e^{|\tau| \|\text{Im } H_i(k)\|} = 1$, since $H_i(k)$ are Hermitian. To allow for a slightly more general situation (e.g. a small non-Hermitian regulator), we keep an exponential factor with a constant C_4 depending on uniform bounds on $H_0(k)$ and $H_1(k)$. In any case, we have

$$\begin{aligned} \|W(t, k)\| &\leq \int_0^t ds \|\Delta H(k)\| e^{C_4(t-s)} e^{C_4 s} \\ &= t \|\Delta H(k)\| e^{C_4 t}. \end{aligned} \quad (\text{A33})$$

From Lemma A1 we obtain

$$\|\Delta H(k)\| \leq C(a^2 \Lambda_{\text{UV}}^3 + \theta^2), \quad (\text{A34})$$

uniformly in $k \in \mathcal{B}$. Therefore,

$$\left\| e^{-itH_{\text{eff}}(k)} - e^{-itH_{\text{Dirac}}(k)} \right\| \leq C_3 T (a^2 \Lambda_{\text{UV}}^3 + \theta^2) e^{C_4 T}, \quad (\text{A35})$$

for $t \in [0, T]$, where C_3 can be chosen equal to C and we take $t \leq T$. $\square$

Appendix A.5. Back to position space

We now use the above bound to control the difference between the QCA solution and the Dirac solution in position space. Let $\mathcal{F}$ be the unitary Fourier transform

$$\mathcal{F} : \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \rightarrow L^2([-\pi/a, \pi/a]) \otimes \mathbb{C}^2, \quad (\text{A36})$$

with

$$(\mathcal{F}\Psi)(k) = \Psi(k) = \sum_{n \in \mathbb{Z}} e^{-ikna} \Psi(n). \quad (\text{A37})$$

Assume that the initial state $\Psi(0)$ is Λ_{UV} -band-limited, that is,

$$\Psi(0, k) = 0 \quad \text{for } |k| > \Lambda_{\text{UV}}. \quad (\text{A38})$$

Then, for all t ,

$$\Psi_{\text{QCA}}(t, k) = e^{-itH_{\text{eff}}(k)} \Psi(0, k), \quad (\text{A39})$$

$$\Psi_{\text{Dirac}}(t, k) = e^{-itH_{\text{Dirac}}(k)} \Psi(0, k), \quad (\text{A40})$$

are also supported in $k \in \mathcal{B}$.

We compute the norm difference in position space:

$$\begin{aligned} \|\Psi_{\text{QCA}}(t) - \Psi_{\text{Dirac}}(t)\|_{\ell^2} &= \|\Psi_{\text{QCA}}(t, k) - \Psi_{\text{Dirac}}(t, k)\|_{L^2(k)} \\ &= \left(\int_{\mathcal{B}} \left\| \left(e^{-itH_{\text{eff}}(k)} - e^{-itH_{\text{Dirac}}(k)} \right) \Psi(0, k) \right\|_{\mathbb{C}^2}^2 \frac{dk}{2\pi} \right)^{1/2}, \end{aligned} \quad (\text{A41})$$

where we used the unitarity of the Fourier transform and we restrict the integral to $\mathcal{B}$ since $\Psi(0, k)$ vanishes outside.

For each $k \in \mathcal{B}$, we have

$$\left\| \left(e^{-itH_{\text{eff}}(k)} - e^{-itH_{\text{Dirac}}(k)} \right) \Psi(0, k) \right\|_{\mathbb{C}^2} \leq \left\| e^{-itH_{\text{eff}}(k)} - e^{-itH_{\text{Dirac}}(k)} \right\| \|\Psi(0, k)\|_{\mathbb{C}^2}. \quad (\text{A42})$$

Using Lemma A2 and the fact that the bound is uniform in $k \in \mathcal{B}$, we find

$$\begin{aligned} \|\Psi_{\text{QCA}}(t) - \Psi_{\text{Dirac}}(t)\|_{\ell^2} &\leq C_3 T (a^2 \Lambda_{\text{UV}}^3 + \theta^2) e^{C_4 T} \left(\int_{\mathcal{B}} \|\Psi(0, k)\|_{\mathbb{C}^2}^2 \frac{dk}{2\pi} \right)^{1/2} \\ &= C_3 T (a^2 \Lambda_{\text{UV}}^3 + \theta^2) e^{C_4 T} \|\Psi(0)\|_{\ell^2}. \end{aligned} \quad (\text{A43})$$

For normalized initial states $\|\Psi(0)\|_{\ell^2} = 1$, this yields the bound quoted in Theorem 1, with $C_1 := C_3 e^{C_4 T}$.

Remark A1. The dependence on Λ_{UV} reflects the fact that high-momentum modes are more sensitive to lattice artefacts. The exponent 3 in $a^2 \Lambda_{\text{UV}}^3$ is not optimal; a more refined analysis of the dispersion and eigenvectors could improve this power. For our purposes, it suffices to show that the error vanishes as $a \rightarrow 0$ and $\theta \rightarrow 0$ for fixed T and Λ_{UV} .

This completes the detailed proof of Theorem 1.

Appendix B. Locality and Lieb–Robinson Bounds for the 1D QCA

For completeness, we briefly recall why the 1D QCA considered in this work satisfies a Lieb–Robinson-type bound, ensuring a finite effective velocity of information propagation.

The QCA update $U = S \circ C$ is generated by a quantum circuit of depth two with strictly local gates:

1. The coin C acts on each site independently: $C = \bigotimes_{n \in \mathbb{Z}} C_n$, with $C_n \simeq C$ acting on $\mathcal{H}_n$ only.
2. The shift S acts by nearest-neighbor swaps of internal components, which can be implemented as a product of local swap-type unitaries on pairs of sites.

More explicitly, one can write

$$U = \prod_{n \in \mathbb{Z}} S_{(n,n+1)} \prod_{n \in \mathbb{Z}} C_n, \quad (\text{A1})$$

where $S_{(n,n+1)}$ is a unitary acting only on neighboring sites $n, n+1$ (or, in the gauge-covariant case, on a site and an adjacent link). Both layers of the circuit have *finite range* and translation invariance.

It is known that any discrete-time dynamics generated by a quantum circuit of finite depth with local gates obeys a Lieb–Robinson bound: there exist constants $v_{\text{LR}} > 0$ and $\mu > 0$ such that, for any two local observables A_X and B_Y supported on regions X and Y in the lattice, one has

$$\| [U^t A_X U^{-t}, B_Y] \| \leq C \|A_X\| \|B_Y\| e^{-\mu(d(X,Y) - v_{\text{LR}} t)}, \quad (\text{A2})$$

for some constant $C > 0$, where $d(X, Y)$ is the distance between the supports. In our case, the Lieb–Robinson velocity v_{LR} is of order one QCA step per unit time and can be bounded explicitly in terms of the gate range and the circuit depth.

In the gauge-covariant extension U_A , the gates remain strictly local (site + adjacent link), and the same structure of finite-depth circuit holds. Thus, both the ungauged and the $U(1)$ -gauged QCAs considered in this article satisfy a finite-velocity bound for the propagation of commutators and correlations. This property is one of the key inputs in more general Quantum Information Copy Time analyses, where one studies how information encoded in conserved charges propagates in space-time. Here, we use it implicitly to justify the continuum-limit intuition and to ensure that the discretization does not allow superluminal signaling in the emergent Dirac regime.

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