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Article

Well-Posedness of the Fisher–KPP Equation with Neumann, Dirichlet, and Robin Boundary Conditions on the Real Half Line

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Abstract

We consider the Fisher–KPP equation with Neumann boundary conditions on the real half line. We claim that the Fisher-KPP equation with Neumann boundary conditions is well-posed only for odd positive stationary solutions. We begin by proving that the Fisher-KPP equation with a Dirichlet boundary condition is stable, and with a Robin condition is stable only for odd positive stationary solutions. Then we inferred and proved that the Fisher-KPP equation with Neumann boundary conditions is stable only for odd positive stationary solutions. We solved the Fisher–KPP equation with Neumann boundary conditions to demonstrate the existence of the solution. In addition, we proved the uniqueness of the solution. Moreover, we proved the the solution of Fisher-Kpp equation with Dirichlet condition is stable. We also showed that the Fisher-KPP equation with Robin boundary conditions is stable only for odd positive stationary solutions. The uniqueness and existence proof of the Fisher-KPP equation with Robin condition are similar to the Neumann condition. Hence, we conclude that the Fisher-KPP equation on the real line is well-posed for the Dirichlet condition, and well-posed only for odd positive stationary solutions for both the Neumann condition and the Robin condition.

Keywords: the Fisher-KPP equations; well-posedness; boundary conditions

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1. Introduction

The Fisher–Kolmogorov–Petrovsky–Piskunov (Fisher–KPP) equation is a one-dimensional semi-linear reaction–diffusion equation, written as [1,2]:

$$u_t = u_{xx} + f(u). \tag{1}$$

The Fisher-KPP equation was introduced by Fisher, Kolmogorov, Petrovsky, and Piskunov introduced in 1937, [3,4].

Let $u = u(t, x)$ be the population at time t and location x to generate the Fisher-KPP equation. Then

$$u_t - u_{xx} = f(u)$$

models the population size, where $f(u)$ is the diffusion function. Here, we consider the Fisher-KPP equation to be defined on $[0, 1]$. Note that there is no difference if the Fisher-KPP equation is defined arbitrarily $[n, n+1]$, where n is a positive integer. In addition, $\bigcup_{n=0}^{\infty}[n, n+1] = [0, \infty)$ which is exactly the real half line. Hence, the Fisher-KPP equation is defined on the real half-line.

Since the diffusion of the biological population is inhomogeneous, we need to determine the inhomogeneous term in this diffusion equation to measure population diffusion. If $f(0) = 0$, the population is extinct in this case, since there is no diffusion. If the population is at the maximal capacity, there is also no diffusion. Hence, $f(1) = 0$. If $0 < u < 1$, then there exists diffusion. During the extinction, the diffusion rate increased as the density increased, and the diffusion rate decreased during

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the maximal capacity. We have $f'(1) < 0 < f'(0)$. Consider the biological meaning of the diffusion rate; we find that during the extinction is the upper bound of general $f'(u)$. Then we generate the inhomogeneous term $f(u)$ have the following properties as desired [1,5]:

$$f(0) = f(1) = 0, f'(1) < 0 < f'(0), f'(u) \leq f'(0), f(u) > 0$$

for $u \in (0,1)$.

A nonnegative or positive domain is an example of a semi-infinite domain (objects that are infinite or unbounded in some but not all possible ways [8]). Hence, the Fisher-KPP equation is defined in a semi-infinite domain.

The solutions of the Fisher-KPP equation is on a semi-infinite domain and evolve from initial conditions with compact support to a traveling wave with a minimum wave speed. Let λ be the population density. $c_{min} = 2\sqrt{D\lambda}$ with $t \to \infty$ [8]. With $D = 1$ in this case, we get that $c = 2\sqrt{\lambda}$ is the minimum wave speed for traveling wave solutions of the Fisher-KPP equation, since the general expression of the inhomogeneous term of the Fisher-KPP equation is [2,6]

$$f(u) = ru(1-u), \tag{2}$$

where r is constant.

To generate this solution, by letting $z = x - ct$, we get $u_t = -c\frac{du}{dz}$ and $u_{xx} = \frac{d^2u}{dz^2}$. Thus, we have

$$-\frac{d^2u}{dz^2} - c\frac{du}{dz} = \lambda u(1-u),$$

which is equal to

$$\frac{d^2u}{dz^2} + c\frac{du}{dz} + \lambda u(1-u) = 0. \tag{3}$$

The minimal wave speed happens only if $u \to 1$, and since $1 - u$ is small enough so that we can ignore it.

Using $u = e^{rt}$ in (3), we get

$$r^2 + rc + \lambda = 0. \tag{4}$$

Then

$$r = \frac{-c \pm \sqrt{c^2 - 4\lambda}}{2}.$$

To have $c^2 - 4\lambda \geq 0$, then $c_{min} = 2\sqrt{\lambda}$. For the case $u \to 0$, it cannot describe the traveling wave because it is a homogeneous diffusion equation. Then we can not get a solution of traveling waves since $r^2 + rc = 0$, $r = 0$ or $r = -c$.

Since c is positive, r cannot be negative because the Fisher-KKP equation describes the diffusion of population, so $r = 0$. There are no traveling waves. Although we care about the traveling wave with minimum wave speed more, we can also find the solutions of traveling waves with speed greater than c_{min}.

Consider the entire solution(both solutions) of the Fisher-KPP equation defined in the real interval [0,1], where f is continuously differentiable on [0,1] to make sure both f and f' are well-defined and satisfy the following conditions [1]:

$$f(0) = f(1) = 0, f'(1) < 0 < f'(0), f'(u) \leq f'(0), f(u) > 0$$

for $u \in (0,1)$.

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In this paper, we consider the Fisher-Kpp equation with Neumann condition in addition to the Dirichlet condition and Robin condition. The Fisher-Kpp equation with Neumann boundary conditions is given as:

$$\begin{cases} u_t = u_{xx} + f(u), & 0 \leq x \leq 1, t > 0, \\ u_x(0,t) = 0 = u_x(1,t), & t > 0. \end{cases}$$

Similarly, by taking the union, we get $x \in \mathbb{R}^+$:

$$\bigcup_{n=0}^{\infty} [n, n+1] = [0, \infty).$$

2. Well-Posedness of the Fisher-KPP Equation

Definition 1 (Well-posedness [7])**.**

Existence: There exists at least one solution u(x, t) satisfying all these conditions.

Uniqueness: There is at most one solution.

Stability: The unique solution u(x, t) depends in a stable manner on the data of the problem. This means that if the data are changed a little, the corresponding solution changes only a little.

The Bouandray value problem for the Fisher-KPP equation is well-posed if it has a unique and stable solution.

To show that the Fisher–KPP equation with Neumann boundary conditions is well-posed, we first show stability.

We begin with the proof of stability of the solution.

2.1. Stability

Theorem 1 (Maximum principle for diffusion equation [8] The maximal value is attained only on the boundary and the initial part of the region)**.** *Suppose we are given an open spatial domain Ω and a time interval $I \equiv (t_0, t_f]$, where Ω may be unbounded and t_f may be infinite.*

We define a parabolic cylinder $\Omega \times I$ *(the Cartesian product representing space–time) and a* parabolic boundary

$$\Gamma := \left(\overline{\Omega} \times \{t_0\}\right) \cup \left(\partial\Omega \times \overline{I}\right),$$

which includes the spatial boundary and the initial time boundary (but not the final time boundary).

If u satisfies the diffusion equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{in } \Omega \times I,$$

then

$$\max_{(x,t)\in\overline{\Omega}\times\overline{I}} u(x,t) = \max_{(x,t)\in\Gamma} u(x,t).$$

In words, the solution to the heat equation attains the maximum value somewhere on the parabolic boundary Γ.

Theorem 2 (Stability, cf. Theorem 5.8 in [8] and [7])**.** *Let u_1 and u_2 be solutions to the initial-boundary value problems associated with two different sets of boundary and initial data (h_1, g_1) and (h_2, g_2), respectively. For $i = 1, 2$:*

$$\frac{\partial u_i}{\partial t} - \Delta u_i = 0 \quad \text{in } \Omega \times I,$$
$$u_i = h_i \quad \text{on } \partial\Omega \times I,$$
$$u_i = g_i \quad \text{on } \Omega \times \{t_0\}.$$

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Here Ω is an open spatial domain and $I = (t_0, t_f]$ is a time interval. $\Omega \times I$ is the parabolic cylinder. Then the solution depends continuously on the data in the sense that

$$\max_{(x,t)\in\Omega\times I} |u_1(x,t) - u_2(x,t)| \leq \max\left\{ \max_{(x,t)\in\partial\Omega\times\overline{I}} |h_1(x,t) - h_2(x,t)|, \max_{x\in\overline{\Omega}} |g_1(x) - g_2(x)| \right\}.$$

Proof: See *Theorem 5.8* [8] and Maximum Principle Theorem in [7, Page-42].

Before, we state the stability of the Fisher-Kpp equation with Dirichlet condition, we write the following definitions.

Definition 2 (periodic function)**.** *If f:$\mathbb{R} \to \mathbb{R}$, Then f is periodic if and only if $\exists L \in \mathbb{R} \setminus \{0\} : \forall x \in \mathbb{R} : f(x) = f(x+L)$*

Definition 3 (periodic solution)**.** *A cycle, or periodic solution, is a solution of a differential equation that is a periodic function.*

Theorem 3. *The Fisher-KPP equation with Dirichlet boundary conditions is stable.*

Proof: Let $u(t,x)$ be the entire solution and $V(t,x)$ be the unique positive periodic solution(a solution both positive and periodic) of the Fisher-KPP equation with Dirichlet boundary conditions. According to Cai et al. [9, Theorem 1.1],

$$limsup_{t\to\infty}|u(t,x) - V(t,x)| = 0.$$

Let $u_1(t,x), u_2(t,x)$ be two distinct solution of this equation. We have

$$limsup_{t\to\infty}|u_1(t,x) - V(t,x)| = 0.$$

and

$$limsup_{t\to\infty}|u_2(t,x) - V(t,x)| = 0.$$

By the triangle inequality,

$$\begin{aligned} & limsup_{t\to\infty}(|u_1(t,x) - u_2(t,x)|) \\ & = limsup_{t\to\infty}(|u_1(t,x) - V(t,x) + V(t,x) - u_2(t,x)|) \\ & \leq limsup_{t\to\infty}(|u_1(t,x) - V(t,x)| + |u_2(t,x) - V(t,x)|) = 0. \\ & = 0. \end{aligned}$$

■

We only consider that $t \to \infty$ because, in our model, which is in the real world, time can not be negative.

Theorem 4. *The Fisher-KPP equation with Robin boundary conditions is stable only for odd positive stationary solutions.*

Proof: According to the proof of Theorem 1.1 by Suo et al. [1], let $u(x,t;\phi_A(x))$ be the solution of the Fisher-KPP equation with Robin boundary conditions with initial condition $\phi_A(x)$ and $V_i(x)$ be positive stationary solutions. Similarly, let $u_1(x,t;\phi_A(x)), u_2(x,t;\phi_A(x))$ be distinct solutions of this equation. For i is an odd integer, we have

$$limsup_{t\to\infty}|u(x,t;\phi_A(x)) - V_i(x)| = 0.$$

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We also have

$$limsup_{t\to\infty}|u_1(x,t;\phi_A(x)) - V_i(x)| = 0.$$

and

$$limsup_{t\to\infty}|u_2(x,t;\phi_A(x)) - V_i(x)| = 0.$$

By the triangle inequality, for an odd i integer, we have

$$\begin{aligned}
&limsup_{t\to\infty}|u_1(x,t;\phi_A(x)) - u_2(x,t;\phi_A(x))| \\
&\leq limsup_{t\to\infty}|u_2(x,t;\phi_A(x)) - V_i(x) + V_i(x) - u_1(x,t;\phi_A(x))| \\
&= limsup_{t\to\infty}(|u_2(x,t;\phi_A(x)) - V_i(x)| + |V_i(x) - u_1(x,t;\phi_A(x))|) \\
&= 0.
\end{aligned}$$

Hence by Theorem 2.1 above, the proof completes.■

Definition 4 (stationary solution)**.** *u are an equilibrium point or a stationary point*

$$\frac{dx}{dt} = f(x(t))$$

if and only if $f(u) = 0$. in this case, it $x(t) = u$ is called equlibrium solution or a stationary solution. Remark: if u is an equilibrium solution, then it is a constant solution of $\frac{dx}{dt} = f(x)$. i.e., $\frac{du}{dt} = 0$.

Theorem 5. *The Fisher-KPP equation with Neumann boundary conditions is also stable only for odd positive stationary solutions.*

Proof: According to Suo et al. [1] and Cai et al. [9], the solution is stable if the Fisher-KPP equation with Dirichlet conditions or the Fisher-KPP equation with Robin boundary conditions is stable only for odd positive stationary solutions. Hence, a positive periodic solution is close to the odd positive stationary solutions in our model, since any two solutions are arbitrarily close to each other. By definition of stable, we have

$$limsup_{t\to\infty}|V(t,x) - V_i(x)| = 0.$$

We also have

$$limsup_{t\to\infty}|u(t,x) - V(t,x)| = 0.$$

and

$$limsup_{t\to\infty}|u(x,t;\phi_A(x)) - V_i(x)| = 0.$$

for i is an odd integer. Then by triangle inequality, we have

$$\begin{aligned}
&limsup_{t\to\infty}|u(t,x) - u(x,t;\phi_A(x))| \\
&\leq limsup_{t\to\infty}|V(t,x) - V_i(x) + u(t,x) - V(t,x) + V_i(x) - u(x,t;\phi_A(x))| \\
&\leq limsup_{t\to\infty}|V(t,x) - V_i(x)| + limsup_{t\to\infty}|u(t,x) - V(t,x)| + limsup_{t\to\infty}|u(x,t;\phi_A(x)) - V_i(x)| \\
&= 0.
\end{aligned}$$

Since

$$0 \leq limsup_{t\to\infty}|u(t,x) - u(x,t;\phi_A(x))| \leq 0,$$

$$limsup_{t\to\infty}|u(t,x) - u(x,t;\phi_A(x))| = 0$$

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Since $u(t,x)$ is the entire solution of the Fisher-KPP equation with the Dirichlet condition and $u(x,t;\phi_A(x))$ is the solution of the Fisher-KPP equation with the Robin condition. The Dirichlet boundary condition is [9]

$$u(t,0) = 0$$

$$u(t,x) = 0,$$

consider the steady-state solution (stationary solution) of the Fisher-KPP equation, $u = 0$ or $u = 1$. We can consider this case because

$$\begin{aligned}
&2limsup_{t\to\infty}|u(t,x) - V_i(x)| \\
&\le limsup_{t\to\infty}|V_i(x) - V(t,x) + V(t,x) - u(t,x) + V_i(x) - u(x,t;\phi_A(x)) + u(x,t;\phi_A(x)) - u(t,x)| \\
&\le limsup_{t\to\infty}|V(t,x) - V_i(x)| + limsup_{t\to\infty}|u(t,x) - \\
&V(t,x)| + limsup_{t\to\infty}|u(x,t;\phi_A(x)) - V_i(x)| + limsup_{t\to\infty}|u(t,x) - u(x,t;\phi_A(x))| \\
&= 0.
\end{aligned}$$

Hence

$$limsup_{t\to\infty}|u(t,x) - V_i(x)| = 0.$$

Similarly, Robin's condition is $u_x(0,t) = g((u(0,t))$ [7] where $g(u) > 0$ for $u > 0$ and $g(0) = 0$ [1]. In the latter case, we have $u(0,t) = 0$ and are getting back to the Dirichlet condition. So we only consider the case where $u > 0$. Hence $u(x,t;\phi_A(x)) = 1$. Since 0 to 1 is the greatest distance between two solutions, and the distance between the Fisher-KPP equation with Robin condition and with Dirichlet condition is arbitrarily small. Hence, the solution of the Fisher-KPP equation with the Neumann condition is between them, hence close to them. Let $U(t,x)$ be the solution of the Fisher-KPP equation with the Neumann boundary conditions. We have,

$$limsup_{t\to\infty}|U(t,x) - u(x,t;\phi_A(x))| = 0.$$

By the triangle inequality, for i an odd integer,

$$\begin{aligned}
&limsup_{t\to\infty}|U(t,x) - V_i(x)| \\
&\le limsup_{t\to\infty}|U(t,x) - u(x,t;\phi_A(x)) + u(x,t;\phi_A(x)) - V_i(x)| \\
&\le limsup_{t\to\infty}|U(t,x) - u(x,t;\phi_A(x))| + limsup_{t\to\infty}|u(x,t;\phi_A(x)) - V_i(x)| \\
&= 0.
\end{aligned}$$

Hence

$$limsup_{t\to\infty}|U(t,x) - V_i(x)| = 0.$$

For the same reason as above, Theorem 2.5 is proved. ■

Hence, we proved the stability of the Fisher-KPP equation with Neumann, Dirichlet, and Robin conditions.

2.2. Existence

By Section 2.1, we showed that we can only consider the steady state solution. Hence, for the purpose of this study, the Fisher-Kpp equation becomes

$$u_t = u_{xx}$$

with Neumann boundary condition

$$u_x(0,t) = 0 = u_x(1,t)$$

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We solve it by the separation of variables. Let $u(x,t) = X(x)T(t)$ and take into the equation we get

$$X(x)T'(t) = X''(x)T(t)$$

Then

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Then we have

$$T'(t) = -\lambda T(t)$$

and

$$X''(x) = -\lambda X(x)$$

Then

$$X''(x) + \lambda X(x) = 0$$

$X(x) =$

$$\begin{cases} c_1 e^{xr_1} + c_2 e^{xr_2}, & r_1, r_2 \in \mathbb{R}, r_1 \neq r_2, \\ c_1 e^{xr} + c_2 x e^{xr}, & r \in \mathbb{R} \\ e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x), & r_1 = \alpha + i\beta, r_2 = \alpha - i\beta. \end{cases}$$

Let $X(x) = e^{rx}$. Then $r^2 + \lambda = 0$.
Case 1:$\lambda > 0 : r = \pm i\sqrt{\lambda} = \alpha \pm i\beta$.
We have

$$X(x) = c_1 cos\sqrt{\lambda}x + c_2 sin\sqrt{\lambda}x$$

According to our boundary conditions, we have

$$0 = u_x(0,t) = X'(0)T(t) = X'(1)T(t) = u_x(1,t)$$

Hence

$$X'(0) = X'(1) = 0$$

Since

$$X'(x) = -\sqrt{\lambda}c_1 sin\sqrt{\lambda}x + c_2\sqrt{\lambda}cos\sqrt{\lambda}x,$$

take 0 and 1 in it, we get

$$X'(0) = c_2\sqrt{\lambda} = 0$$

Hence $c_2 = 0$.

$$X'(1) = -\sqrt{\lambda}c_1 sin\sqrt{\lambda} = 0$$

The trivial solution is $c_1 = 0$. We only consider the nontrivial solution

$$sin\sqrt{\lambda} = 0$$

. We have

$$\sqrt{\lambda} = n\pi$$

hence

$$\lambda = n^2\pi^2$$

Hence

$$X(x) = X_n(x) = c_1 cos n\pi x$$

by taking, $c_1 = 1$ we get

$$X_n(x) = c_1 cos n\pi x$$

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Case 2:$\lambda = 0$.Then

$$X(x) = c_1 e^{rx} + c_2 x e^{rx}$$

$$X'(x) = rc_1 e^{rx} + rc_2 x e^{rx} + c_2 e^{rx}$$

$$X'(0) = rc_1 + c_2 = 0$$

and

$$X'(1) = rc_1 e^{r} + rc_2 e^{r} + c_2 e^{r} = 0$$

Then we have

$$r(c_1 + c_2) = -c_2$$

and

$$rc_1 = -c_2$$

We have

$$c_1 + c_2 = c_1 \rightarrow c_2 = 0$$

Hence

$$c_1 = c_2 = 0,$$

no eigenfunctions.
Case 3:$\lambda < 0$

$$X(x) = c_1 e^{xr_1} + c_2 e^{xr_2}$$

$$X'(x) = r_1 c_1 e^{xr_1} + r_2 c_2 e^{xr_2}$$

$$X'(0) = r_1 c_1 + r_2 c_2 = 0$$

and

$$X'(1) = r_1 c_1 e^{r_1} + r_2 c_2 e^{r_2} = 0$$

$$r_1 c_1 = -r_2 c_2$$

hence

$$-r_2 c_2 e^{r_1} + r_2 c_2 e^{r_2} = 0 \rightarrow r_2 c_2 (e^{r_2} - e^{r_1}) = 0$$

Since $e^{r_2} - e^{r_1} \neq 0, c_2 = 0$. Hence $c_1 = 0$ No eigenfunctions.
We also have

$$T'(t) + \lambda T(t) = 0$$

Then

$$\frac{dT}{dt} = -\lambda T$$

Then

$$\int \frac{1}{T} dT = -\int \lambda dt \rightarrow lnT = -\lambda t + c_3$$

hence

$$T_n(t) = c_3 e^{-\lambda t} = c_3 e^{-n^2 \pi^2 t}$$

Hence

$$u_n(x,t) = X_n T_n = cos(n\pi x) e^{-n^2 \pi^2 t}$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} cos(n\pi x) e^{-n^2 \pi^2 t}$$

Since we are able to solve $u(x,t)$, the solution exists. ■

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Similarly, by Section 2.1, we proved that we can consider the steady state solution in our paper. Hence, our equation becomes

$$u_t = u_{xx}$$

with Dirichlet boundary condition

$$u(0,t) = 0 = u(1,t)$$

We solve it by the separation of variables. Let $u(x,t) = X(x)T(t)$ and take into the equation we get

$$X(x)T'(t) = X''(x)T(t)$$

Then

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Then we have

$$T'(t) = -\lambda T(t)$$

and

$$X''(x) = -\lambda X(x)$$

Then

$$X''(x) + \lambda X(x) = 0$$

$X(x) =$

$$\begin{cases} c_1 e^{xr_1} + c_2 e^{xr_2}, & r_1, r_2 \in \mathbb{R}, r_1 \neq r_2, \\ c_1 e^{xr} + c_2 x e^{xr}, & r \in \mathbb{R} \\ e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x), & r_1 = \alpha + i\beta, r_2 = \alpha - i\beta. \end{cases}$$

Let $X(x) = e^{rx}$. Then $r^2 + \lambda = 0$.
Case 1:$\lambda > 0 : r = \pm i\sqrt{\lambda} = \alpha \pm i\beta$.
We have

$$X(x) = c_1 cos\sqrt{\lambda}x + c_2 sin\sqrt{\lambda}x$$

According to our boundary conditions, we have

$$0 = u(0,t) = X(0)T(t) = X(1)T(t) = u(1,t)$$

Hence

$$X(0) = X(1) = 0$$

$$X(0) = c_1 cos0 = 0$$

Hence $c_1 = 0$.

$$X(1) = c_2 sin\sqrt{\lambda} = 0$$

The trivial solution is $c_2 = 0$. We only consider the nontrivial solution

$$sin\sqrt{\lambda} = 0.$$

We have

$$\sqrt{\lambda} = n\pi$$

hence

$$\lambda = n^2\pi^2$$

Hence

$$X(x) = X_n(x) = c_2 sin(n\pi x)$$

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by taking, $c_1 = 1$ we get
$$X_n(x) = c_2 sin(n\pi x)$$
Case 2: $\lambda = 0$.Then
$$X(x) = c_1 e^{rx} + c_2 x e^{rx}$$
$$X(0) = c_1 = 0$$
and
$$X(1) = c_2 e^{r} = 0$$
Hence
$$c_1 = c_2 = 0,$$
no eigenfunctions.
Case 3: $\lambda < 0$
$$X(x) = c_1 e^{xr_1} + c_2 e^{xr_2}$$
$$X(0) = c_1 + c_2 = 0$$
and
$$X(1) = c_1 e^{r_1} + c_2 e^{r_2}$$
Since
$$c_1 = -c_2,$$
$$c_1(e^{r_1} - e^{r_2}) = 0$$
Since $e^{r_1} - e^{r_2} \neq 0, c_1 = 0$. Hence $c_2 = 0$ No eigenfunctions.
We also have
$$T'(t) + \lambda T(t) = 0$$
Then
$$\frac{dT}{dt} = -\lambda T$$
Then
$$\int \frac{1}{T} dT = -\int \lambda dt \rightarrow lnT = -\lambda t + c_3$$
hence
$$T_n(t) = c_3 e^{-\lambda t} = c_3 e^{-n^2\pi^2 t}$$
Hence
$$u_n(x,t) = X_n T_n = sin(n\pi x) e^{-n^2\pi^2 t}$$
Hence
$$u(x,t) = \sum_{n=1}^{\infty} sin(n\pi x) e^{-n^2\pi^2 t}$$
Since we are able to solve $u(x,t)$, the solution exists. ■

Similarly, by Section 2.1, we proved that we can consider the steady state solution in our paper. Hence, our equation becomes
$$u_t = u_{xx}$$
with Robin boundary condition
$$u(0,t) = 0 = u_x(1,t)$$
and
$$u_x(0,t) = 0 = u(1,t)$$

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Now we only prove with the boundary condition

$$u(0,t) = 0 = u_x(1,t)$$

because the other one is similar. We solve it by the separation of variables. Let $u(x,t) = X(x)T(t)$ and take into the equation we get

$$X(x)T'(t) = X''(x)T(t)$$

Then

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Then we have

$$T'(t) = -\lambda T(t)$$

and

$$X''(x) = -\lambda X(x)$$

Then

$$X''(x) + \lambda X(x) = 0$$

$X(x) =$

$$\begin{cases} c_1 e^{xr_1} + c_2 e^{xr_2}, & r_1, r_2 \in \mathbb{R}, r_1 \neq r_2, \\ c_1 e^{xr} + c_2 x e^{xr}, & r \in \mathbb{R} \\ e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x), & r_1 = \alpha + i\beta, r_2 = \alpha - i\beta. \end{cases}$$

Let $X(x) = e^{rx}$. Then $r^2 + \lambda = 0$.
Case 1:$\lambda > 0 : r = \pm i\sqrt{\lambda} = \alpha \pm i\beta$.
We have

$$X(x) = c_1 cos\sqrt{\lambda}x + c_2 sin\sqrt{\lambda}x$$

According to our boundary conditions, we have

$$0 = u(0,t) = X(0)T(t) = X'(1)T(t) = u_x(1,t)$$

Hence

$$X(0) = X'(1) = 0$$

$$X(0) = c_1 cos0 = 0$$

Hence $c_1 = 0$. Since

$$X'(x) = -\sqrt{\lambda}c_1 sin\sqrt{\lambda}x + c_2\sqrt{\lambda}cos\sqrt{\lambda}x,$$

$$X'(1) = c_2\sqrt{\lambda}cos\sqrt{\lambda} = 0$$

The trivial solution is $c_2 = 0$. We only consider the nontrivial solution

$$cos\sqrt{\lambda} = 0$$

. We have

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2}$$

hence

$$\lambda = (\frac{(2n-1)\pi}{2})^2$$

Hence

$$X(x) = X_n(x) = c_2 sin(\frac{(2n-1)\pi}{2}x)$$

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by taking, $c_1 = 1$ we get

$$X_n(x) = c_2 sin(\frac{(2n-1)\pi}{2}x)$$

Case 2: $\lambda = 0$.Then

$$X(x) = c_1 e^{rx} + c_2 x e^{rx}$$

$$X(0) = c_1 = 0$$

and

$$X'(x) = rc_2 x e^{rx} + c_2 e^{rx}$$

Then

$$X'(1) = rc_2 e^r + c_2 e^r = c_2(re^r + e^r) = 0$$

Hence

$$c_1 = c_2 = 0,$$

no eigenfunctions.
Case 3: $\lambda < 0$

$$X(x) = c_1 e^{xr_1} + c_2 e^{xr_2}$$

$$X(0) = c_1 + c_2 = 0$$

and

$$X'(x) = r_1 c_1 e^{xr_1} + r_2 c_2 e^{xr_2}$$

Then

$$X'(1) = r_1 c_1 e^{r_1} + r_2 c_2 e^{r_2}$$

Since

$$c_1 = -c_2,$$

$$c_1(r_1 e^{r_1} - r_2 e^{r_2}) = 0$$

Since $r_1 e^{r_1} - r_2 e^{r_2} \neq 0, c_1 = 0$. Hence $c_2 = 0$ No eigenfunctions.
We also have

$$T'(t) + \lambda T(t) = 0$$

Then

$$\frac{dT}{dt} = -\lambda T$$

Then

$$\int \frac{1}{T} dT = -\int \lambda dt \rightarrow lnT = -\lambda t + c_3$$

hence

$$T_n(t) = c_3 e^{-\lambda t} = c_3 e^{-n^2\pi^2 t}$$

Hence

$$u_n(x,t) = X_n T_n = sin(\frac{(2n-1)\pi}{2}x)e^{-n^2\pi^2 t}$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} sin(n\pi x)e^{-n^2\pi^2 t}$$

Since we are able to solve $u(x,t)$, the solution exists. ■

Hence, we proved the existence of solutions of the Fisher-KPP equation with Neumann, Dirichlet, and Robin conditions.

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2.3. Uniqueness

Since we proved in section 2.1 that we can consider the steady state solution, our equation becomes:

$$u_t = u_{xx}$$

with Neumann boundary condition

$$u_x(0,t) = 0 = u_x(1,t)$$

According to Strauss [7], assume there are u_1, u_2 that are two distinct solutions of the Fisher-KPP equation with Neumann conditions. Let $w = u_1 - u_2$.Then

$$w_t - w_{xx} = (u_1 - u_2)_t - [(u_1) - (u_2)]_{xx}$$

$$= (u_1)_t - (u_2)_t - [(u_1)_{xx} - (u_2)_{xx}] = (u_1)_t - (u_1)_{xx} - [(u_2)_t - (u_2)_{xx}] = 0 - 0 = 0$$

and

$$w_x(0,t) = (u_1)_x(0,t) - (u_2)_x(0,t) = 0 - 0 = 0$$

$$w_x(1,t) = (u_1)_x(1,t) - (u_2)_x(1,t) = 0 - 0 = 0$$

Hence $w_x = 0$ on the boundary $R = \{(x,t) | x \in [0,1], t > 0\}$ Hence, w does not depend on x. Then $w_{xx} = (u_1 - u_2)_{xx} = 0$. Then $(u_1)_{xx} = (u_2)_{xx}$. Since there are no constant term in the Fisher-KPP equation, we can integrate both side and get $u_1 = u_2$, which also contradicts our assumption. Hence $w = 0$ on the boundary of R. By Theorem 2.1, $w \leq 0$ on the boundary R and by the minimal principal, $w \geq 0$. on teh boundary R. Hence $w = 0$ on R. Then we get $u_1 = u_2$, the uniqueness as desired. ■

Since we proved in section 2.1 that we can consider the steady state solution, our equation becomes:

$$u_t = u_{xx}$$

with Dirichlet boundary condition

$$u(0,t) = 0 = u(1,t)$$

According to Strauss [7], assume there are u_1, u_2 that are two distinct solutions of the Fisher-KPP equation with Neumann conditions. Let $w = u_1 - u_2$.Then

$$w_t - w_{xx} = (u_1 - u_2)_t - [(u_1) - (u_2)]_{xx}$$

$$= (u_1)_t - (u_2)_t - [(u_1)_{xx} - (u_2)_{xx}] = (u_1)_t - (u_1)_{xx} - [(u_2)_t - (u_2)_{xx}] = 0 - 0 = 0$$

and

$$w(0,t) = u_1(0,t) - u_2(0,t) = 0 - 0 = 0$$

$$w(1,t) = u_1(1,t) - u_2(1,t) = 0 - 0 = 0$$

Hence $w = 0$ on the boundary $R = \{(x,t) | x \in [0,1], t > 0\}$ By Theorem 2.1, $w \leq 0$ on the boundary R and by the minimal principal, $w \geq 0$. on R. Hence $w = 0$ on R. Then we get $u_1 = u_2$, the uniqueness as desired. ■

Since we proved in section 3.1 that we can consider the steady state solution, our equation becomes:

$$u_t = u_{xx}$$

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with Robin boundary condition

$$u(0,t) = 0 = u_x(1,t)$$

or

$$u_x(0,t) = 0 = u(1,t)$$

According to Strauss [7], assume there are u_1, u_2 that are two distinct solutions of the Fisher-KPP equation with Neumann conditions. Let $w = u_1 - u_2$.Then

$$w_t - w_{xx} = (u_1 - u_2)_t - [(u_1) - (u_2)]_{xx}$$

$$= (u_1)_t - (u_2)_t - [(u_1)_{xx} - (u_2)_{xx}] = (u_1)_t - (u_1)_{xx} - [(u_2)_t - (u_2)_{xx}] = 0 - 0 = 0$$

and

$$w(0,t) = u_1(0,t) - u_2(0,t) = 0 - 0 = 0$$

$$w_x(1,t) = (u_1)_x(1,t) - (u_2)_x(1,t) = 0 - 0 = 0$$

and

$$w_x(0,t) = (u_1)_x(0,t) - (u_2)_x(0,t) = 0 - 0 = 0$$

$$w(1,t) = u_1(1,t) - u_2(1,t) = 0 - 0 = 0$$

Hence $w_x = 0$ and $w = 0$ on the boundary $R = \{(x,t)|x \in [0,1], t > 0\}$ By Theorem 2.1, $w \leq 0$ on the boundary R and by the minimal principal, $w \geq 0$. on R. Hence $w = 0$ on R. Then we get $u_1 = u_2$, the uniqueness as desired. ■

3. Conclusions

The main results hold for a general $f(u)$ of the Fisher-KPP equation. The solution of the Fisher-KPP equation with Neumann, Robin and Dirichelet conditions all can approximate to steady state condition in our process. Because if $u_t = 0$,

$$u_{xx} + f(u) = 0.$$

then $f(u) = 0$. By properties of $f(u)$, $u = 0, 1$ since $f(0) = f(1) = 0$ and $f(u) > 0$ for $u \in (0,1)$. Then they would form the greatest distance to cover the solution of the Fisher-KPP equation with the Neumann condition. Also, for the proof of existence and uniqueness, we can also consider a steady state solution, then ignore the inhomogeneous term, and write a similar proof. Hence, this paper proved the general case of the Fisher-KPP equation with the Neumann, Dirichlet, and Robin conditions.

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