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Explicit Discrete Solution for Some Optimization Problems and Estimations with Respect to the Exact Solution

Julieta Bollati [1], Mariela C. Olguin [2] and Domingo A. Tarzia [1,*]

[1] CONICET-Depto. Matemática, FCE, Univ. Austral, Paraguay 1950, Rosario, Argentina
[2] Departamento de Matemática, EFB-FCEIA, Univ. Nacional de Rosario, Rosario, Argentina
* Correspondence: dtarzia@austral.edu.ar

Abstract

We consider two steady state heat conduction systems called, S and S_α, in a multidimensional bounded domain D for the Poisson equation with source energy g. In one system we impose mixed boundary conditions (temperature b on the boundary Γ_1, heat flux q on Γ_2 and an adiabatic condition on Γ_3). In the other system, the condition on Γ_1 is replaced by a convective heat flux condition with coefficient α. For each of these systems, we consider three associated optimization problems (P_i) and $(P_{i\alpha})$, $i = 1, 2, 3$ where the variable will be the source energy g, the heat flux q and the environmental temperature b, respectively. In the particular case that D is a rectangle, the explicit continuous optimization variables and the corresponding state of the systems are known. In the present work, by using a finite difference scheme, we obtain the discrete systems (S_h) and $(S_{h\alpha})$ and discrete optimization problems (P_{ih}) and $(P_{ih\alpha})$, $i = 1, 2, 3$, where h is the space step in the discretization. Explicit discrete solutions are found, and convergence and estimation errors results are proved when h goes to zero and when α goes to infinite. Moreover, some numerical simulations are provided in order to test theoretical results. Finally, we note that the use of a three–point finite–difference approximation for the Neumann or Robin boundary condition at the boundary improves the global order of convergence from $O(h)$ to $O(h^2)$.

Keywords: optimal control; finite difference method; explicit discrete solutions; estimation error

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1. Introduction

We consider a multidimensional bounded domain $\Omega \subset \mathbb{R}^n$ whose regular boundary Γ consists of three disjoint portions Γ_i with $meas(\Gamma_i) > 0$, for $i = 1, 2, 3$. We define two stationary heat conduction problems (S) and (S_α) with mixed boundary conditions which are given by (1)-(2) and (1)-(3) respectively:

$$-\Delta u = g \quad in \quad \Omega, \tag{1}$$

$$u\Big|_{\Gamma_1} = b, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_3} = 0, \tag{2}$$

$$-\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha\,(u - b), \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_3} = 0, \tag{3}$$

where g is the internal energy of the system in Ω, $b > 0$ the environmental temperature on Γ_1, q is the heat flux on Γ_2 and $\alpha > 0$ is the convective heat coefficient on Γ_1. We assume that: $g \in H = L^2(\Omega)$, $q \in Q = L^2(\Gamma_2)$ and $b \in B = H^{1/2}(\Gamma_1)$. These problems can be considered as Stefan's stationary problems [8,19–22]. Notice that mixed boundary conditions play an important role in several applications, e.g. heat conduction and electric potential problems [12].

The variational formulation of the elliptic problems (1)-(2) and (1)-(3) can be found in [8–10,19]. It can be seen that they have unique solution [20]. In general, the solutions of mixed elliptic boundary problems is not so regular [11] but there are cases in which they are regular [4,15,18]. Other theoretical optimization problems in the subject have been done in [13,25].

We define the optimization problems (P_i) and $(P_{i\,\alpha})$ $i = 1,2,3$, associated to the systems (S) and (S_α), respectively (see [5,9,16,17,24]).

The distributed optimization problems (P_1) and $(P_{1\alpha})$ on the constant internal energy g are formulated as:

$$\text{find} \quad g_{op} \in \mathbb{R} \quad \text{such that} \quad J_1(g_{op}) = \min_{g\in\mathbb{R}} J_1(g) \tag{4}$$

$$\text{find} \quad g_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{1\alpha}(g_{\alpha_{op}}) = \min_{g\in\mathbb{R}} J_{1\alpha}(g) \tag{5}$$

where $J_1 : \mathbb{R}\to\mathbb{R}_0^+$ and $J_{1\alpha} : \mathbb{R}\to\mathbb{R}_0^+$ are given by

$$J_1(g) = \frac{1}{2}\|u_g - z_d\|_H^2 + \frac{M_1}{2}\|g\|_H^2, \qquad J_{1\alpha}(g) = \frac{1}{2}\|u_{\alpha g} - z_d\|_H^2 + \frac{M_1}{2}\|g\|_H^2 \tag{6}$$

with $M_1 \in \mathbb{R}^+$ and $z_d \in \mathbb{R}$. For each $g \in \mathbb{R}$, the functions u_g and $u_{\alpha g}$ denote the unique solutions to the systems (S) and (S_α) respectively, for data $q \in \mathbb{R}$, $b \in \mathbb{R}$.

The boundary optimization problems (P_2) and $(P_{2\alpha})$ on the constant heat flux q on Γ_2 are defined as:

$$\text{find} \quad q_{op} \in \mathbb{R} \quad \text{such that} \quad J_2(q_{op}) = \min_{q\in\mathbb{R}} J_2(q) \tag{7}$$

$$\text{find} \quad q_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{2\alpha}(q_{\alpha_{op}}) = \min_{q\in\mathbb{R}} J_{2\alpha}(q) \tag{8}$$

where $J_2 : \mathbb{R}\to\mathbb{R}_0^+$ and $J_{2\alpha} : \mathbb{R}\to\mathbb{R}_0^+$ are given by

$$J_2(q) = \frac{1}{2}\|u_q - z_d\|_H^2 + \frac{M_2}{2}\|q\|_Q^2, \qquad J_{2\alpha}(q) = \frac{1}{2}\|u_{\alpha q} - z_d\|_H^2 + \frac{M_2}{2}\|q\|_Q^2 \tag{9}$$

with $M_2 \in \mathbb{R}^+$ and $z_d \in \mathbb{R}$. For each $q \in \mathbb{R}$, we denote with u_q and $u_{\alpha q}$ the unique solutions to the systems (S) and (S_α) respectively, for data $g \in \mathbb{R}$ and $b \in \mathbb{R}$.

The boundary optimization problems (P_3) and $(P_{3\alpha})$ on the constant temperature b in an external neighborhood of Γ_1 are set as

$$\text{find} \quad b_{op} \in \mathbb{R} \quad \text{such that} \quad J_3(b_{op}) = \min_{b\in\mathbb{R}} J_3(b) \tag{10}$$

$$\text{find} \quad b_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{3\alpha}(b_{\alpha_{op}}) = \min_{b\in\mathbb{R}} J_{3\alpha}(q) \tag{11}$$

where $J_3 : \mathbb{R}\to\mathbb{R}_0^+$ and $J_{3\alpha} : \mathbb{R}\to\mathbb{R}_0^+$, given by

$$J_3(b) = \frac{1}{2}\|u_b - z_d\|_H^2 + \frac{M_3}{2}\|b\|_B^2, \qquad J_{3\alpha}(b) = \frac{1}{2}\|u_{\alpha b} - z_d\|_H^2 + \frac{M_3}{2}\|b\|_B^2 \tag{12}$$

with $M_3 \in \mathbb{R}^+$ and $z_d \in \mathbb{R}$. For every $b \in \mathbb{R}$, the functions u_b and $u_{\alpha b}$ are the unique solutions of the systems (S) and (S_α) respectively, for data $g \in \mathbb{R}$ and $q \in \mathbb{R}$, .

In [6] the explicit solutions to the continuous systems (S) and (S_α) were obtained, as well as the associated optimization problems (P_i) and $(P_{i\alpha})$ for $i = 1,2,3$, in the particular case when the domain is a rectangle. These explicit solutions provide a benchmark for testing the accuracy of numerical methods.

The goal of this paper is to find the discrete explicit solutions to the systems (S) and (S_α), through the finite difference method, in a rectangular domain. In addition, we aim to obtain the discrete explicit solutions to the optimization problems (P_i) and $(P_{i\alpha})$ for $i = 1,2,3$. Taking advantage of the exact

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explicit solutions, we will estimate the order of convergence of the approximate solutions as a function of the step discretization.

It is worth to mention that there are several articles available in the literature that obtain explicit discrete solutions of some optimization problems [1–3]. For example, in [14], exact formulas are derived for the solution of an optimal boundary control problem governed by the one dimensional heat equation where the control function measures the distance of the final state from the target. In [26] a finite element approximation is applied for some kind of parabolic optimal control problems with Neumann boundary conditions. Some numerical experiments are carried out setting a rectangular domain.

This paper is organized as follows: in Section 2 we obtain the discrete explicit solution to the systems (S) and (S_α) by the finite difference method. In Section 3, we obtain explicit discrete solutions to the discrete distributed optimization problems associated to (P_1) and $(P_{1\alpha})$, respectively where the variable is the internal energy g. In Section 4, we define discrete boundary optimization problems where the variable is the heat flux q, associated to (P_2) and $(P_{2\alpha})$, respectively obtaining the discrete explicit solutions. In the same manner, in Section 5, we derive explicit discrete solutions to the discrete boundary optimal control problems associated to (P_3) and $(P_{3\alpha})$, respectively where the optimization variable is b. In all cases, when the step discretization goes to zero, convergence results are obtained estimating also the order of convergence of the approximate solutions. In Section 6, we carry out some numerical simulations in order to illustrate the convergence theoretical results obtained in the previous sections. Finally, in Section 7, we analyze the order of convergence of the discrete systems associated with (S) and (S_α) by considering a modified approximation of the Neumann boundary condition on Γ_2, which leads to an improved convergence order.

2. Discrete Systems for (S) and (S_α)

In this section we obtain the discrete explicit solutions to the systems (S) and (S_α) in a particular domain.

Let us consider a rectangular domain in the plane $\Omega = (0, x_0) \times (0, y_0)$ with $x_0 > 0$ and $y_0 > 0$. Its boundaries Γ_i for $i = 1, 2, 3$ are defined by:

$$\Gamma_1 = \{(0, y) : y \in (0, y_0]\}, \quad \Gamma_2 = \{(x_0, y) : y \in (0, y_0]\}$$

and

$$\Gamma_3 = \{(x, 0) : x \in [0, x_0]\} \cup \{(x, y_0) : x \in [0, x_0]\}.$$

According to [6] the continuous solutions, in Ω, for the systems (S) and (S_α) defined by (1)-(2) and (1)-(3) are given by:

$$\begin{aligned} u(x,y) &= -\tfrac{1}{2} g x^2 + (g x_0 - q)x + b, \qquad \forall\ (x,y) \in \Omega \\ u_\alpha(x,y) &= -\tfrac{1}{2} g x^2 + (g x_0 - q)x + \tfrac{1}{\alpha}(g\, x_0 - q) + b, \qquad \forall\ (x,y) \in \Omega. \end{aligned} \tag{13}$$

As consequence of the symmetry of the domain Ω, the solution u and u_α of the systems (S) and (S_α) are independent of the variable y, and therefore we will work with one-dimensional problems.

Given $n \in \mathbb{N}$, we define:

$$h = \tfrac{x_0}{n}; \quad x_i = (i-1)\, h, \text{ for } i = 1, \dots, n+1, \quad \mathrm{u}_i \approx u(x_i, y) \text{ for } i = 2, \dots, n+1, \tag{14}$$

where, n is the number of subintervals to consider in $[0, x_0]$, h is the constant size of these subintervals, $\mathrm{u}_1 = b$ and u_i is the approximate value of the function u in $(x_i, y) \in \Omega$ for $i = 2, \dots, n+1$.

We apply the classical finite–difference method to the system (S) described by equations (1)–(2). Since the boundary condition on Γ_1 prescribes $u(0, y) = b$, we immediately obtain $\mathrm{u}_1 = b$.

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For the interior nodes, we use the classical centered second–order finite difference approximation:

$$\frac{\partial^2 u}{\partial x^2}(x_i, y) \approx \frac{u(x_{i+1}, y) - 2u(x_i, y) + u(x_{i-1}, y)}{h^2}, \qquad i = 2, 3, \dots, n, \tag{15}$$

and from (1) it follows that

$$-gh^2 = \mathrm{u}_{i+1} - 2\mathrm{u}_i + \mathrm{u}_{i-1}, \qquad i = 2, 3, \dots, n. \tag{16}$$

To incorporate the Neumann boundary condition on Γ_2, we use a backward finite difference for the first derivative:

$$\frac{\partial u}{\partial x}(x_{n+1}, y) \approx \frac{u(x_{n+1}, y) - u(x_n, y)}{h}, \tag{17}$$

which, using the boundary condition $\frac{\partial u}{\partial x}(x_{n+1}, y) = q$, leads to

$$-qh = \mathrm{u}_{n+1} - \mathrm{u}_n. \tag{18}$$

As a consequence, we obtain the discrete linear system (S_h)

$$A\,\overline{\mathrm{v}} = T, \tag{19}$$

where $\overline{\mathrm{v}} = (\mathrm{u}_i)_{i=2,\dots,n+1} \in \mathbb{R}^n$ is the vector of unknowns, A is the associated tridiagonal coefficient matrix:

$$A = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & & 0 \\ 1 & -2 & 1 & 0 & \dots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 1 \end{pmatrix} \tag{20}$$

and $T \in \mathbb{R}^n$ is the vector of independent terms:

$$T = \Big(-g\,h^2 - b, -gh^2, \dots, -gh^2, -q\,h\Big)^t. \tag{21}$$

The square matrix A is invertible and its inverse matrix is given by

$$A^{-1} = \begin{pmatrix} -1 & -1 & -1 & \dots & \dots & -1 & 1 \\ -1 & -2 & -2 & \dots & \dots & -2 & 2 \\ -1 & -2 & -3 & \dots & \dots & -3 & 3 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ -1 & -2 & -3 & \dots & & -(n-1) & n-1 \\ -1 & -2 & -3 & \dots & & -(n-1) & n \end{pmatrix}$$

Then the system (S_h) has a unique solution:

$$\mathrm{u}_1 = b \qquad \mathrm{u}_i = b + h^2 g\Big((i-1)\,n - \tfrac{i(i-1)}{2}\Big) - (i-1)hq, \quad i = 2, \dots, n+1.$$

As $n = \frac{x_0}{h}$, it follows that:

$$\mathrm{u}_i = b + (i-1)\,h\,(g\,x_0 - q) - h^2 g\,\frac{i(i-1)}{2}, \quad i = 1, \dots, n+1.$$

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Then, the continuous solution $u(x,y)$ of system (S) can be approximated by the piecewise linear interpolant $u_h(x,y)$ obtained from the nodal values computed by the finite difference scheme. More precisely, we define

$$u_h(x,y) = (gx_0 - q - h\,g\,i)x + h^2 g\left(\tfrac{i(i-1)}{2}\right) + b, \quad x \in [x_i,\, x_{i+1}],\ \ y \in [0, y_0], \tag{22}$$

with $i = 1, \ldots, n$.

In the following lemma, we give some bounds for the approximate function u_h :

Lemma 1.

a) *For every (x_i, y) with $i = 1, \ldots, n+1$, $y \in [0, y_0]$, we have that:*
   i) *if $g > 0$ then $u_h(x_i, y) \leq u(x_i, y)$.*
   ii) *if $g < 0$, then $u_h(x_i, y) \geq u(x_i, y)$.*

b) *The following bounds hold:*

$$\|u - u_h\|_H \leq C_1\, h, \qquad \text{and} \qquad \|\tfrac{\partial u}{\partial x} - \tfrac{\partial u_h}{\partial x}\|_H \leq \widetilde{C_1}\, h,$$

where $C_1 = x_0\,|g|\,\sqrt{\frac{2}{15} x_0\, y_0}$ and $\widetilde{C_1} = |g|\sqrt{\frac{1}{3} x_0\, y_0}$.

Proof.

a) From functions u and u_h, we have that

$$u(x_i, y) - u_h(x_i, y) = u(x_i, y) - \mathrm{u}_i = \tfrac{g\,h^2\,(i-1)}{2}, \quad i = 1, \ldots, n+1.$$

b) From the definition of norm in space H and the formulas (13) and (22) for the functions u and u_h, respectively, it follows that:

$$\begin{aligned} ||u - u_h||_H^2 &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \big(u(x,y) - u_h(x,y)\big)^2 dx \\ &= \tfrac{1}{120} y_0\, h^5\, g^2\, n^3\, \left(\tfrac{1}{n^2} + \tfrac{5}{n} + 10\right) \\ &\leq \tfrac{2}{15}\, y_0\, h^5\, g^2\, n^3 = \tfrac{2}{15}\, h^2 g^2\, x_0^3\, y_0 = C_1^2 h^2. \end{aligned}$$

The norm $||\frac{\partial u}{\partial x} - \frac{\partial u_h}{\partial x}||_H$ can be computed analogously.
□

We apply the classical finite–difference method to the system (S_α) defined by equations (1)–(3) where $\mathrm{u}_{\alpha,i} \approx u_\alpha(x_i, y)$ for $i = 1, \cdots, n+1$.

The Robin boundary condition on Γ_1 is approximated by a forward finite–difference scheme, namely,

$$\frac{\mathrm{u}_{\alpha,2} - \mathrm{u}_{\alpha,1}}{h} = \alpha\big(\mathrm{u}_{\alpha,1} - b\big). \tag{23}$$

Moreover, at the interior nodes we use the approximation given in (16), while the Neumann boundary condition at x_{n+1} is discretized according to (17).

Then we obtain a linear system $(S_{h\,\alpha})$:

$$A_\alpha\, \overline{\mathrm{v}}_\alpha = T_\alpha$$

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where the vector of unknowns $\overline{v}_\alpha \in \mathbb{R}^{n+1}$ is given by $\overline{v}_\alpha = (u_{\alpha,i})_{i=1,\ldots n+1}$, the tridiagonal coefficient matrix A_α of order $n+1$, is defined as:

$$A_\alpha = \begin{pmatrix} -(1+\alpha\, h) & 1 & 0 & \ldots & \ldots & & 0 \\ 1 & -2 & 1 & & & & \\ \vdots & & \ldots & \ldots & & 1 & -2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{pmatrix} \tag{24}$$

and

$$T_\alpha = \Big(-\alpha\, b\, h, -gh^2, \ldots, -gh^2, -q\, h \Big)^t \in \mathbb{R}^{n+1}. \tag{25}$$

It can be seen that the square matrix A_α is invertible and its inverse matrix is given by

$$A_\alpha^{-1} = \frac{1}{\alpha h} \begin{pmatrix} -1 & -1 & -1 & \ldots & \ldots & -1 & 1 \\ -1 & -(1+\alpha h) & -(1+\alpha h) & \ldots & \ldots & -(1+\alpha h) & 1+\alpha h \\ -1 & -(1+\alpha h) & -(1+2\alpha h) & \ldots & \ldots & -(1+2\alpha h) & -(1+2\alpha h) \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ -1 & -(1+\alpha h) & -(1+2\alpha h) & \ldots & & -(1+(n-1)\alpha h) & 1+(n-1)\alpha h \\ -1 & -(1+\alpha h) & -(1+2\alpha h) & \ldots & & -(1+(n-1)\alpha h) & 1+n\alpha h \end{pmatrix}$$

Then the linear system $(S_{h\alpha})$ has a unique solution:

$$u_{\alpha,i} = (b + \tfrac{g\, x_0 - q}{\alpha}) + (i-1)\,(g\, x_0 - q)\, h - \tfrac{g}{\alpha}\, h - g\, \tfrac{i(i-1)}{2}\, h^2, \quad i = 1, \ldots, n+1.$$

As a consequence, the continuous solution $u_\alpha(x,y)$ of the system (S_α), can be approximated by the discrete function $u_{h\alpha}(x,y)$ in $\overline{\Omega}$ given by

$$u_{h\alpha}(x,y) = (g\, x_0 - q - h\, g\, i)x + b + \tfrac{g\, x_0 - q}{\alpha} - \tfrac{g\, h}{\alpha} + \tfrac{i^2 - i}{2} g\, h^2, \tag{26}$$

for $x \in [x_i,\, x_{i+1}], y \in [0, y_0], i = 1, \ldots, n$.

Notice that $u_{\alpha,i} \to u_i$, when $\alpha \to \infty$ for each $i = 1, 2, \cdots, n+1$. In addition, $u_{h\,\alpha}(x,y) \to u_h(x,y)$ when $\alpha \to \infty$ for every $(x,y) \in \overline{\Omega}$.

Lemma 2. *If u_α is the solution of (S_α) and $u_{h\,\alpha}$ is the function given in (26) for each h, it follows that:*

$$\|u_\alpha - u_{h\,\alpha}\|_H \le C_{1\alpha}\, h, \qquad \text{and} \qquad \|\tfrac{\partial u_\alpha}{\partial x} - \tfrac{\partial u_{h\alpha}}{\partial x}\|_H \le \widetilde{C_1}\, h,$$

with $C_{1\alpha} = |\, g| x_0 \sqrt{x_0\, y_0 \Big(\frac{2}{15} + \frac{2}{3} \frac{1}{\alpha x_0} + \frac{1}{\alpha^2 x_0^2} \Big)}$ and $\widetilde{C_1} = |g| \sqrt{\frac{1}{3} x_0\, y_0}$.

Proof. From the definition of u_α and $u_{h\alpha}$, we have

$$\begin{aligned} ||u_\alpha - u_{h\alpha}||_H^2 &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \big(u_\alpha(x,y) - u_{h\alpha}(x,y)\big)^2 dx \\ &= y_0\, g^2 \sum_{i=1}^{n} \frac{h^5 i^2}{4} - \frac{h^5 i}{6} + \frac{h^5}{20} - \frac{h^4}{3\alpha} + \frac{h^4 i}{\alpha} + \frac{h^3}{\alpha^2} \\ &= y_0\, g^2 \left[h^5 n^3 \Big(\tfrac{1}{12} + \tfrac{1}{24n} + \tfrac{1}{120n^2} \Big) + \tfrac{h^4 n^2}{\alpha} \Big(\tfrac{1}{2} + \tfrac{1}{6n} \Big) + \tfrac{h^3 n}{\alpha^2} \right] \\ &\le x_0\, y_0\, g^2\, h^2 \Big(\tfrac{2}{15} x_0^2 + \tfrac{2}{3} \tfrac{x_0}{\alpha} + \tfrac{1}{\alpha^2} \Big) = C_{1\alpha}^2 h^2. \end{aligned}$$

In addition, the derivatives of the functions u_α and $u_{h\alpha}$ are given by:

$$\tfrac{\partial u_\alpha}{\partial x}(x,y) = \tfrac{\partial u}{\partial x}(x,y) = -gx + gx_0 - q, \qquad \forall (x,y) \in \Omega$$

and

$$\tfrac{\partial u_{h\alpha}}{\partial x}(x,y) = \tfrac{\partial u_h}{\partial x}(x,y) = g\, x_0 - q - h\, g\, i, \qquad x \in [x_i,\, x_{i+1}], \quad y \in [0, y_0].$$

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Then, the bound for $\|\frac{\partial u_\alpha}{\partial x} - \frac{\partial u_{h\alpha}}{\partial x}\|_H$ coincides with the bound for $\|\frac{\partial u}{\partial x} - \frac{\partial u_h}{\partial x}\|_H$, obtained in Lemma 21. □

Remark 1. *Notice that $C_{1\alpha} \to C_1$ when $\alpha \to \infty$, where C_1 is given in Lemma 21.*

3. Distributed Optimization Problem with Variable g

In this section we are going to obtain discrete optimal solutions to the continuous optimization problem (P_1) and $(P_{1\alpha})$ in the rectangular domain Ω for the case when the optimization variable is g.

3.1. Discrete Problem (P_{1h}) Associated to (P_1)

Taking into account that b, g, q and the desired state z_d are constants in the expression (6), according to [6], the continuous quadratic functional cost for the problem (P_1) is explicitly given by:

$$J_1(g) = \tfrac{1}{2} x_0^3\, y_0\, q^2 \Big\{ g^2\, \frac{x_0^2}{q^2} \Big(\frac{2}{15} + \frac{M_1}{x_0^4} \Big) + g\, \frac{x_0}{q} \Big(-\frac{5}{12} + \frac{2}{3} \frac{(b-z_d)}{q x_0} \Big) + \frac{1}{3} - \frac{(b-z_d)}{q x_0} + \frac{(b-z_d)^2}{q^2 x_0^2} \Big\}. \tag{27}$$

Since the function $J_1(g)$ is a polynomial function of the variable g, the analytic expresssion for $J_1'(g)$ is given by

$$J_1'(g) = \tfrac{1}{2} x_0^3 y_0 q^2 \Big\{ \frac{2 x_0^2}{q^2}\, g \Big(\frac{2}{15} + \frac{M_1}{x_0^4} \Big) + \frac{x_0}{q} \Big(-\frac{5}{12} + \frac{2}{3} \frac{(b-z_d)}{q x_0} \Big) \Big\} \tag{28}$$

Then, the continuous solution to the distributed optimization problem (P_1) is g_{op} defined by:

$$g_{op} = \frac{q}{3\, x_0} \frac{\Big(\frac{5}{8} - \frac{(b-z_d)}{q\, x_0} \Big)}{\Big(\frac{2}{15} + \frac{M_1}{x_0^4} \Big)} \tag{29}$$

and the continuous optimization state is

$$u_{g_{op}}(x,y) = -\tfrac{1}{2} g_{op}\, x^2 + (g_{op}\, x_0 - q)x + b. \tag{30}$$

We define the discrete distributed optimization problem (P_{1h}) on the constant internal energy g as

$$\text{find} \quad g_{h_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{1h}(g_{h_{op}}) = \min_{g \in \mathbb{R}} J_{1h}(g)$$

where the discrete cost function J_{1h} is given by:

$$J_{1h}(g) = \tfrac{1}{2} \|u_h - z_d\|_H^2 + \tfrac{1}{2} M_1 \|g\|_H^2$$

with h defined in (14) and u_h given in (22).

Taking into account that the variable g is constant it results that:

$$J_{1h}(g) = \tfrac{1}{2} y_0 \Big\{ M_1\, g^2\, x_0 + \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} (u_h(x,y) - z_d)^2\, dx \Big\} \tag{31}$$

and from algebraic work it follows that

$$\begin{aligned} J_{1h}(g) \quad &= \tfrac{1}{2} x_0^3\, y_0\, q^2 \Big\{ g^2 \frac{x_0^2}{q^2} \Big[\frac{2}{15} + \frac{M_1}{x_0^4} + \frac{1}{180} \frac{h^4}{x_0^4} + \frac{1}{24} \frac{h^3}{x_0^3} + \frac{1}{36} \frac{h^2}{x_0^2} - \frac{5}{24} \frac{h}{x_0} \Big] \\ &+ g\, \frac{x_0}{q} \Big[-\frac{5}{12} + \frac{2}{3} \frac{(b-z_d)}{q x_0} + \frac{h}{x_0} \Big(\frac{1}{3} + \frac{1}{12} \frac{h}{x_0} \Big) - \frac{h}{x_0} \frac{(b-z_d)}{q x_0} \Big(\frac{1}{2} + \frac{1}{6} \frac{h}{x_0} \Big) \Big] \\ &+ \frac{1}{3} - \frac{(b-z_d)}{q\, x_0} + \frac{(b-z_d)^2}{q^2\, x_0^2} \Big\}. \end{aligned} \tag{32}$$

Lemma 3. *The following estimate holds*

$$|J_1(g) - J_{1h}(g)| \approx C_2\, h \tag{33}$$

where $C_2 = \frac{1}{2}x_0^3\, y_0\, g\, q \left| -\frac{5}{24}\frac{g x_0}{q} + \frac{1}{3} - \frac{1}{2}\frac{(b-z_d)}{q x_0} \right|$ *does not depend on* h*.*

Proof. From (27) and (32) we get

$$\begin{aligned} J_{1h}(g) - J_1(g) &= \frac{1}{2}x_0^3\, y_0\, q^2 \Big\{ g^2 \frac{x_0^2}{q^2} \Big[\frac{1}{180}\frac{h^4}{x_0^4} + \frac{1}{24}\frac{h^3}{x_0^3} + \frac{1}{36}\frac{h^2}{x_0^2} - \frac{5}{24}\frac{h}{x_0} \Big] \\ &+ g\,\frac{h}{q} \Big[\frac{1}{3} + \frac{1}{12}\frac{h}{x_0} - \frac{(b-z_d)}{q x_0}\Big(\frac{1}{2} + \frac{1}{6}\frac{h}{x_0}\Big)\Big] \Big\} \\ &\approx \frac{1}{2}x_0^3\, y_0\, q^2 \Big[-\frac{5}{24}\frac{g^2 x_0}{q^2} + \frac{g}{q}\Big(\frac{1}{3} - \frac{1}{2}\frac{(b-z_d)}{q x_0}\Big)\Big] h \\ &= \frac{1}{2}x_0^3\, y_0\, g q \Big[-\frac{5}{24}\frac{g x_0}{q} + \frac{1}{3} - \frac{1}{2}\frac{(b-z_d)}{q x_0}\Big] h. \end{aligned}$$

Therefore, we obtain (33).
□

Since the function $J_{1h}(g)$ is a polynomial function of the variable g, it is easy to obtain the following analytic expression for the function J'_{1h}:

$$\begin{aligned} J'_{1h}(g) &= \frac{1}{2}x_0^3\, y_0\, q^2 \Big\{ 2g \frac{x_0^2}{q^2} \Big[\frac{2}{15} + \frac{M_1}{x_0^4} + \frac{1}{180}\frac{h^4}{x_0^4} + \frac{1}{24}\frac{h^3}{x_0^3} + \frac{1}{36}\frac{h^2}{x_0^2} - \frac{5}{24}\frac{h}{x_0} \Big] \\ &+ \frac{x_0}{q} \Big[-\frac{5}{12} + \frac{2}{3}\frac{(b-z_d)}{q x_0} + \frac{h}{x_0}\Big(\frac{1}{3} + \frac{1}{12}\frac{h}{x_0}\Big) - \frac{h}{x_0}\frac{(b-z_d)}{q x_0}\Big(\frac{1}{2} + \frac{1}{6}\frac{h}{x_0}\Big)\Big] \Big\}. \end{aligned} \tag{34}$$

From the optimality condition we obtain the following result:

Lemma 4.

a) *The explicit expression for the optimal variable* $g_{h_{op}}$ *is given by:*

$$g_{h_{op}} = \frac{q}{3\, x_0} \frac{A_1 + \frac{h}{x_0} A_2 + \frac{h^2}{x_0^2} A_3}{A_4 + A_5(h)}, \tag{35}$$

where

$$\begin{aligned} A_1 &= \frac{5}{8} - \frac{b-z_d}{q x_0}, & A_4 &= \frac{2}{15} + \frac{M_1}{x_0^4}, \\ A_2 &= \frac{3(b-z_d)}{4 q x_0} - \frac{1}{2}, & A_5(h) &= \frac{h}{12\, x_0}\Big(\frac{h^3}{15\, x_0^3} + \frac{h^2}{2\, x_0^2} + \frac{h}{3\, x_0} - \frac{5}{2}\Big). \\ A_3 &= \frac{b-z_d}{4\, q x_0} - \frac{1}{8}, \end{aligned} \tag{36}$$

b) *In addition, the following error estimates hold:*

$$|g_{op} - g_{h_{op}}| \approx C_3\, h, \tag{37}$$

$$\Big| J_1(g_{op}) - J_{1h}(g_{h_{op}}) \Big| \approx C_4\, h, \tag{38}$$

where C_3 *and* C_4 *do not depend on* h*.*

Proof.

a) It follows immediately from the expression of $J'_{1h}(g)$ given by (34).

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b) Rewriting g_{op} given by (29) as: $g_{op} = \frac{q}{3\,x_0}\frac{A_1}{A_4}$, it follows that:

$$g_{h_{op}} - g_{op} = \frac{q}{3\,x_0}\frac{-A_1\,A_5(h) + \frac{h}{x_0}A_2\,A_4 + \frac{h^2}{x_0^2}A_3\,A_4}{A_4^2 + A_4\,A_5(h)} \approx \frac{q}{3\,x_0^2}\frac{A_2A_4 + \frac{5}{24}A_1}{A_4^2}h + o(h^2),$$

and we obtain (37) with $C_3 = |C_3^*|$ where

$$C_3^* = \frac{q}{3\,x_0^2}\frac{A_2\,A_4 + \frac{5}{24}A_1}{A_4^2}. \tag{39}$$

From the expression for $J_1(g)$ with $g = g_{op}$ and $J_{1h}(g)$ with $g = g_{h_{op}}$, it results that:

$$J_1(g_{op}) - J_{1h}(g_{h_{op}}) = \tfrac{1}{2}x_0^3\,y_0\,q^2\Big[\tfrac{x_0^2}{q^2}A_4(g_{op}^2 - g_{h_{op}}^2) - \tfrac{2}{3}\tfrac{x_0}{q}A_1(g_{op} - g_{h_{op}})$$
$$-g_{h_{op}}^2\tfrac{x_0^2}{q^2}A_5(h) + \tfrac{2}{3}g_{h_{op}}\tfrac{h}{q}\Big(A_2 + \tfrac{h}{x_0}A_3\Big)\Big].$$

By using (37) we get (38) where

$$C_4 = \tfrac{1}{2}x_0^2y_0q^2\left|-2A_4C_3^*g_{op}\tfrac{x_0^3}{q^2} + \tfrac{5}{24}g_{op}^2\tfrac{x_0^2}{q^2} + \tfrac{2}{3}A_1C_3^*\tfrac{x_0^2}{q} + \tfrac{2}{3}A_2g_{op}\tfrac{x_0}{q}\right|$$
$$= \tfrac{1}{2}x_0^2y_0q^2\left|\frac{A_1(5A_1 + 48A_2A_4)}{216A_4^2}\right|.$$

□

Lemma 5. *Let us consider $u_{g_{op}}$ the solution of (1)-(2) for $g = g_{op}$ and $u_{hg_{hop}}$ the discrete solution defined as in (22) for $h > 0$, where $g = g_{h_{op}}$ is the optimal variable of $(P_{1\,h})$ given by (35). We have that:*

$$a)\quad ||u_{g_{op}} - u_{hg_{hop}}||_H \approx C_5\,h, \qquad\qquad b)\quad ||\frac{\partial u_{g_{op}}}{\partial x} - \frac{\partial u_{hg_{hop}}}{\partial x}||_H \approx C_6\,h$$

where C_5 and C_6 are positive constants independent of the parameter h.

Proof:

a) From definition of norm in H, we obtain

$$||u_{g_{op}} - u_{hg_{hop}}||_H^2 = y_0\sum_{i=1}^{n}\int_{x_i}^{x_{i+1}}\Big(u_{g_{op}}(x,y) - u_{hg_{hop}}(x,y)\Big)^2 dx$$
$$= y_0\sum_{i=1}^{n}\int_{x_i}^{x_{i+1}}\Big[-\tfrac{1}{2}g_{op}x^2 + x_0x(g_{op} - g_{h_{op}}) + hg_{h_{op}}(ix - h\tfrac{i(i-1)}{2})\Big]^2 dx$$
$$= \tfrac{1}{120}x_0^3\,y_0\,g_{op}^2\Big[10 - 25\ \tfrac{x_0\,C_3^*}{g_{op}} + 16\Big(\tfrac{x_0\,C_3^*}{g_{op}}\Big)^2\Big]h^2 + o(h^3).$$

where C_3^* is given by (39). Therefore it follows that

$$||u_{g_{op}} - u_{hg_{hop}}||_H \approx C_5\,h,$$

with

$$C_5 = |g_{op}|\sqrt{\tfrac{1}{120}x_0^3\,y_0\ \Big[10 - 25\ \tfrac{x_0\,C_3^*}{g_{op}} + 16\Big(\tfrac{x_0\,C_3^*}{g_{op}}\Big)^2\Big]}.$$

b) We have that

$$\begin{aligned}
&\left\|\frac{\partial u_{g_{op}}}{\partial x} - \frac{\partial u_{hg_{hop}}}{\partial x}\right\|_H^2 \\
&= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \Big(-g_{op} x + h g_{h_{op}} i + x_0 (g_{op} - g_{h_{op}}) \Big)^2 dx \\
&= \tfrac{x_0 y_0}{6} \left[2 g_{op}^2 x_0^2 + g_{op}\, g_{h_{op}} (h^2 + 3 h x_0 - 4 x_0^2) + g_{h_{op}}^2 (h^2 - 3 h x_0 + 2 x_0^2) \right] \\
&= \tfrac{x_0 y_0}{6} \left[2 x_0^2 (g_{op} - g_{h_{op}})^2 + 3h\, x_0\, g_{h_{op}}\, (g_{op} - g_{h_{op}}) + h^2 g_{h_{op}} (g_{op} + g_{h_{op}}) \right].
\end{aligned}$$

Taking into account Lemma 4, we get

$$\left\|\frac{\partial u_{g_{op}}}{\partial x} - \frac{\partial u_{h\,g_{op_h}}}{\partial x}\right\|_H^2 = \frac{x_0 y_0}{6}\, g_{op}^2 \left[2 + 3\, \frac{x_0 C_3^*}{g_{op}} + 2 \left(\frac{x_0 C_3^*}{g_{op}} \right)^2 \right] h^2 + o(h^3).$$

Then

$$\left\|\frac{\partial u_{g_{op}}}{\partial x} - \frac{\partial u_{h\,g_{op_h}}}{\partial x}\right\|_H \approx C_6 h,$$

with

$$C_6 = |g_{op}| \sqrt{\frac{x_0 y_0}{6} \left[2 - 3\, \frac{x_0 C_3^*}{g_{op}} + 2 \left(\frac{x_0 C_3^*}{g_{op}} \right)^2 \right]},$$

where C_3^* is given by (39).

3.2. Discrete Problem $(P_{1h\alpha})$ Associated to $(P_{1\alpha})$

From [6], we know that the continuous quadratic functional cost in (6) for the optimization problem $(P_{1\alpha})$ is explicitly given by:

$$\begin{aligned}
&J_{1\alpha}(g) = J_1(g) \\
&+ \frac{x_0^2 y_0 q^2}{2\alpha} \left\{ \frac{g^2 x_0^2}{q^2} \left(\frac{2}{3} + \frac{1}{\alpha x_0} \right) + \frac{g x_0}{q} \left(-\frac{5}{3} - \frac{2}{\alpha x_0} + \frac{2(b - z_d)}{q x_0} \right) + 1 + \frac{1}{\alpha x_0} - \frac{2(b - z_d)}{q x_0} \right\},
\end{aligned} \tag{40}$$

where J_1 is defined by (27). Moreover, the continuous optimal distributed variable denoted by $g_{\alpha_{op}}$ is

$$g_{\alpha_{op}} = \frac{q}{3\, x_0} \frac{\left(\frac{5}{8} - \frac{(b - z_d)}{q\, x_0} + \frac{5}{2\alpha x_0} + \frac{3}{\alpha^2 x_0^2} - \frac{3(b - z_d)}{\alpha q x_0^2} \right)}{\left(\frac{2}{15} + \frac{M_1}{x_0^4} + \frac{2}{3\alpha x_0} + \frac{1}{\alpha^2 x_0^2} \right)}. \tag{41}$$

The continuous associated state is established by:

$$u_{\alpha g_{\alpha_{op}}}(x, y) = -\tfrac{1}{2} g_{\alpha_{op}} x^2 + (g_{\alpha_{op}} x_0 - q) x + \tfrac{1}{\alpha} (g_{\alpha_{op}}\, x_0 - q) + b. \tag{42}$$

Defining the discrete cost function as:

$$J_{1h\alpha}(g) = \tfrac{1}{2} \| u_{h\alpha} - z_d \|_H^2 + \tfrac{1}{2} M_1 \| g \|_H^2 \tag{43}$$

where the function $u_{h\alpha}$ is given in (26) we set the following discrete optimization problem $(P_{1h\alpha})$ on the constant internal energy g as

$$\text{find} \quad g_{h\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{1h\alpha}(g_{h\alpha_{op}}) = \min_{g \in \mathbb{R}} J_{1\alpha h}(g).$$

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The discrete cost function $J_{1h\alpha}$ is explicitly given by

$$\begin{aligned}
&J_{1h\alpha}(g) = J_{1\alpha}(g) \\
&+\tfrac{1}{2}x_0^3\, y_0\, g\, q\, h \Bigg\{ \frac{g x_0}{q}\Bigg[-\frac{5}{24} + \frac{1}{36}\frac{h}{x_0} + \frac{1}{24}\frac{h^2}{x_0^2} + \frac{1}{180}\frac{h^3}{x_0^3} + \frac{1}{\alpha x_0}\left(-\frac{7}{6} + \frac{1}{3}\frac{h}{x_0} + \frac{1}{6}\frac{h^2}{x_0^2} \right) \\
&+\frac{1}{\alpha^2 x_0^2}\left(-2 + \frac{h}{x_0} \right) \Bigg] + \frac{1}{3} + \frac{1}{12}\frac{h}{x_0} + \frac{1}{\alpha x_0}\left(\frac{3}{2} + \frac{1}{6}\frac{h}{x_0} \right) + \frac{2}{\alpha^2 x_0^2} \\
&+\frac{(b-z_d)}{q x_0}\left(-\frac{1}{2} - \frac{1}{6}\frac{h}{x_0} - \frac{2}{\alpha x_0} \right) \Bigg\},
\end{aligned} \tag{44}$$

where $J_{1\alpha}$ is given by (40).

Lemma 6. *For $g \in H$, $h > 0$ and $\alpha > 0$ the following estimate holds*

$$|J_{1h\,\alpha}(g) - J_{1\alpha}(g)| \approx C_{2\alpha}\, h$$

with

$$C_{2\alpha} = \tfrac{1}{2}x_0^3\, y_0\, |g| \left| \frac{g x_0}{q}\left(-\frac{5}{24} - \frac{7}{6}\frac{1}{\alpha x_0} - \frac{2}{\alpha^2 x_0^2} \right) + \frac{1}{3} + \frac{3}{2}\frac{1}{\alpha x_0} + \frac{2}{\alpha^2 x_0^2} + \frac{(b-z_d)}{q x_0}\left(-\frac{1}{2} - \frac{2}{\alpha x_0} \right) \right|,$$

a constant independent of h.

Proof. It follows immediately from expression (44). □

Remark 2. *$C_{2\alpha} \to C_2$ when $\alpha \to \infty$, where C_2 is given in Lemma 3.*

Lemma 7.

a) *The explicit expression for the optimal variable $g_{h\alpha_{op}}$ is given by:*

$$g_{h\alpha_{op}} = \frac{q}{3\, x_0} \frac{A_{1\alpha} + \frac{h}{x_0} A_{2\alpha} + \frac{h^2}{x_0^2} A_{3\alpha}}{A_{4\alpha} + A_{5\alpha}(h)}, \tag{45}$$

where

$$\begin{aligned}
A_{1\alpha} &= \frac{5}{8} - \frac{b-z_d}{q x_0} + \frac{1}{\alpha x_0}\left(\frac{5}{2} + \frac{3}{\alpha x_0} - \frac{3(b-z_d)}{q x_0} \right), \\
A_{2\alpha} &= \frac{3(b-z_d)}{4 q x_0} - \frac{1}{2} + \frac{1}{\alpha x_0}\left(-\frac{9}{4} - \frac{3}{\alpha x_0} + \frac{3(b-z_d)}{q x_0} \right), \\
A_{3\alpha} &= \frac{b-z_d}{4\, q x_0} - \frac{1}{8} - \frac{1}{4\alpha x_0}, \\
A_{4\alpha} &= \frac{2}{15} + \frac{M_1}{x_0^4} + \frac{1}{\alpha x_0}\left(\frac{2}{3} + \frac{1}{\alpha x_0} \right), \\
A_{5\alpha}(h) &= \frac{h}{12\, x_0}\left(\frac{h^3}{15\, x_0^3} + \frac{h^2}{2\, x_0^2} + \frac{h}{3\, x_0} - \frac{5}{2} \right) \\
&+ \frac{h}{\alpha x_0^2}\left(-\frac{7}{6} + \frac{h}{3 x_0} + \frac{h^2}{6 x_0^2} + \frac{1}{\alpha x_0}\left(-2 + \frac{h}{x_0} \right) \right).
\end{aligned} \tag{46}$$

b) *In addition, the following error estimates hold:*

$$|g_{\alpha_{op}} - g_{h\alpha_{op}}| \approx C_{3\alpha}\, h, \tag{47}$$

$$\left| J_{1\alpha}(g_{\alpha_{op}}) - J_{1h\alpha}(g_{h\alpha_{op}}) \right| \approx C_{4\alpha}\, h, \tag{48}$$

where $C_{3\alpha}$ and $C_{4\alpha}$ do not depend on h.

Proof.

a) It follows immediately from the fact that

$$J_{1h\alpha}(g) = \tfrac{1}{2}x_0^3 y_0 q^2 \Big[2g\tfrac{x_0^2}{q^2}\Big(A_{4\alpha} + A_{5\alpha}(h)\Big) - \tfrac{2}{3}\tfrac{x_0}{q}\Big(A_{1\alpha} + \tfrac{h}{x_0}A_{2\alpha} + \tfrac{h^2}{x_0^2}A_{3\alpha}\Big)\Big].$$

b) Notice that $g_{\alpha_{op}}$ given by (41) can be rewriten as $g_{\alpha_{op}} = \frac{q}{3x_0}\frac{A_{1\alpha}}{A_{4\alpha}}$. Then,

$$\begin{aligned} g_{h\alpha_{op}} - g_{\alpha_{op}} &= \frac{q}{3\,x_0}\frac{-A_{1\alpha}\,A_{5\alpha}(h) + \frac{h}{x_0}A_{2\alpha}\,A_{4\alpha} + \frac{h^2}{x_0^2}A_{3\alpha}\,A_{4\alpha}}{A_{4\alpha}^2 + A_{4\alpha}\,A_{5\alpha}(h)} \\ &\approx \frac{q}{3\,x_0^2}\frac{A_{2\alpha}A_{4\alpha} + \frac{5}{24}A_{1\alpha}}{A_{4\alpha}^2}h + o(h^2). \end{aligned}$$

and we obtain (47) with $C_{3\alpha} = |C^*_{3\alpha}|$ where

$$C^*_{3\alpha} = \frac{q}{3\,x_0^2}\frac{A_{2\alpha}\,A_{4\alpha} + \Big(\frac{5}{24} + \frac{7}{6\alpha x_0}\Big)A_{1\alpha}}{A_{4\alpha}^2}. \tag{49}$$

Following Lemma 4 it is obtained formula (48) with

$$\begin{aligned} C_{4\alpha} &= \tfrac{1}{2}x_0^2 y_0 q^2 \Big| -2A_{4\alpha}C^*_{3\alpha}g_{\alpha_{op}}\tfrac{x_0^3}{q^2} + \tfrac{5}{24}g^2_{\alpha_{op}}\tfrac{x_0^2}{q^2} \\ &\quad + \tfrac{7}{6}g^2_{\alpha_{op}}\tfrac{x_0}{\alpha q^2} + \tfrac{2}{3}A_{1\alpha}C^*_{3\alpha}\tfrac{x_0^2}{q} + \tfrac{2}{3}A_{2\alpha}g_{\alpha_{op}}\tfrac{x_0}{q}\Big|. \end{aligned}$$

□

Remark 3. *When $\alpha \to \infty$, we have that $A_{i\alpha} \to A_i$, where A_i and $A_{i\alpha}$ are given by (36) and (46), respectively for $i = 1, 2, \cdots, 5$. As an immediate consequence it follows that, $C_{3\alpha} \to C_3$ and $C_{4\alpha} \to C_4$ when $\alpha \to \infty$, where C_3 and C_4 are defined in Lemma 4.*

Lemma 8. *Let us consider $u_{\alpha g_{\alpha_{op}}}$ the function given by (14) for $g = g_{\alpha_{op}}$ where $g_{\alpha_{op}}$ is the optimal variable of problem $(P_{1\alpha})$ given by (41) and $u_{h\alpha g_{h\alpha_{op}}}$ the function defined by (26) for $h > 0$ where $g = g_{h\alpha_{op}}$ is the optimal variable of $(P_{1h\alpha})$ given by (35). We have that:*

$$a)\quad \|u_{\alpha g_{\alpha_{op}}} - u_{h\alpha g_{h\alpha_{op}}}\|_H \approx C_{5\alpha}\,h, \qquad b)\quad \Big\|\frac{\partial u_{\alpha g_{\alpha_{op}}}}{\partial x} - \frac{\partial u_{h\alpha g_{h\alpha_{op}}}}{\partial x}\Big\|_H \approx C_{6\alpha}\,h,$$

where $C_{5\alpha}$ and $C_{6\alpha}$ are positive constants independent of the parameter h.

Proof. Working algebraically we can obtain that

$$\begin{aligned} C_{5\alpha} = |g_{\alpha_{op}}|\Bigg\{ &\tfrac{1}{120}x_0^3\,y_0\,\Big[10 - 25\frac{x_0\,C^*_{3\alpha}}{g_{op}} + 16\Big(\frac{x_0\,C^*_{3\alpha}}{g_{op}}\Big)^2 \\ &+ \tfrac{1}{\alpha x_0}\Big(60 + \frac{120}{\alpha x_0} - 240\frac{C^*_{3\alpha}}{\alpha g_{\alpha_{op}}} + 120\frac{C^*_{3\alpha}x_0}{\alpha g^2_{\alpha_{op}}} - 140\frac{C^*_{3\alpha}x_0}{g_{\alpha_{op}}} + 80\frac{C^*_{3\alpha}x_0^2}{g^2_{\alpha_{op}}}\Big)\Big]\Bigg\}^{1/2}, \end{aligned}$$

and

$$C_{6\alpha} = |g_{\alpha_{op}}|\sqrt{\frac{x_0 y_0}{6}\Big[2 - 3\,\frac{x_0 C^*_{3\alpha}}{g_{\alpha_{op}}} + 2\Big(\frac{x_0 C^*_{3\alpha}}{g_{\alpha_{op}}}\Big)^2\Big]},$$

where $C^*_{3\alpha}$ is given by (49). □

Remark 4. *$C_{5\alpha} \to C_5$ and $C_{6\alpha} \to C_6$ when $\alpha \to \infty$ where C_5 and C_6 are given in Lemma 5.*

Remark 5. *In [23] the double convergence when $(h, \alpha) \to (0, +\infty)$ of the optimal control problem $(P_{1h\alpha})$ has been studied, obtaining a commutative diagram that relates the continuous and discrete optimal control problems (P_1), $(P_{1\alpha})$, (P_{1h}) and $(P_{1h\alpha})$ as in the following scheme:*

$$\begin{tikzcd}
\begin{array}{c}\text{Problem } (P_{1h}) \\ g_{hop}, J_{1h}(g_{hop}), u_{hg_{hop}}\end{array} \arrow[r, "h \to 0"'] & \begin{array}{c}\text{Problem } (P_1) \\ g_{op}, J_1(g_{op}), u_{g_{op}}\end{array} \\
\begin{array}{c}\text{Problem } (P_{1h\alpha}) \\ g_{h\alpha op}, J_{1h\alpha}(g_{h\alpha op}), u_{hg_{h\alpha op}}\end{array} \arrow[u, "\alpha \to \infty"] \arrow[r, "h \to 0"'] \arrow[ur, dashed, "{(h,\alpha) \to (0,\infty)}"'] & \begin{array}{c}\text{Problem } (P_{1\alpha}) \\ g_{\alpha op}, J_{1\alpha}(g_{\alpha op}), u_{\alpha g_{\alpha op}}\end{array} \arrow[u, "\alpha \to \infty"']
\end{tikzcd}$$

4. Boundary Optimization Problem with Variable q

4.1. Discrete Problem (P_{2h}) Associated to (P_2)

Under the same considerations given in Section 3.1 and taking into account formula (9), for a given $q \in Q$, we obtain the following quadratic cost function:

$$\begin{aligned} J_2(q) &= \tfrac{x_0 y_0}{2}\Big\{q^2 x_0^2\Big(\tfrac{1}{3} + \tfrac{M_2}{x_0^3}\Big) + q x_0\Big(-\tfrac{5}{12} g x_0^2 - (b - z_d)\Big) \\ &\quad + \tfrac{2}{15} g^2 x_0^4 + (b - z_d)^2 + \tfrac{2}{3} g x_0^2 (b - z_d)\Big\}. \end{aligned} \tag{50}$$

Then, the boundary optimal control of the problem (P_2), called q_{op}, and the associated continuous optimal state are given by:

$$q_{op} = \frac{\frac{5}{12} g x_0^2 + (b - z_d)}{2x_0\left(\frac{1}{3} + \frac{M_2}{x_0^3}\right)}, \qquad u_{q_{op}}(x, y) = -\tfrac{1}{2} g\, x^2 + (g\, x_0 - q_{op})x + b. \tag{51}$$

Associated to (P_2), we define the approximate discrete distributed optimal control problem (P_{2h}) on the constant heat flux q as

$$\text{find} \quad q_{h_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{2h}(q_{h_{op}}) = \min_{q \in \mathbb{R}} J_{2h}(q)$$

where the discrete cost function J_{2h} is defined by

$$J_{2h}(q) = \tfrac{1}{2}\|u_h - z_d\|_H^2 + \tfrac{1}{2} M_2 \|q\|_Q^2,$$

where u_h is given in (22), h is the spatial step, and z_d (the desired state) and the control q are constant. From the definition of norm over Q, it results that:

$$J_{2h}(q) = \tfrac{1}{2} y_0 \Big\{ M_2\, q^2\, x_0 + \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} [u_h(x, y) - z_d]^2\, dx \Big\}$$

and working algebraically, we get

$$\begin{aligned} J_{2h}(q) &= J_2(q) + \tfrac{x_0 y_0}{2}\Big\{h\, g\, x_0\Big[\tfrac{1}{3} q x_0 - \tfrac{5}{24} g x_0^2 - \tfrac{1}{2}(b - z_d)\Big] \\ &\quad + h^2 g\Big[\tfrac{1}{36} x_0^2 g - \tfrac{1}{6}(b - z_d) + \tfrac{1}{12} q\, x_0\Big] + \tfrac{1}{24} h^3 g^2 x_0 + \tfrac{1}{180} h^4 g^2\Big\}. \end{aligned} \tag{52}$$

Lemma 9. *Given $q \in Q$ and $h > 0$, we have that*

$$|J_{2h}(q) - J_2(q)| \approx C_7\, h, \tag{53}$$

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where $C_7 = \frac{1}{2}x_0^2\, y_0\, |g| \left| \frac{1}{3} q x_0 - \frac{5}{24} g x_0^2 - \frac{1}{2}(b - z_d) \right|$ *is a constant independent of* h.

proof: It follows immediately from the expression (52) for J_{2h}.

Lemma 10. *Let us consider* $h > 0$.

a) *The explicit expression for the optimal variable* $q_{h_{op}}$ *is given by:*

$$q_{h_{op}} = q_{op} - B_1 \frac{gh}{6} - B_2 \frac{gh^2}{24\, x_0}, \tag{54}$$

with $B_1 = B_2 = \left(\frac{1}{3} + \frac{M_2}{x_0^3} \right)^{-1}$.

b) *The following error estimates hold:*

$$|q_{h_{op}} - q_{op}| \approx C_8\, h, \tag{55}$$

$$\left| J_{2h}(q_{h_{op}}) - J_2(q_{op}) \right| \approx C_9\, h, \tag{56}$$

where C_8 *and* C_9 *are constants independent of* h.

Proof.

a) From the expression (52) for J_{2h} we have that

$$J'_{2h}(q) = \frac{1}{2}x_0^2\, y_0 \left\{ 2\, x_0\, q \left(\frac{1}{3} + \frac{M_2}{x_0^3} \right) - \frac{5}{12} g\, x_0^2 - (b - z_d) + \frac{1}{3}\, g\, x_0\, h + \frac{1}{12}\, g\, h^2 \right\}.$$

Therefore

$$q_{h_{op}} = \frac{\frac{5}{12} g x_0^2 + (b - z_d) - \frac{1}{12} g h^2 - \frac{1}{3} g x_0 h}{2x_0 \left(\frac{1}{3} + \frac{M_2}{x_0^3} \right)}.$$

Taking into account that q_{op} is given by (51), we obtain formula (54).

b) On one hand, expression (55) is a direct consequence of expression (54) with $C_8 = \left| \frac{g\, B_1}{6} \right|$. On the other hand, taking into account formula (52) for J_{2h} it follows that

$$\begin{aligned} J_{2h}(q_{h_{op}}) - J_2(q_{op}) &= J_2(q_{h_{op}}) - J_2(q_{op}) \\ &+ \frac{x_0^2 y_0}{2} h\, g \left[\frac{1}{3} q_{h_{op}} x_0 - \frac{5}{24} g x_0^2 - \frac{1}{2}(b - z_d) \right] + o(h^2). \end{aligned} \tag{57}$$

From the definition of J_2 given by (50) and the explicit expression (54) for $q_{h_{op}}$ we get

$$\begin{aligned} J_2(q_{h_{op}}) - J_2(q_{op}) &= \frac{x_0^2 y_0}{2} \left\{ \frac{x_0}{B_1} (q_{h_{op}}^2 - q_{op}^2) \right. \\ &\left. + (q_{h_{op}} - q_{op}) \left(- \frac{5}{12} g x_0^2 - (b - z_d) \right) \right\} \\ &= \frac{x_0^2 y_0}{2} (q_{h_{op}} - q_{op}) \left(\frac{x_0}{B_1} (q_{h_{op}} + q_{op}) - \frac{5}{12} g x_0^2 - (b - z_d) \right) \\ &= \frac{x_0^2 y_0}{12} g h B_1 \left(- \frac{2x_0}{B_1} q_{op} + \frac{5}{12} g x_0^2 + b - z_d \right) + o(h^2). \end{aligned} \tag{58}$$

Taking into account (57) and (58), we obtain the estimate (56) with

$$C_9 = \frac{|g| x_0^2 y_0}{2} \left| \frac{B_1}{6} \left(- \frac{2x_0}{B_1} q_{op} + \frac{5}{12} g x_0^2 + b - z_d \right) + \frac{1}{3} q_{op} x_0 - \frac{5}{24} g x_0^2 - \frac{1}{2}(b - z_d) \right|.$$

□

Lemma 11. *Consider $u_{q_{op}}$ the solution of (1)-(2) for $q = q_{op}$ and $u_{h\,q_{hop}}$ the discrete solution given by (22) for each $h > 0$ where $q = q_{h_{op}}$ is the optimal variable of the problem $(P_{2\,h})$ given by (54). Then we have that:*

$$\text{a)}\quad \|u_{h\,q_{hop}} - u_{q_{op}}\|_H \approx C_{10}\, h, \qquad \text{b)}\quad \left\| \frac{\partial u_{hq_{hop}}}{\partial x} - \frac{\partial u_{q_{op}}}{\partial x} \right\|_H \approx C_{11}\, h, \tag{59}$$

where C_{10} and C_{11} do not depend on the parameter h.

Proof.

a) From the definition of u and u_h given by (13) and (22), respectively, it follows for $i = 1, \ldots, n$ that

$$\begin{aligned} u_{h\,q_{hop}}(x,y) - u_{q_{op}}(x,y) &= \tfrac{1}{2}\, g\, x^2 - (q_{h_{op}} - q_{op} + g\, i\, h)x + g\, \tfrac{i^2 - i}{2}\, h^2, \\ &= \tfrac{1}{2}\, g\, x^2 + hgx\left(\tfrac{B_1}{6} - i\right) + h^2 g\left(\tfrac{B_1}{24}\tfrac{x}{x_0} + \tfrac{i(i-1)}{2}\right), \quad \forall x \in [x_i,\, x_{i+1}]. \end{aligned} \tag{60}$$

Then

$$\begin{aligned} \|u_{h\,q_{hop}} - u_{q_{op}}\|_H^2 &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \left(u_{h\,q_{hop}}(x) - u_{q_{op}}(x)\right)^2 dx \\ &= x_0\, y_0\, g^2 \Big[\tfrac{1}{108}(B_1 - 3)^2 h^2\, x_0^2 + \tfrac{1}{216} h^3\, x_0 (B_1 - 3)^2 \\ &+ \tfrac{1}{8640} h^4\, (72 - 30B_1 + 5B_1^2)\Big]. \end{aligned}$$

Therefore, we get formula (59) a) with $C_{10} = |g(B_1 - 3)|x_0\sqrt{\frac{x_0 y_0}{108}}$.

b) In the same manner, we get

$$\begin{aligned} \left\| \frac{\partial u_{hq_{hop}}}{\partial x} - \frac{\partial u_{q_{op}}}{\partial x} \right\|_H^2 &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \left(g\, x - (q_{op_h} - q_{op}) - h\, g\, i\right)^2 dx \\ &= y_0\, g^2 \Big[\tfrac{1}{36}(12 - 6B_1 + B_1^2)h^2 x_0 + \tfrac{1}{72}(B_1 - 3)B_1 h^3 + \tfrac{1}{576} B_1^2 \tfrac{h^4}{x_0}\Big]. \end{aligned}$$

Then we get (59) b) with

$$C_{11} = \frac{|g|}{6}\sqrt{x_0 y_0 (12 - 6B_1 + B_1^2)}.$$

□

4.2. Discrete Problem $(P_{2h\alpha})$ Associated to $(P_{2\alpha})$

If we suppose that the desired state z_d is constant in (9), the quadratic cost function $J_{2\alpha}$ for the optimal control problem $(P_{2\alpha})$ is explicitly given by:

$$\begin{aligned} J_{2\alpha}(q) = \frac{x_0 y_0}{2} &\Big[q^2 x_0^2 (D_{1\alpha} + \tfrac{M_2}{x_0^3}) + q x_0 \left(D_{2\alpha} g x_0^2 + D_{3\alpha}(b - z_d)\right) \\ &+ D_{4\alpha} g^2 x_0^4 + D_{5\alpha}(b - z_d)^2 + D_{6\alpha} g x_0^2 (b - z_d)\Big], \end{aligned} \tag{61}$$

where

$$\begin{aligned} D_{1\alpha} &= \tfrac{1}{3} + \tfrac{1}{\alpha x_0} + \tfrac{1}{\alpha^2 x_0^2}, & D_{2\alpha} &= -\tfrac{5}{12} - \tfrac{5}{3\alpha x_0} - \tfrac{2}{\alpha^2 x_0^2}, \\ D_{3\alpha} &= -1 - \tfrac{2}{\alpha x_0}, & D_{4\alpha} &= \tfrac{2}{15} + \tfrac{2}{3\alpha x_0} + \tfrac{1}{\alpha^2 x_0^2}, \\ D_{5\alpha} &= 1, & D_{6\alpha} &= \tfrac{2}{3} + \tfrac{2}{\alpha x_0}. \end{aligned} \tag{62}$$

Then, the continuous boundary optimization control, called $q_{\alpha_{op}}$, and the associated state are:

$$\begin{aligned} q_{\alpha_{op}} &= -\frac{D_{2\alpha} g x_0^2 + D_{3\alpha}(b - z_d)}{2x_0\left(D_{1\alpha} + \frac{M_2}{x_0^3}\right)}, \\ u_{\alpha q_{\alpha_{op}}}(x, y) &= -\tfrac{1}{2} g\, x^2 + (g\, x_0 - q_{\alpha_{op}})x + \tfrac{1}{\alpha}(g\, x_0 - q_{\alpha_{op}}) + b. \end{aligned} \tag{63}$$

Remark 6. *Notice that $J_{2\alpha}(q) \to J_2(q)$ for all $q \in Q$ and $q_{\alpha_{op}} \to q_{op}$ when $\alpha \to \infty$.*

Defining the discrete cost function as:

$$J_{2h\alpha}(q) = \tfrac{1}{2}\|u_{h\,\alpha} - z_d\|_H^2 + \tfrac{1}{2} M_2 \|q\|_Q^2, \tag{64}$$

where $u_{h\,\alpha}$ is the solution of $(S_{h\,\alpha})$ given in (26), we set the following discrete optimization problem $(P_{2h\alpha})$ on the constant heat flux q as

$$\text{find} \quad q_{h\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{2h\alpha}(q_{h\alpha_{op}}) = \min_{q \in \mathbb{R}} J_{2h\alpha}(g).$$

Working algebraically, the cost function $J_{2h\,\alpha}$ can be written explicitly as:

$$\begin{aligned} J_{2h\,\alpha}(q) &= J_{2\alpha}(q) \\ &+ \tfrac{1}{2} x_0\, y_0\, g\, h \Big\{ \tfrac{q\, x_0^2}{3} - \tfrac{5\, g x_0^3}{24} - \tfrac{(b - z_d)\, x_0}{2} + \tfrac{3 x_0 q}{2\,\alpha} - \tfrac{7 g\, x_0^2}{6\alpha} - \tfrac{2(b - z_d)}{\alpha} + \tfrac{2(q - g\, x_0)}{\alpha^2} \\ &+ h\left(\tfrac{x_0^2\, g}{36} - \tfrac{(b - z_d)}{6} + \tfrac{q\, x_0}{12} + \tfrac{2 g\, x_0 + q}{6\,\alpha} + \tfrac{g}{\alpha^2} \right) \\ &+ h^2\, g \left(\tfrac{x_0}{24} + \tfrac{1}{6\alpha} \right) + \tfrac{1}{180} g\, h^3 \Big\}. \end{aligned} \tag{65}$$

Lemma 12. *For each $q \in Q$ and $h > 0$, we have that:*

$$|J_{2h\,\alpha}(q) - J_{2\alpha}(q)| \approx C_{7\alpha}\, h, \tag{66}$$

with

$$C_{7\alpha} = \tfrac{1}{2} x_0^2\, y_0 \left| g \left(\tfrac{q\, x_0}{3} - \tfrac{5\, g x_0^2}{24} - \tfrac{(b - z_d)}{2} + \tfrac{3q}{2\,\alpha} - \tfrac{7 g\, x_0}{6\alpha} - \tfrac{2(b - z_d)}{x_0\,\alpha} + \tfrac{2(q - g\, x_0)}{x_0\,\alpha^2} \right) \right|,$$

a constant independent of h.

Proof. It follows immediately from expression (65). □

Lemma 13. *Let us consider $h > 0$.*

a) *The explicit expression for the optimal control $q_{h_{op}}$ is given by:*

$$q_{h\alpha_{op}} = q_{\alpha_{op}}, - B_{1\alpha} \frac{gh}{6} - B_{2\alpha} \frac{gh^2}{24 x_0} \tag{67}$$

with

$$\begin{aligned} B_{1\alpha} &= \left(1 + \tfrac{9}{2\alpha x_0} + \tfrac{6}{\alpha^2 x_0^2}\right)\left(D_{1\alpha} + \tfrac{M_2}{x_0^3}\right)^{-1}, \\ B_{2\alpha} &= \left(1 - \tfrac{2}{\alpha x_0}\right)\left(D_{1\alpha} + \tfrac{M_2}{x_0^3}\right)^{-1}. \end{aligned}$$

b) *The following error estimates hold:*

$$|q_{h\alpha_{op}} - q_{\alpha_{op}}| \approx C_{8\alpha}\, h, \tag{68}$$

$$\left| J_{2h\alpha}(q_{h\alpha_{op}}) - J_{2\alpha}(q_{\alpha_{op}}) \right| \approx C_{9\alpha}\, h, \tag{69}$$

where $C_{8\alpha}$ and $C_{9\alpha}$ does not depend on h.

Proof.

a) From the derivative of the control function $J_{2h\alpha}$ given by

$$J'_{2h\,\alpha}(q) = J'_{2\alpha}(q) + \tfrac{1}{2}x_0\, y_0\, g\, h \left\{ \tfrac{x_0^2}{3} + \tfrac{3x_0}{2\,\alpha} + \tfrac{2}{\alpha^2} + h\left(\tfrac{x_0}{12} + \tfrac{1}{6\,\alpha}\right)\right\}$$
$$= \tfrac{x_0 y_0}{2}\left\{2qx_0^2\left(D_{1\alpha} + \tfrac{M_2}{x_0^3}\right) + x_0\left(D_{2\alpha}gx_0^2 + D_{3\alpha}(b - zd)\right)\right.$$
$$\left. +gh\left(\tfrac{x_0^2}{3} + \tfrac{3x_0}{2\,\alpha} + \tfrac{2}{\alpha^2} + h\left(\tfrac{x_0}{12} + \tfrac{1}{6\,\alpha}\right)\right)\right\},$$

it follows that

$$q_{h\alpha_{op}} = \frac{-gh\left(\frac{x_0^2}{3}+\frac{3x_0}{2\,\alpha}+\frac{2}{\alpha^2}\right)-gh^2\left(\frac{x_0}{12}+\frac{1}{6\,\alpha}\right)-x_0\left(D_{2\alpha}gx_0^2+D_{3\alpha}(b-zd)\right)}{2x_0^2\left(D_{1\alpha}+\frac{M_2}{x_0^3}\right)}.$$

Working algebraically we get formula (67).

b) The estimate (68) follows straightforwardly from (67) with

$$C_{8\alpha} = \left|\frac{gB_{1\alpha}}{6}\right|.$$

From formula (65) we obtain that

$$\begin{aligned} J_{2h\alpha}(q_{h\alpha_{op}}) - J_{2\alpha}(q_{\alpha_{op}}) &= J_{2\alpha}(q_{h\alpha_{op}}) - J_{2\alpha}(q_{\alpha_{op}}) \\ &+\tfrac{1}{2}x_0\, y_0\, g\, h\left(\tfrac{q_{h\alpha_{op}}\, x_0^2}{3} - \tfrac{5\, g x_0^3}{24} - \tfrac{(b-z_d)\, x_0}{2} + \tfrac{3x_0 q_{h\alpha_{op}}}{2\,\alpha}\right. \\ &\left. -\tfrac{7g\, x_0^2}{6\alpha} - \tfrac{2(b-z_d)}{\alpha} + \tfrac{2(q_{h\alpha_{op}} - g\, x_0)}{\alpha^2}\right) + o(h^2). \end{aligned} \tag{70}$$

Moreover taking into account the explicit expression for $J_{2\alpha}$ given by (65) and formula (67) it follows that

$$\begin{aligned} &J_{2\alpha}(q_{h\alpha_{op}}) - J_{2\alpha}(q_{\alpha_{op}}) = \tfrac{x_0^2 y_0}{2}(q_{h\alpha_{op}} - q_{\alpha_{op}})\left[(q_{h\alpha_{op}} + q_{\alpha_{op}})x_0\left(D_{1\alpha} + \tfrac{M_2}{x_0^3}\right)\right. \\ &\left.+D_{2\alpha}gx_0^2 + D_{3\alpha}(b - z_d)\right] \\ &= -\tfrac{x_0^2 y_0}{12} gB_{1\alpha}\left(2q_1 x_0\left(D_{1\alpha} + \tfrac{M_2}{x_0^3}\right) + D_{2\alpha}gx_0^2 + D_{3\alpha}(b - z_d)\right) + o(h^2). \end{aligned} \tag{71}$$

Combining (70) and (71) we get estimate (69) with

$$\begin{aligned} C_{9\alpha} \;&= \tfrac{x_0^2 y_0}{2}|g|\left| -\tfrac{B_{1\alpha}}{6}\left(2q_{\alpha_{op}}x_0\left(D_{1\alpha} + \tfrac{M_2}{x_0^3}\right) + D_{2\alpha}gx_0^2 + D_{3\alpha}(b - z_d)\right)\right. \\ &\left. +\tfrac{q_{\alpha_{op}}\, x_0}{3} - \tfrac{5\, g x_0^2}{24} - \tfrac{(b-z_d)}{2} + \tfrac{3q_{\alpha_{op}}}{2\,\alpha} - \tfrac{7g\, x_0}{6\alpha} - \tfrac{2(b-z_d)}{\alpha x_0} + \tfrac{2(q_{\alpha_{op}} - g\, x_0)}{\alpha^2 x_0}\right|. \end{aligned}$$

□

Lemma 14. *Let us consider $u_{q_{\alpha_{op}}}$ the solution of* (1) *-*(3) *for $q = q_{\alpha_{op}}$ and $u_{h\, q_{h\alpha_{op}}}$ the discrete solution given in* (26) *for $h > 0$ and $q = q_{h\alpha_{op}}$. Then we have that:*

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$$
\begin{aligned}
&\text{a)} \quad \| u_{h\alpha\, q_{h\alpha op}} - u_{\alpha q_{\alpha op}} \|_H \approx C_{10\,\alpha}\, h, \\
&\text{b)} \quad \left\| \frac{\partial u_{h\alpha q_{h\alpha op}}}{\partial x} - \frac{\partial u_{\alpha q_{\alpha op}}}{\partial x} \right\|_H \approx C_{11\,\alpha}\, h,
\end{aligned} \tag{72}
$$

Proof. Similarly to what was done in Lemma 12 it is obtained that

$$
\begin{aligned}
C_{10\,\alpha} &= |g| x_0 \\
&\sqrt{\frac{x_0 y_0}{108}} \sqrt{B_{1\alpha}^2 \left(\frac{3}{\alpha^2 x_0^2} + \frac{3}{\alpha x_0} + 1 \right) + 9 \left(\frac{12}{\alpha^2 x_0^2} + \frac{6}{\alpha x_0} + 1 \right) - 3 B_{1\alpha} \left(\frac{12}{\alpha^2 x_0^2} + \frac{9}{\alpha x_0} + 2 \right)}
\end{aligned}
$$

$$
C_{11\,\alpha} = \frac{|g|}{6} \sqrt{x_0 y_0 (12 - 6 B_{1\alpha} + B_{1\alpha}^2)}.
$$

□

Remark 7. *The constants verify that* $C_{i\alpha} \to C_i$, *when* $\alpha \to \infty$, *for each* $i = 7, \dots, 11$.

Remark 8. *The double convergence when* $(h, \alpha) \to (0, +\infty)$ *of the optimal control of the problem* $(P_{2h\alpha})$ *holds. The relationship of the optimal control problems* (P_2), $(P_{2\alpha})$, (P_{2h}) *and* $(P_{2h\alpha})$ *is given by the following diagram:*

$$
\begin{tikzcd}
\begin{array}{c}\text{Problem } (P_{2h}) \\ q_{hop}, J_{2h}(q_{hop}), u_{hq_{hop}}\end{array} \arrow[r, "h \to 0"'] & \begin{array}{c}\text{Problem } (P_{2}) \\ q_{op}, J_{q}(q_{op}), u_{q_{op}}\end{array} \\
\begin{array}{c}\text{Problem } (P_{2h\alpha}) \\ q_{h\alpha op}, J_{2h\alpha}(q_{h\alpha op}), u_{hq_{h\alpha op}}\end{array} \arrow[u, "\alpha \to \infty"] \arrow[r, "h \to 0"'] \arrow[ur, dashed, "(h,\alpha) \to (0,\infty)"'] & \begin{array}{c}\text{Problem } (P_{2\alpha}) \\ q_{\alpha op}, J_{2\alpha}(q_{\alpha op}), u_{\alpha q_{\alpha op}}\end{array} \arrow[u, "\alpha \to \infty"']
\end{tikzcd}
$$

5. Boundary Optimization Problem with Variable b

5.1. Discrete Problem (P_{3h}) Associated to (P_3)

In this section we consider the boundary optimal control problem (P_3) given by (10). Taking into account expression (12), for a given constant b we get that

$$
\begin{aligned}
J_3(b) = \frac{x_0 y_0}{2} \Big\{ & b^2 \left(1 + \frac{M_3}{x_0} \right) + b \left(\frac{2 g x_0^2}{3} - q\, x_0 - 2\, z_d \right) \\
& + \left[\frac{2 g^2 x_0^4}{15} - \frac{5 g q x_0^3}{12} + \frac{x_0^2}{3} (q^2 - 2 g z_d) + z_d (z_d + q x_0) \right] \Big\}.
\end{aligned} \tag{73}
$$

Then the boundary optimal variable of the problem (P_3), called b_{op}, and the associated continuous optimal state, are given respectively by:

$$
b_{op} = \frac{-\frac{g x_0^2}{3} + \frac{q x_0}{2} + z_d}{1 + \frac{M}{x_0}}, \qquad u_{b_{op}} = -\tfrac{1}{2} g\, x^2 + (g\, x_0 - q) x + b_{op}. \tag{74}
$$

We define the discrete optimal control problem (P_{3h}) on the constant temperature b as

$$
\text{find} \quad b_{h_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{3h}(b_{h_{op}}) = \min_{b \in \mathbb{R}} J_{3h}(b)
$$

where the discrete cost function $J_{3h}(b)$ is defined as:

$$
J_{3h}(b) = \tfrac{1}{2} \| u_h - z_d \|_H^2 + \tfrac{1}{2} M_3 \| b \|_B^2,
$$

where u_h is given in (22), h is the spatial step, and z_d (the desired state) is constant.

Notice that the cost function J_{3h} can be explicitly written as:

$$J_{3h}(b) = J_3(b) + \frac{x_0 y_0 g}{2}\Big\{ -bh\Big(\frac{x_0}{2}+\frac{h}{6}\Big) + hx_0\big(\frac{1}{3}qx_0 - \frac{5}{24}gx_0^2 + \frac{z_d}{2}\big) + \frac{1}{6}h^2\Big(\frac{qx_0}{2}+\frac{1}{6}gx_0^2+z_d\Big) + \frac{1}{24}gh^3x_0 + \frac{1}{180}gh^4\Big\}. \tag{75}$$

Lemma 15. *Let $b \in R$ and $h > 0$ we have that:*

$$|J_{3h}(b) - J_3(b)| \approx C_{12}\, h, \tag{76}$$

where

$$C_{12} = \frac{1}{2}x_0^2\, y_0\, |g| \Big| -\frac{b}{2} + \frac{qx_0}{3} - \frac{5gx_0^2}{24} + \frac{z_d}{2}\Big|,$$

does not depend on h.

Proof. It follows from expression (75) for J_{3h}. □

Lemma 16. *Let us consider $h > 0$.*

a) *The explicit expression for the optimal variable $b_{h_{op}}$ is given by:*

$$b_{h_{op}} = b_{op} + E_1 g x_0\, h\Big(1+\frac{h}{3x_0}\Big), \qquad E_1 = \frac{1}{4\Big(1+\frac{M_3}{x_0}\Big)}. \tag{77}$$

b) *The following error estimates hold:*

$$|b_{h_{op}} - b_{op}| \approx C_{13}\, h, \tag{78}$$

$$\Big|J_{3h}(b_{h_{op}}) - J_3(b_{op})\Big| \approx C_{14}\, h, \tag{79}$$

where C_{13} and C_{14} do not depend on h.

Proof.

a) According to (75) we have that

$$J'_{3h}(b) = J'_3(b) - \frac{x_0 y_0}{2} gbh\Big(\frac{x_0}{2}+\frac{h}{6}\Big). \tag{80}$$

Then, formula (77) for $b_{h_{op}}$ follows immediately.

b) Estimate (78) is a direct consequence of (77) with $C_{13} = |E_1 g x_0|$.
Moreover, taking into account formulas (73) and (75) for J_3 and J_{3h} and formulas (74) and (77) for b_{op} and $b_{h_{op}}$, respectively, it follows that

$$\begin{aligned} &J_{3h}(b_{h_{op}}) - J_3(b_{op}) \\ &= J_3(b_{h_{op}}) - J_3(b_{op}) + \frac{x_0^2 y_0 g}{2}\Big\{ -\frac{b_{h_{op}}}{2} + \frac{1}{3}qx_0 - \frac{5}{24}gx_0^2 + \frac{z_d}{2}\Big\}h + o(h^2). \end{aligned}$$

In addition, the expression $J_3(b_{h_{op}}) - J_3(b_{op})$ can be rewritten as

$$J_3(b_{h_{op}}) - J_3(b_{op}) = \frac{x_0^2 y_0 g}{2} E_1\Big(2b_{op}\Big(1+\frac{M_3}{x_0}\Big) + \frac{2}{3}gx_0^2 - qx_0 - 2z_d\Big)h + o(h^2).$$

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Therefore it follows estimate (16) with

$$C_{14} = \frac{x_0^2 y_0 g}{2}\left|E_1\left(2b_{op}\left(1+\frac{M_3}{x_0}\right)+\frac{2}{3}g x_0^2 - q x_0 - 2z_d\right) - \frac{b_{op}}{2} + \frac{q x_0}{3} - \frac{5 g x_0^2}{24} + \frac{z_d}{2}\right|.$$

□

Lemma 17. *Let us consider $u_{b_{op}}$ the solution of (1) -(3) for $b = b_{op}$ and $u_{h\, b_{hop}}$ the discrete solution given in (26) for $h > 0$ and $b = b_{h_{op}}$. Then we have that:*

$$\text{a)} \quad \|u_{hb_{hop}} - u_{b_{op}}\|_H \approx C_{15}\, h, \qquad \text{b)} \quad \left\|\frac{\partial u_{hb_{hop}}}{\partial x} - \frac{\partial u_{b_{op}}}{\partial x}\right\|_H \approx C_{16}\, h, \tag{81}$$

where C_{15} and C_{16} are constants that do not depend on h.

Proof. Working algebraically it is obtained that

$$\|u_{hb_{hop}} - u_{b_{op}}\|_H^2 = \frac{x_0^3 y_0 g^2}{2}\left(2E_1^2 - E_1 + \frac{1}{6}\right)h^2 + o(h^3).$$

Then it is obtained estimate a) with

$$C_{15} = x_0|g|\sqrt{\frac{x_0 y_0}{2}}\sqrt{2E_1^2 - E_1 + \frac{1}{6}}.$$

In a similar manner we get that estimate b) holds with

$$C_{16} = |g|\sqrt{\frac{x_0 y_0}{2}}.$$

□

5.2. Discrete Problem $(P_{3h\alpha})$ Associated to $(P_{3\alpha})$

From [6], we know that the continuous quadratic functional cost in (6) for the optimization problem $(P_{3\alpha})$ is explicitly given by:

$$J_{3\alpha}(b) = J_3(b) + \frac{x_0 y_0}{2}\left\{\frac{1}{3\alpha}(q - g x_0)(-6b + 3q x_0 - 2g x_0^2 + 6z_d) + \frac{1}{\alpha^2}(q - g x_0)^2\right\} \tag{82}$$

where J_3 is defined by (73). Moreover, the continuous optimal boundary control $b_{\alpha_{op}}$ is given by

$$b_{\alpha_{op}} = b_{op} - \frac{g x_0 - q}{\alpha(1 + \frac{M_3}{x_0})}. \tag{83}$$

The continuous associated state is established by:

$$u_{b_{\alpha_{op}}}(x,y) = -\frac{1}{2}g\,x^2 + (g\,x_0 - q)x + \frac{1}{\alpha}(g\,x_0 - q) + b_{\alpha_{op}}. \tag{84}$$

Defining the discrete cost function as:

$$J_{3h\alpha}(b) = \frac{1}{2}\|u_{h\,\alpha} - z_d\|_H^2 + \frac{1}{2}M_3\|b\|_B^2 \tag{85}$$

where $u_{h\,\alpha}$ is the solution of $(S_{h\,\alpha})$ given in (26), we set the following discrete optimization problem $(P_{3h\alpha})$ as

$$\text{find} \quad b_{h\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{3h\alpha}(b_{h\alpha_{op}}) = \min_{b\in\mathbb{R}} J_{3h\alpha}(b).$$

Working algebraically lead us to write $J_{3h\alpha}$ as follows:

$$J_{3h\,\alpha}(b) = J_{3\alpha}(b) + \tfrac{1}{2}x_0\,y_0\,g\,h\Big\{-b\Big(\tfrac{x_0}{2}+\tfrac{2}{\alpha}+\tfrac{h}{6}\Big)+g\Big(\tfrac{-5x_0^3}{24}-\tfrac{7x_0^2}{6\alpha}-\tfrac{2x_0}{\alpha^2}\Big)$$
$$+q\Big(\tfrac{x_0^2}{3}+\tfrac{3x_0}{2\alpha}+\tfrac{2}{\alpha^2}\Big)+z_d\Big(\tfrac{x_0}{2}+\tfrac{2}{\alpha}\Big) \tag{86}$$
$$+h\Big[g\Big(\tfrac{x_0^2}{36}+\tfrac{x_0}{3\alpha}+\tfrac{1}{\alpha^2}\Big)+q\Big(\tfrac{x_0}{12}+\tfrac{1}{6\alpha}\Big)+\tfrac{z_d}{6}\Big]+h^2 g\Big(\tfrac{x_0}{24}+\tfrac{1}{6\alpha}\Big)+h^3\tfrac{g}{180}\Big\}.$$

Lemma 18. *For $b \in B$ and $h > 0$ we have that*

$$|J_{3h\,\alpha}(b) - J_{3\alpha}(b)| \approx C_{12\alpha}\,h, \tag{87}$$

with

$$C_{12\alpha} = \tfrac{x_0\,y_0}{2}|g|\Big|-b\Big(\tfrac{x_0}{2}+\tfrac{2}{\alpha}+\tfrac{h}{6}\Big)+g\Big(\tfrac{-5x_0^3}{24}-\tfrac{7x_0^2}{6\alpha}-\tfrac{2x_0}{\alpha^2}\Big)+q\Big(\tfrac{x_0^2}{3}+\tfrac{3x_0}{2\alpha}+\tfrac{2}{\alpha^2}\Big)+z_d\Big(\tfrac{x_0}{2}+\tfrac{2}{\alpha}\Big)\Big|.$$

Proof. It arises immediately from (86). □

Lemma 19. *Let us consider $h > 0$.*

a) *The explicit expression for the optimal control $b_{h\alpha_{op}}$ is given by:*

$$b_{h\alpha_{op}} = b_{\alpha_{op}} + E_1 g x_0\,h\Big(1+\tfrac{4}{\alpha x_0}+\tfrac{h}{3x_0}\Big), \tag{88}$$

where E_1 is given in (77).

b) *The following error estimates hold:*

$$|b_{h\alpha_{op}} - b_{\alpha_{op}}| \approx C_{13\alpha}\,h, \tag{89}$$

$$\Big|J_{3h\alpha}(b_{h\alpha_{op}}) - J_{3\alpha}(b_{\alpha_{op}})\Big| \approx C_{14\alpha}\,h, \tag{90}$$

where $C_{13\alpha}$ and $C_{14\alpha}$ do not depend on h.

Proof.

a) It follows from the expression $J_{3h\alpha}$ given by (86).

b) The estimate in (89) is obtained immediately from item a) with

$$C_{13\alpha} = |E_1||g|x_0\Big|1+\tfrac{4}{\alpha x_0}\Big|.$$

Taking into account (86) and (82) it results that

$$J_{3h\alpha}(b_{h\alpha_{op}}) - J_{3\alpha}(b_{\alpha_{op}}) = J_{3\alpha}(b_{h\alpha_{op}}) - J_{3\alpha}(b_{\alpha_{op}}) + F_{1\alpha}h + o(h^2).$$

with

$$F_{1\alpha} = \tfrac{1}{2}x_0^2\,y_0\,g\Big(-(b_{\alpha_{op}}-z_d)\Big(\tfrac{1}{2}+\tfrac{2}{\alpha x_0}\Big)+g x_0^2\Big(\tfrac{-5}{24}-\tfrac{7}{6\alpha x_0}-\tfrac{2}{\alpha^2 x_0^2}\Big)+q x_0\Big(\tfrac{1}{3}+\tfrac{3}{2\alpha x_0}+\tfrac{2}{\alpha^2 x_0^2}\Big)\Big).$$

In addition, from the definition of $J_{3\alpha}$ and $b_{h\alpha_{op}}$ we have that

$$J_{3\alpha}(b_{h\alpha_{op}}) - J_{3\alpha}(b_{\alpha_{op}}) = J_3(b_{h\alpha_{op}}) - J_3(b_{\alpha_{op}}) + F_{2\alpha}h + o(h^2)$$

with
$$F_{2\alpha} = \frac{x_0^2 y_0 g}{\alpha} E_1(-q + gx_0)\left(1 + \frac{4}{\alpha x_0}\right).$$
Finally, according to formula (73) for J_3 we get that
$$J_3(b_{h\alpha_{op}}) - J_3(b_{\alpha_{op}}) = F_{3\alpha} h + o(h^2),$$
with
$$F_{3\alpha} = \frac{x_0^2 y_0 g}{2} E_1\left(1 + \frac{4}{\alpha x_0}\right)\left(2b_{\alpha_{op}}\left(1 + \frac{M_3}{x_0}\right) + \frac{2}{3} g x_0^2 - q x_0 - 2z_d\right).$$
Therefore, estimate (19) holds for
$$C_{14\alpha} = \left|F_{1\alpha} + F_{2\alpha} + F_{3\alpha}\right|.$$
□

Lemma 20. *Let us consider $u_{b_{\alpha_{op}}}$ the solution of* (1) *-*(3) *for $b = b_{\alpha_{op}}$ and $u_{h\, b_{h\alpha_{op}}}$ the discrete solution given in* (26) *for $h > 0$ and $b = b_{h\alpha_{op}}$. Then we have that:*
$$\text{a)} \quad \|u_{h\alpha\, b_{h\alpha_{op}}} - u_{\alpha b_{\alpha_{op}}}\|_H \approx C_{15\,\alpha}\, h, \quad \text{b)} \quad \left\|\frac{\partial u_{h\alpha b_{h\alpha_{op}}}}{\partial x} - \frac{\partial u_{\alpha b_{\alpha_{op}}}}{\partial x}\right\|_H \approx C_{16\,\alpha}\, h, \tag{91}$$

Proof. Similarly to what was done in Lemma 12 it is obtained that
$$C_{15\,\alpha} = x_0|g|\sqrt{\frac{x_0 y_0}{2}}\mathcal{A},$$
$$\mathcal{A} = \sqrt{E_1^2\left(2 + \frac{16}{\alpha x_0} + \frac{32}{\alpha^2 x_0^2}\right) + E_1\left(-1 - \frac{8}{\alpha x_0} - \frac{16}{\alpha^2 x_0^2}\right) + \frac{1}{6} + \frac{1}{\alpha x_0} + \frac{2}{\alpha^2 x_0^2}},$$
$$C_{16\,\alpha} = C_{16}.$$
□

Remark 9. *The constants obtained in the estimates of the previous lemmas verify that $C_{i\alpha} \to C_i$, when $\alpha \to \infty$ for $i = 12, \cdots, 16$.*

Remark 10. *The double convergence when $(h,\alpha) \to (0,+\infty)$ of the optimal control of the problem $(P_{3h\alpha})$ holds. The relationship of the optimal control of the problems (P_3), $(P_{3\alpha})$, (P_{3h}) and $(P_{3h\alpha})$ is given by the following diagram:*

$$\begin{tikzcd}
\begin{array}{c}\text{Problem } (P_{3h}) \\ b_{h_{op}}, J_{3h}(b_{h_{op}}), u_{hb_{h_{op}}}\end{array} \arrow[r, "h \to 0"'] & \begin{array}{c}\text{Problem } (P_3) \\ b_{op}, J_b(b_{op}), u_{b_{op}}\end{array} \\
\begin{array}{c}\text{Problem } (P_{3h\alpha}) \\ b_{h\alpha_{op}}, J_{3h\alpha}(b_{h\alpha_{op}}), u_{hb_{h\alpha_{op}}}\end{array} \arrow[u, "\alpha \to \infty"] \arrow[r, "h \to 0"'] \arrow[ur, dashed, "{(h,\alpha) \to (0,\infty)}"'] & \begin{array}{c}\text{Problem } (P_{3\alpha}) \\ b_{\alpha_{op}}, J_{3\alpha}(b_{\alpha_{op}}), u_{\alpha b_{\alpha_{op}}}\end{array} \arrow[u, "\alpha \to \infty"']
\end{tikzcd}$$

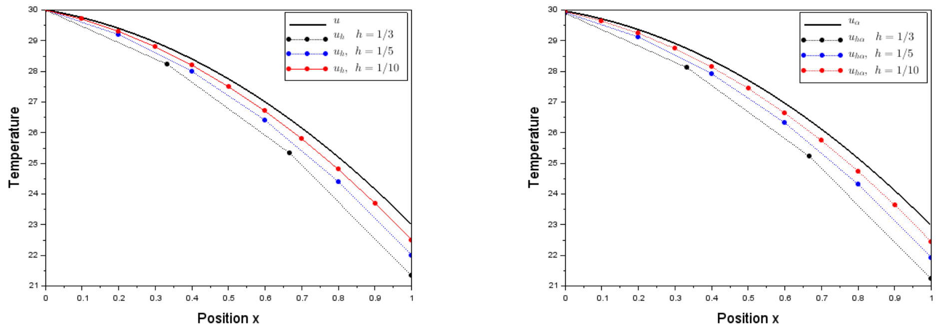
6. Numerical Results

We will carry out some numerical simulations in order to illustrate the theoretical results obtained in the previous sections for the optimal control problems (P_{ih}) and $(P_{ih\alpha})$ for $i = 1, 2, 3$.

Throughout this section we consider the domain $\Omega = [0,1] \times [0,1]$, i.e, $x_0 = y_0 = 1$.

Before analyzing the optimal control problems we illustrate the behavior of the continuous state of the systems (S) and (S_α) and the discrete state of the systems (S_h) and $(S_{h\alpha})$.

In Figure 1 a) we plot the state of the system u given by (13) and the approximate discrete function u_h defined by (22) against the position x for $h = 1/3, 1/5, 1/10$. As we saw in Lemma 21 for each fixed x, the $u_h(x)$ increase and get closer to the limit $u(x)$ as h decreases. In a similar manner, in 1 b), for $\alpha = 50$ we obtain the system u_α given by (13) and the approximate discrete function $u_{h\alpha}$ defined by (26) against the position x for $h = 1/3, 1/5, 1/10$. Notice that as h decreases, the functions $\{u_{h\alpha}\}$ increase and get closer to the limit u_α as it was proved in Lemma 2.



a) u and u_h for different values of h

b) u_α and $u_{h\alpha}$ for different values of h with $\alpha = 50$

Figure 1. State of the systems (S), (S_h), (S_α) and $(S_{h\alpha})$ taking $q = 12$, $b = 30$, $z_d = 40$ and $g = 10$.

In addition in order to visualize the double convergence of $u_{h\alpha} \to u$ when $(h, \alpha) \to (0, \infty)$, in Figure 2 we plot u and $u_{h\alpha}$ for $(h, \alpha) = (1/3, 10), (1/5, 50)$ and $(1/10, 500)$.

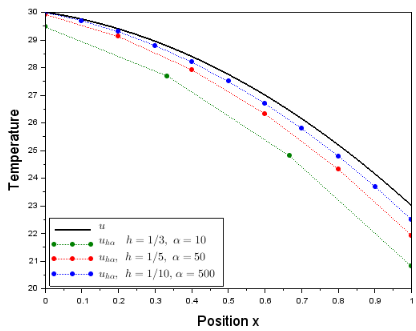


Figure 2. Plot of u and $u_{h\alpha}$ against $n = 1/h$ for different values of (h, α).

6.1. Control Variable g

In this subsection we obtain some computational examples for the optimal distributed control problems (P_1), (P_{1h}), $(P_{1\alpha})$ and $(P_{1h\alpha})$. For each plot we set $q = 12$, $b = 30$, $z_d = 40$ and $M_1 = 1$.

In Figure 3 we plot the continuous quadratic cost function J_1 given by (27) and the discrete cost function J_{1h} obtained in (32) against g for $h = 1/10$, $1/50$ and $1/100$. Notice that as h decreases the function $J_{1h} = J_{1h}(g)$ also decreases to the limit function $J_1 = J_1(g)$ in agreement with Lemma 3. In a similar manner in Figure 4, for $\alpha = 50$ we obtain the continuous function $J_{1\alpha}$ and the discrete functions $J_{1h\alpha}$ for $h = 1/10, 1/50$ and $1/100$ observing the convergence of $J_{1h\alpha} \to J_{1\alpha}$ as h decreases to 0. Moreover, Figure 5 shows the double convergence of $J_{1h\alpha} \to J_1$ when $(h, \alpha) \to (0, \infty)$. We illustrate how $J_{1h\alpha}$ gets closer to J_1 as the value of h decreases and the value of α increases.

In Figure 6 we plot the continuous optimal control g_{op} for the problem (P_1) given by (29) and the optimal control $g_{\alpha_{op}}$ given by (41) for $\alpha = 15, 50, 100$. Notice that as α increases, $g_{\alpha_{op}}$ decreases to the limit g_{op}. In addition, we set different values of n between $n = 10$ and $n = 100$. Recalling that $h = \frac{x_0}{n} = \frac{1}{n}$, for each h we obtain the optimal discrete control $g_{h_{op}}$ to the problem (P_{1h}) defined by (4) and the optimal discrete control $g_{h\alpha_{op}}$ to the problem $(P_{1h\alpha})$ given by (41) for $\alpha = 15, 50, 100$. For each α fixed, we have that the discrete solution $g_{h\alpha_{op}} \to g_{\alpha_{op}}$ when $h \to 0$, i.e. $n \to \infty$.

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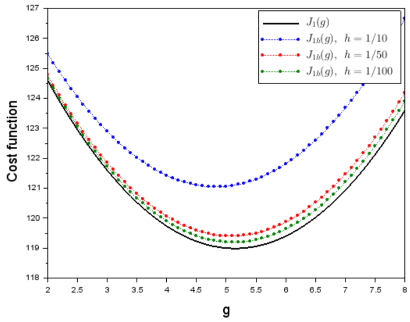

Figure 3. Plot of J_1 and J_{1h}

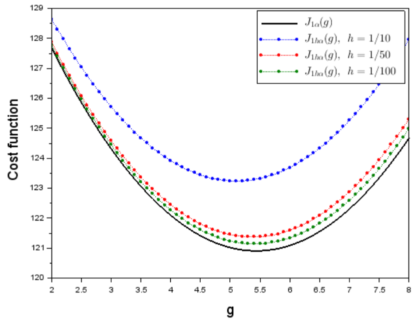

Figure 4. Plot of $J_{1\alpha}$ and $J_{1h\alpha}$ for $\alpha = 50$.

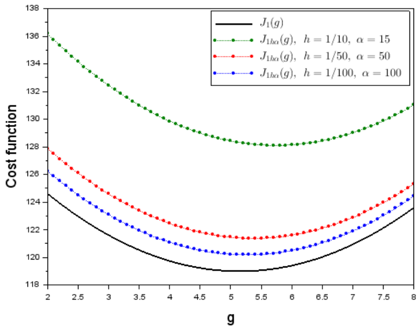

Figure 5. Plot of J_1 and $J_{1h\alpha}$

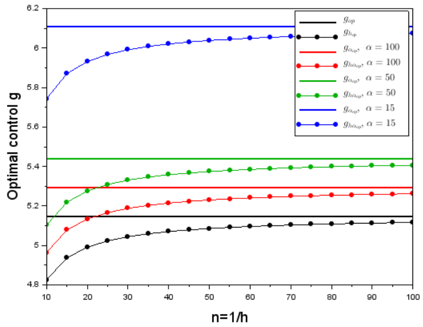

Figure 6. Plot of g_{op}, $g_{h_{op}}$, $g_{\alpha_{op}}$ and $g_{h\alpha_{op}}$

6.2. Control Variable q

In this subsection we run some computational examples for the optimal boundary control problems (P_2), (P_{2h}), $(P_{2\alpha})$ and $(P_{2h\alpha})$. For each plot we set $g = 10, b = 50, z_d = 40$ and $M_2 = 1$.

In Figure 7 we plot the continuous quadratic cost function J_2 given by (50) and the discrete cost function J_{2h} obtained in (52) against q for $h = 1/10, 1/25$ and $1/50$. Observe that as h decreases the function $J_{2h} = J_{2h}(q)$ also decreases to the limit function $J_2 = J_2(q)$. In a similar way, in Figure 8, for $\alpha = 100$ we obtain the continuous function $J_{2\alpha}$ and the discrete functions $J_{2h\alpha}$ for $h = 1/10, 1/25$ and $1/50$. The convergences $J_{2h} \to J_2$ and $J_{2h\alpha} \to J_{2\alpha}$ when $h \to 0$ are in agreement with Lemmas 9 and 12, respectively.

Moreover, Figure 9 shows the double convergence of $J_{2h\alpha} \to J_2$ when $(h, \alpha) \to (0, \infty)$. We illustrate how $J_{2h\alpha}$ gets closer to J_2 as the value of h decreases and the value of α increases.

In Figure 10 we plot the continuous optimal control q_{op} for the problem (P_2) given by (51) and the optimal control $q_{\alpha_{op}}$ given by (63) for $\alpha = 50, 100, 200$. Notice that as α increases, $q_{\alpha_{op}}$ decreases to the limit q_{op}. In addition, we set different values of n between $n = 10$ and $n = 100$. Recalling that $h = \frac{x_0}{n} = \frac{1}{n}$, for each h we obtain the optimal discrete control $q_{h_{op}}$ to the problem (P_{2h}) defined by (54) and the optimal discrete control $q_{h\alpha_{op}}$ to the problem $(P_{2h\alpha})$ given by (67) for $\alpha = 50, 100, 200$. For each α fixed, we have that the discrete solution $q_{h\alpha_{op}} \to q_{\alpha_{op}}$ when $h \to 0$, i.e. $n \to \infty$.

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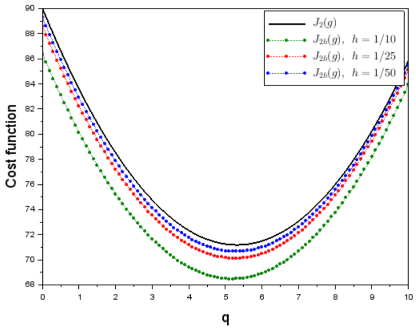

Figure 7. Plot of J_2 and J_{2h}

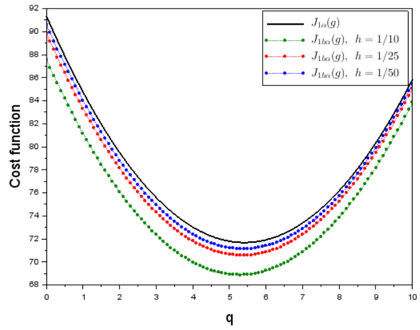

Figure 8. Plot of $J_{2\alpha}$ and $J_{2h\alpha}$ for $\alpha = 100$.

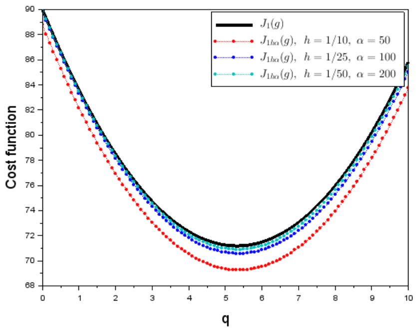

Figure 9. Plot of J_2 and $J_{2h\alpha}$

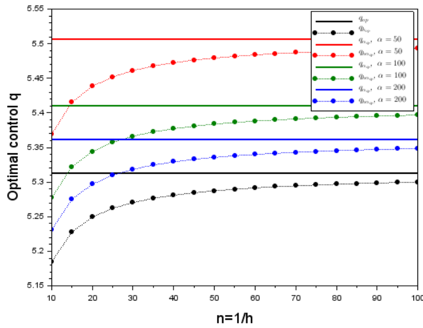

Figure 10. Plot of $q_{op}, q_{h_{op}}, q_{\alpha op}$ and $q_{h\alpha_{op}}$

6.3. Control Variable b

In this section we obtain some computational examples for the optimal distributed control problems (P_3), (P_{3h}), $(P_{3\alpha})$ and $(P_{3h\alpha})$. For each plot we set $q = 12, g = 10, z_d = 40$ and $M_3 = 1$.

In Figure 11 we plot the continuous quadratic cost function J_3 given by (73) and the discrete cost function J_{3h} obtained in (75) against g for $h = 1/10$, $1/25$ and $1/100$. Notice that as h decreases the function $J_{3h} = J_{3h}(b)$ also decreases to the limit function $J_3 = J_3(b)$ in agreement with Lemma 15. In a similar manner in Figure 12, for $\alpha = 50$ we obtain the continuous function $J_{3\alpha}$ and the discrete functions $J_{3h\alpha}$ for $h = 1/10, 1/25$ and $1/100$. Observe the convergence of $J_{3h\alpha} \to J_{3\alpha}$ as $h \to 0$. Moreover, Figure 13 shows the double convergence of $J_{3h\alpha} \to J_3$ when $(h, \alpha) \to (0, \infty)$. We illustrate how $J_{3h\alpha}$ gets closer to J_3 as the value of h decreases and the value of α increases.

In Figure 14 we plot the continuous optimal control b_{op} for the problem (P_3) given by (74) and the optimal control $b_{\alpha_{op}}$ given by (83) for $\alpha = 15, 50, 100$. Notice that as α increases, $b_{\alpha_{op}}$ decreases to the limit b_{op}. In addition, we set different values of n between $n = 10$ and $n = 100$. Recalling that $h = \frac{x_0}{n} = \frac{1}{n}$, for each h we obtain the optimal discrete control $b_{h_{op}}$ to the problem (P_{3h}) defined by (77) and the optimal discrete control $b_{h\alpha_{op}}$ to the problem $(P_{3h\alpha})$ given by (88) for $\alpha = 15, 50, 100$. For each α fixed, we have that the discrete solution $b_{h\alpha_{op}}$ decreases to $b_{\alpha_{op}}$ when $h \to 0$.

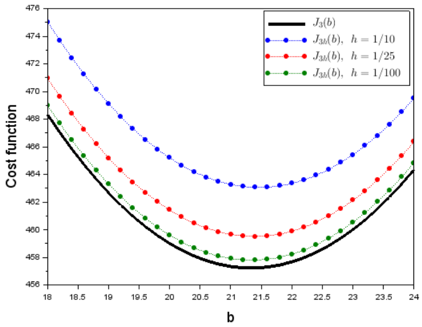

Figure 11. Plot of J_3 and J_{3h}

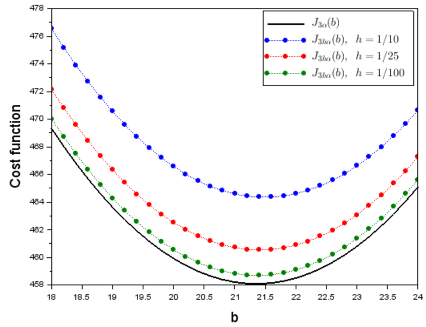

Figure 12. Plot of $J_{3\alpha}$ and $J_{3h\alpha}$ for $\alpha = 100$.

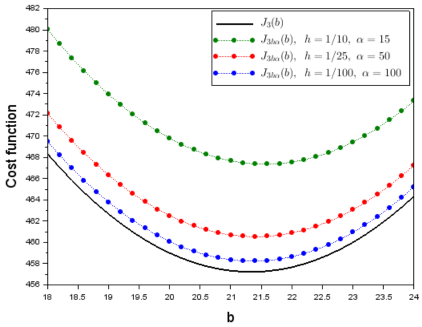

Figure 13. Plot of J_3 and $J_{3\alpha}$

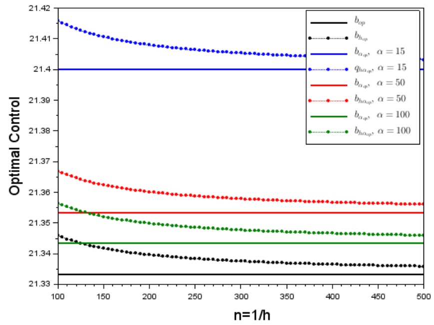

Figure 14. Plot of b_{op}, $b_{h_{op}}$, $b_{\alpha op}$ and $b_{h\alpha op}$

7. Improvement of the Order of Convergence

In this section, we introduce alternative discrete solutions $\tilde{u}_h$ and $\tilde{u}_{h\alpha}$ associated with the systems (S) and (S_α), respectively, and analyze the order of convergence of $\tilde{u}_h$ to u and of $\tilde{u}_{h\alpha}$ to u_α as $h \to 0^+$. The Neumann boundary condition on Γ_2 is approximated by a three–point backward finite–difference scheme. Moreover, for the discrete solution $\tilde{u}_{h\alpha}$, the Robin boundary condition on Γ_1 is approximated by a three–point forward finite–difference scheme. These higher–order boundary approximations lead to an improved order of accuracy.

We consider the system (S) defined by equations (1)–(2). From this system, we define the discrete problem $(\tilde{S}_h)$, where $\tilde{u}_i$ approximates $u(x_i, y)$, for $i = 1, \cdots, n+1$. Notice that, from the Dirichlet condition on Γ_1 it follows immediately that $\tilde{u}_1 = b$.

For the interior nodes, we employ the classical centered second–order finite–difference approximation given in (15), which leads to the discrete system (16) for $\tilde{u}_i$, $i = 2, ..., n$.

For the Neumann boundary condition on Γ_2, we use the three–point backward approximation

$$\frac{\partial u}{\partial x}(x_{n+1}, y) \approx \frac{3u(x_{n+1}, y) - 4u(x_n, y) + u(x_{n-1}, y)}{2h}. \tag{92}$$

Thus, the discrete Neumann condition can be written as

$$-2qh = 3\tilde{u}_{n+1} - 4\tilde{u}_n + \tilde{u}_{n-1}. \tag{93}$$

In addition, from (16) for $i = n$, we obtain

$$-gh^2 = \tilde{u}_{n+1} - 2\tilde{u}_n + \tilde{u}_{n-1}. \tag{94}$$

Subtracting the two previous equations, it follows that

$$-\tilde{u}_n + \tilde{u}_{n+1} = \frac{gh^2}{2} - qh. \tag{95}$$

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Therefore the system given by (16) together with (95) can be written as

$$A\overline{\mathrm{w}} = T^* \tag{96}$$

where $\overline{\mathrm{w}} = (\widetilde{\mathrm{u}}_i)_{i=2,\dots,n+1} \in \mathbb{R}^n$ is the vector of unknowns, A is the matrix given by (20) and $T^* \in \mathbb{R}^n$ is the vector of independent terms:

$$T^* = \Big(-g\,h^2 - b, -gh^2, \dots, -gh^2, -q\,h + \tfrac{gh^2}{2} \Big)^t. \tag{97}$$

Notice that the system (96) differs from (19) in the last component of the vector of independent terms. Solving the linear system gives

$$\widetilde{\mathrm{u}}_i = b + (i-1)h(gx_0 - q) - \frac{gh^2}{2}(i-1)^2. \tag{98}$$

Taking into account that for $i = 1, \cdots, n$

$$\widetilde{m}_i = \frac{\widetilde{\mathrm{u}}_{i+1} - \widetilde{\mathrm{u}}_i}{x_{i+1} - x_i} = gx_0 - q - (2i-1)\frac{gh}{2}, \tag{99}$$

and

$$\widetilde{h}_i = \widetilde{\mathrm{u}}_i - \widetilde{m}_i x_i = b + \frac{gh^2}{2} i(i-1), \tag{100}$$

the linear approximation is given by $\widetilde{u}_h(x,y) = \widetilde{m}_i x + \widetilde{h}_i$, i.e.

$$\widetilde{u}_h(x,y) = \left(gx_0 - q - (2i-1)\frac{gh}{2} \right) x + \frac{gh^2}{2} i(i-1) + b, \qquad x \in [x_i, x_{i+1}], \quad i = 1, \cdots, n \tag{101}$$

In the following lemma, we give some bounds for the approximate function $\widetilde{u}_h$:

Lemma 21. *The following bounds hold:*

$$\|u - \widetilde{u}_h\|_H \leq D_1 h^2, \qquad \text{and} \qquad \|\tfrac{\partial u}{\partial x} - \tfrac{\partial \widetilde{u}_h}{\partial x}\|_H \leq \widetilde{D}_1\, h,$$

where $D_1 = \sqrt{\frac{x_0 y_0}{120}}\, g$ *and* $\widetilde{D}_1 = \sqrt{\frac{x_0 y_0}{12}}\, g$.

Proof. From the definition of the norm in the space H and using the expressions (13) and (101) for the functions u and $\widetilde{u}_h$, respectively, it follows that

$$\begin{aligned} \|u - \widetilde{u}_h\|_H^2 &= \int_0^{y_0} \int_0^{x_0} \big(u(x,y) - \widetilde{u}_h(x,y) \big)^2 \, dx\, dy \\ &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} E_i^2(x)\, dx, \end{aligned} \tag{102}$$

where

$$E_i(x) = u(x,y) - \widetilde{u}_h(x,y), \quad x \in [x_i, x_{i+1}],\ y \in [0, y_0].$$

Note that, within each subinterval, $E_i(x)$ depends only on x and the index i, but not on y, since both u and $\widetilde{u}_h$ are constant along the y-direction.

A direct computation yields

$$E_i(x) = \frac{g}{2}\Big(-x^2 + (2i-1)hx - h^2 i(i-1) \Big) = -\frac{g}{2}(x - ih)(x - (i-1)h). \tag{103}$$

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Then

$$\begin{aligned}\int_{x_i}^{x_{i+1}} E_i^2(x)\,dx &= \frac{g^2}{4}\left[\frac{(x-ih)^5}{5} + \frac{h}{2}(x-ih)^4 + \frac{h^2}{3}(x-ih)^3\right]_{x_i}^{x_{i+1}} \\ &= \frac{g^2}{4}\left(\frac{h^5}{5} - \frac{h^5}{2} + \frac{h^5}{3}\right) = \frac{g^2}{120}h^5.\end{aligned} \tag{104}$$

As a consequence, from (102) it follows that

$$\|u - \tilde{u}_h\|_H^2 = y_0\frac{g^2h^5}{120}n = \frac{x_0y_0g^2}{120}h^4,$$

and then

$$\|u - \tilde{u}_h\|_H = \sqrt{\frac{x_0y_0}{120}}gh^2.$$

In addition,

$$\begin{aligned}\left\|\tfrac{\partial u}{\partial x} - \tfrac{\partial \tilde{u}_h}{\partial x}\right\|_H^2 &= \int_0^{y_0}\int_0^{x_0}\left(\tfrac{\partial u}{\partial x}(x,y) - \tfrac{\partial \tilde{u}_h}{\partial x}(x,y)\right)^2 dx\,dy \\ &= y_0\sum_{i=1}^{n}\int_{x_i}^{x_{i+1}} F_i^2(x)\,dx,\end{aligned} \tag{105}$$

where

$$F_i(x) = \tfrac{\partial u}{\partial x}(x,y) - \tfrac{\partial \tilde{u}_h}{\partial x}(x,y) = -g\left(x - \frac{(2i-1)h}{2}\right),$$

for $x \in [x_i, x_{i+1}]$. Then

$$\begin{aligned}\int_{x_i}^{x_{i+1}} F_i^2(x)\,dx &= \int_{x_i}^{x_{i+1}} g^2\left(x - \frac{(2i-1)h}{2}\right)^2 dx \\ &= g^2\left[\frac{1}{3}\left(x - \frac{(2i-1)h}{2}\right)^3\right]_{x_i}^{x_{i+1}} \\ &= g^2\frac{1}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right) = \frac{g^2}{12}h^3.\end{aligned}$$

Therefore, from (105) we have

$$\left\|\tfrac{\partial u}{\partial x} - \tfrac{\partial \tilde{u}_h}{\partial x}\right\|_H^2 = y_0\frac{g^2}{12}nh^3 = \frac{x_0y_0g^2}{12}h^2,$$

and finally

$$\left\|\tfrac{\partial u}{\partial x} - \tfrac{\partial \tilde{u}_h}{\partial x}\right\|_H = \sqrt{\frac{x_0y_0}{12}}\,g\,h.$$

□

Remark 11. *We emphasize that by improving the approximation of the Neumann boundary condition on Γ_2, the convergence order of the error $\|u - \tilde{u}_h\|_H$ is increased to second order, namely $O(h^2)$. This enhancement leads to a more accurate numerical approximation while remaining fully consistent with the theoretical convergence results established in [7,13].*

Remark 12. *The linear system (96) obtained by using the three–point backward finite–difference approximation for the Neumann boundary condition on Γ_2 can be equivalently interpreted by introducing a ghost point x_{n+2} outside the computational domain and assuming that the discrete differential equation holds at the boundary node x_{n+1}. Indeed, assuming that the equation is satisfied at $\tilde{u}_{n+1}$, we have*

$$-gh^2 = \tilde{u}_{n+2} - 2\tilde{u}_{n+1} + \tilde{u}_n,$$

while the Neumann boundary condition is approximated by

$$\frac{\widetilde{u}_{n+2} - \widetilde{u}_n}{2h} = -q.$$

Eliminating the ghost value $\widetilde{u}_{n+2}$ *from these two expressions yields*

$$-\widetilde{u}_n + \widetilde{u}_{n+1} = -qh + \frac{gh^2}{2},$$

which coincides with the boundary equation obtained in (95). Hence, the three–point backward finite–difference approximation of the Neumann condition is consistent with the ghost–point formulation and leads to the same discrete system.

Analogously to the analysis of system (S), we propose a new discrete approximation $\widetilde{u}_{h\alpha}$ for system (S_α) and study the order of convergence of $\widetilde{u}_{h\alpha}$ to u_α as $h \to 0^+$. The associated discrete system $(\widetilde{S}_{h\alpha})$ employs a three–point backward finite–difference approximation for the Neumann boundary condition on Γ_2 and a three–point forward finite–difference approximation for the Robin boundary condition on Γ_1, leading to improved accuracy.

We consider the system (S_α) defined by equations (1)–(3) and define $\widetilde{\mathrm{u}}_{\alpha,i} \approx u_\alpha(x_i, y)$.

For the interior nodes, $i = 2, \ldots, n$, we employ the classical centered second–order finite–difference approximation given in (15):

$$\widetilde{\mathrm{u}}_{\alpha,i+1} - 2\widetilde{\mathrm{u}}_{\alpha,i} + \widetilde{\mathrm{u}}_{\alpha,i-1} = -gh^2. \tag{106}$$

For the Robin boundary at Γ_1, we use the three–point forward approximation:

$$\frac{-3\widetilde{\mathrm{u}}_{\alpha,1} + 4\widetilde{\mathrm{u}}_{\alpha,2} - \widetilde{\mathrm{u}}_{\alpha,3}}{2h} = \alpha(\widetilde{\mathrm{u}}_{\alpha,1} - b). \tag{107}$$

Combining this expression with the interior equation at $i = 2$ yields the simplified discrete condition

$$-(1 + \alpha h)\widetilde{\mathrm{u}}_{\alpha,1} + \widetilde{\mathrm{u}}_{\alpha,2} = -\alpha hb - \frac{gh^2}{2}. \tag{108}$$

For the Neumann boundary at Γ_2 we use the three–point backward approximation:

$$3\widetilde{\mathrm{u}}_{\alpha,n+1} - 4\widetilde{\mathrm{u}}_{\alpha,n} + \widetilde{\mathrm{u}}_{\alpha,n-1} = -2qh. \tag{109}$$

Combining with the interior equation for $i = n$ gives

$$-\widetilde{\mathrm{u}}_{\alpha,n} + \widetilde{\mathrm{u}}_{\alpha,n+1} = -qh + \frac{gh^2}{2}. \tag{110}$$

The system given by (106), (108) and (110) can be rewritten as

$$A_\alpha \overline{\mathrm{w}_\alpha} = T^*_\alpha \tag{111}$$

where $\overline{\mathrm{w}}_\alpha = (\widetilde{\mathrm{u}_{\alpha i}})_{i=1,\ldots,n+1} \in \mathbb{R}^{n+1}$ is the vector of unknowns, A_α is the matrix given by (24) and $T^*_\alpha \in \mathbb{R}^{n+1}$ is the vector of independent terms:

$$T^*_\alpha = \Big(-\alpha\, b\, h - \tfrac{gh^2}{2}, -gh^2, \ldots, -gh^2, -q\, h + \tfrac{gh^2}{2} \Big)^t \in \mathbb{R}^{n+1}. \tag{112}$$

It should be noted that only the first and last components of T^*_α differ from those in T_α given by (25).

The solution of the system (111) is given by

$$\widetilde{u}_{\alpha,i} = b + \frac{1}{\alpha}\Big(gx_0 - q\Big) + (i-1)h\,(gx_0 - q) - \frac{g}{2}\big((i-1)h\big)^2, \quad i = 1, \ldots, n+1. \tag{113}$$

We define the linear interpolation on each subinterval $[x_i, x_{i+1}]$ by

$$\widetilde{u}_{h\alpha}(x,y) = \widetilde{m}_{\alpha,i}x + \widetilde{h}_{\alpha,i}, \quad x \in [x_i, x_{i+1}],\ y \in [0, y_0], \tag{114}$$

where

$$\widetilde{m}_{\alpha,i} = gx_0 - q - gh\left(i - \frac{1}{2}\right), \quad i = 1, \dots, n, \tag{115}$$

$$\widetilde{h}_{\alpha,i} = b + \frac{1}{\alpha}(gx_0 - q) + \frac{gh^2}{2}(i-1)i, \quad i = 1, \dots, n. \tag{116}$$

From the previous expressions, we derive the following lemma.

Lemma 22. *The following bounds hold:*

$$\|u_\alpha - \widetilde{u}_{h\alpha}\|_H \le D_2 h^2, \qquad \|\tfrac{\partial u_\alpha}{\partial x} - \tfrac{\partial \widetilde{u}_{h\alpha}}{\partial x}\|_H \le \widetilde{D}_2 h,$$

where $D_2 = \sqrt{\frac{x_0 y_0}{120}}g$ *and* $\widetilde{D}_2 = \sqrt{\frac{x_0 y_0}{12}}g$.

Proof. By the definition of the H-norm, and using the expression for u_α in (13) as well as the definition of $\widetilde{u}_{h\alpha}$ in (114), it follows that

$$\|u_\alpha - \widetilde{u}_{h\alpha}\|_H^2 = y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} E_{\alpha,i}^2(x)\, dx, \tag{117}$$

where

$$E_{\alpha,i}(x) = u_\alpha(x,y) - \widetilde{u}_{h\alpha}(x,y)$$

$$E_{\alpha,i}(x) = -\frac{g}{2}x^2 + \frac{g}{2}(2i-1)hx - \frac{gh^2}{2}(i^2 - i), \quad x \in [x_i, x_{i+1}] \tag{118}$$

We can notice that $E_{\alpha,i}(x) = E_i(x)$ where $E_i(x)$ is given by (103). Therefore, from (104), it follows immediately that

$$\|u_\alpha - \widetilde{u}_{h\alpha}\|_H^2 \le \frac{x_0 y_0 g^2}{120} h^4.$$

In addition,

$$\begin{aligned}
\|\tfrac{\partial u}{\partial x} - \tfrac{\partial \widetilde{u}_h}{\partial x}\|_H^2 &= \int_0^{y_0}\int_0^{x_0} \left(\tfrac{\partial u}{\partial x}(x,y) - \tfrac{\partial \widetilde{u}_h}{\partial x}(x,y)\right)^2 dx\, dy \\
&= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} g^2 \left(x - \frac{(2i-1)h}{2}\right)^2 dx \qquad (119) \\
&= y_0 \frac{g^2}{12} h^3 n = \frac{x_0 y_0 g^2}{12} h^2.
\end{aligned}$$

□

8. Conclusions

Applying the finite difference method, we have derived the discrete systems (S_h) and $(S_{h\alpha})$ and the discrete optimization problems (P_{ih}) and $(P_{ih\alpha})$, $i = 1, 2, 3$ where $\alpha > 0$ is a parameter that represents the heat transfer coefficient on a portion of the boundary of the domain. Explicit discrete solutions have been found and convergence results when the discrete step h goes to zero and when α goes to infinite have been proved. Error estimations have been also obtained as a function of the step h. Some numerical computations have been provided in order to illustrate the theoretical results.

Finally, for the systems (S) and (S_α), an alternative discretization of the Neumann boundary condition on Γ_2 and of the Robin boundary condition on Γ_1 for (S_α) has been considered. By modifying the approximation of these boundary conditions, the order of convergence of the numerical solution is improved, leading to a more accurate approximation.

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References

1. An, X.; He, Z.; Song, X. The explicit solution to the initial-boundary value problem of Gierer–Meinhardt model. *Applied Mathematics Letters* **2018**, *80*, 59–63.
2. Angel, E. Discrete invariant imbedding and elliptic boundary-value problems over irregular regions. *Journal of Mathematical Analysis and Applications* **1968**, *23*, 471–484.
3. Arnal, A.; Monterde, J.; Ugail, H. Explicit polynomial solutions of fourth order linear elliptic partial differential equations for boundary based smooth surface generation. *Computer Aided Geometric Design* **2011**, *28*, 382–394.
4. Azzam, A.; Kreyszig, E. On solutions of elliptic equations satisfying mixed boundary conditions. *SIAM Journal on Mathematical Analysis* **1982**, *13*, 254–262.
5. Barbu, V. *Optimal Control of Variational Inequalities*; Pitman Advanced Publishing Program: Boston, 1984.
6. Bollati, J.; Gariboldi, C.M.; Tarzia, D.A. Explicit solutions for distributed, boundary and distributed-boundary elliptic optimal control problems. *Journal of Applied Mathematics and Computing* **2020**, *64*, 283–311.
7. Casas, E.; Mateos, M. Uniform convergence of the FEM. Application to state constrained control problems. *Comput. Appl. Math.* **2002**, *21*, 67–100.
8. Garguichevich, G.; Tarzia, D.A. The steady-state two-phase Stefan problem with an internal energy and some related problems. *Atti Sem. Mat. Univ. Modena* **1991**, *39*, 615–634.
9. Gariboldi, C.M.; Tarzia, D.A. Convergence of distributed optimal controls on the internal energy in mixed elliptic problems when the heat transfer coefficient goes to infinity. *Applied Mathematics & Optimization* **2003**, *47*, 213–230.
10. Gariboldi, C.M.; Tarzia, D.A. Convergence of boundary optimal control problems with restrictions in mixed elliptic Stefan-like problems. *Advances in Differential Equations and Control Processes* **2008**, *1*, 113–132.
11. Grisvard, P. *Elliptic Problems in Non-Smooth Domains*; Pitman: London, 1985.
12. Haller-Dintelman, R.; Meyer, C.; Rehberg, J.; Schiela, A. Hölder continuity and optimal control for nonsmooth elliptic problems. *Applied Mathematics and Optimization* **2009**, *60*, 397–428.
13. Hinze, M. A variational discretization concept in control constrained optimization: The linear-quadratic case. *Computational Optimization and Applications* **2005**, *30*, 45–61.
14. Lang, J.; Schmitt, B.A. Exact discrete solutions of boundary control problems for the 1D heat equation. *Journal of Optimization Theory and Applications* **2023**, *196*, 1106–1118.
15. Lanzani, L.; Capogna, L.; Brown, R. The mixed problem in L^p for some two-dimensional Lipschitz domains. *Mathematische Annalen* **2008**, *342*, 91–124.
16. Lions, J. *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*; Dunod: Paris, 1968.
17. Neittaanmäki, P.; Sprekels, J.; Tiba, D. *Optimization of Elliptic Systems. Theory and Applications*; Springer: New York, 2006.
18. Shamir, E. Regularization of mixed second order elliptic problems. *Israel Journal of Mathematics* **1968**, *6*, 150–168.
19. Tabacman, E.D.; Tarzia, D.A. Sufficient and/or necessary condition for the heat transfer coefficient on γ_1 and the heat flux on γ_2 to obtain a steady-state two-phase Stefan problem. *Journal of Differential Equations* **1989**, *77*, 16–37.
20. Tarzia, D.A. Sur le problème de Stefan à deux phases. *C.R. Acad. Sci. Paris* **1979**, *288A*, 941–944.
21. Tarzia, D.A. An inequality for the constant heat flux to obtain a steady-state two-phase Stefan problem. *Engineering Analysis* **1988**, *5*, 177–181.

22. Tarzia, D.A. Numerical analysis for the heat flux in a mixed elliptic problem to obtain a discrete steady-state two-phase Stefan problems. *SIAM Journal on Numerical Analysis* **1996**, *33*, 1257–1265.
23. Tarzia, D.A. Double convergence of a family of discrete distributed mixed elliptic optimal control problems with a parameter. In Proceedings of the 27th IFIP TC 7 Conference on System Modelling and Optimization, CSMO 2015, IFIP AICT 494; Bociu, L.; Desideri, J.; Habbal, A., Eds.; Springer Nature Singapore: Berlin, 2016; pp. 493–504.
24. Tröltzsch, F. *Optimal Control of Partial Differential Equations. Theory, Methods and Applications*; American Mathematical Society: Providence, 2010.
25. Wachsmuth, D.; Wachsmuth, G. Regularization error estimates and discrepancy principle for optimal control problems with inequality constraints. *Control and Cybernetics* **2011**, *40*, 1125–1158.
26. Yang, C.; Sun, T. Second-order time discretization for reaction coefficient estimation of bilinear parabolic optimization problem with Neumann boundary conditions. *Computers & Mathematics with Applications* **2023**, *140*, 211–224.