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Some Properties of Generalized Triangular Fuzzy Numbers and Generalized Trapezoidal Fuzzy Numbers in Terms of Cut Sets

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Abstract

In this paper, we give several properties of the generalized triangular fuzzy numbers and the generalized trapezoidal fuzzy numbers in terms of the cut sets. First, we present a characterization of all cut sets of a fuzzy set in $\mathbb{R}$ when this fuzzy set in $\mathbb{R}$ is a given generalized trapezoidal fuzzy number. Then we show that two generalized trapezoidal fuzzy numbers are equal if and only if there exist two distinct elements λ and τ of $[0,1]$ such that the λ-cuts of these two generalized trapezoidal fuzzy numbers are equal and the τ-cuts of these two generalized trapezoidal fuzzy numbers are equal. As corollaries, we give corresponding conclusions of the above conclusions for the triangular fuzzy numbers, the trapezoidal fuzzy numbers, and the generalized triangular fuzzy numbers, respectively.

Keywords: triangular fuzzy numbers; trapezoidal fuzzy numbers; cut sets generalized triangular fuzzy numbers; generalized trapezoidal fuzzy numbers

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1. Introduction

Let $\mathbb{N}$ be the set of all positive integers and let $\mathbb{R}^m$ be the m-dimensional Euclidean space. $\mathbb{R}^1$ is also written as $\mathbb{R}$.

Triangular fuzzy numbers and trapezoidal fuzzy numbers are often used fuzzy sets in theoretical research and practical applications [1,2].

In [3], we introduced the generalized triangular fuzzy numbers and the generalized trapezoidal fuzzy numbers, which are generalizations of triangular fuzzy numbers and the trapezoidal fuzzy numbers, respectively.

The symbols **Tag**, **Tap**, **Trag** and **Trap** are used to denote the set of triangular fuzzy numbers, the set of trapezoidal fuzzy numbers, the set of generalized triangular fuzzy numbers, and the set of generalized trapezoidal fuzzy numbers, respectively.

It is well known that the cut sets play an important role in the research and applications of fuzzy sets. So it is natural and important to consider properties of the generalized triangular fuzzy numbers and the generalized trapezoidal fuzzy numbers in terms of the cut sets. In this paper, we conduct some such investigations.

First, we give a characterization of all cut sets of a fuzzy set in $\mathbb{R}$ when this fuzzy set in $\mathbb{R}$ is a given generalized trapezoidal fuzzy number. Then we give an equivalent condition for the equality of two generalized trapezoidal fuzzy numbers, which says that two generalized trapezoidal fuzzy numbers are equal if and only if there exist two distinct elements λ and τ of $[0,1]$ such that the λ-cuts of these two generalized trapezoidal fuzzy numbers are equal and the τ-cuts of these two generalized trapezoidal fuzzy numbers are equal. As corollaries, we give corresponding conclusions of the above conclusions for Tag, Tap and Trag, respectively.

The properties in terms of the cut sets given in this paper are convenient to use. The results of this paper provide convenient tools for the analyses and discussions of the generalized triangular fuzzy numbers and the generalized trapezoidal fuzzy numbers.

The remainder of this paper is organized as follows. Section 2 reviews some basic concepts related to the fuzzy sets, the triangular fuzzy numbers and the trapezoidal fuzzy numbers, and some basic properties of the latter two. In Section 3, we recall and discuss the concepts and properties of the generalized triangular fuzzy numbers and the generalized trapezoidal fuzzy numbers. Section 4 gives several properties of Tag, Tap, Trag and Trap, respectively, in terms of the cut sets. At last, we draw our conclusions in Section 5.

2. Fuzzy Sets, Triangular Fuzzy Numbers and Trapezoidal Fuzzy Numbers

In this section, we review some basic concepts related to the fuzzy sets, the triangular fuzzy numbers and the trapezoidal fuzzy numbers, and some basic properties of the latter two. For fuzzy theory and applications, we refer the readers to [1,2,4–14].

Let Y be a nonempty set. The symbol $P(Y)$ denotes the power set of Y, which is the set of all subsets of Y. The symbol $F(Y)$ denotes the set of all fuzzy sets in Y, i.e., functions from Y to $[0,1]$. Given $u \in F(Y)$ and $\alpha \in (0,1]$, the α-cut $[u]_\alpha$ of u is defined by $[u]_\alpha := \{x \in Y : u(x) \geq \alpha\}$.

Let Y be a topological space. The symbol $C(Y)$ denotes the set of all nonempty closed subsets of Y. $K(Y)$ denotes the set of all nonempty compact subsets of Y. For $u \in F(Y)$, the 0-cut $[u]_0$ of u is defined by $[u]_0 := \overline{\{x \in Y : u(x) > 0\}}$, where $\overline{S}$ denotes the topological closure of S in Y. $[u]_0$ is called the support of u, and is also denoted by $\operatorname{supp} u$.

Some properties of distances on fuzzy sets are discussed in [15–18].

Usually, the symbols (a,b,c,d) with a,b,c,d in $\mathbb{R}$ represent the elements in $\mathbb{R}^4$ and the symbols (a,b,c) with a,b,c in $\mathbb{R}$ represent the elements in $\mathbb{R}^3$. In this paper, for each a,b,c,d in $\mathbb{R}$, we use $[a,b,c,d]$ instead of (a,b,c,d) to represent the corresponding element in $\mathbb{R}^4$, and use $[a,b,c]$ instead of (a,b,c) to represent the corresponding element in $\mathbb{R}^3$.

We use T to denote the set $\{[a,b,c,d] \in \mathbb{R}^4 : a \leq b \leq c \leq d\}$ and T_0 to denote the set $\{[a,b,c,d] \in \mathbb{R}^4 : a < b \leq c < d\}$. Clearly $T_0 \subsetneqq T$.

We use G to denote the set $\{[a,b,c] \in \mathbb{R}^3 : a \leq b \leq c\}$ and G_0 to denote the set $\{[a,b,c] \in \mathbb{R}^3 : a < b < c\}$. Clearly $G_0 \subsetneqq G$.

Definition 1. *We use* ***Tag*** *to denote the set of all triangular fuzzy numbers. Tag* $:= \{(a,b,c) : [a,b,c] \text{ in } G_0\}$, *where, for any* $[a,b,c]$ *in* G_0, *the triangular fuzzy number* (a,b,c) *is defined to be the fuzzy set u in* $F(\mathbb{R})$ *given by*

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, \text{ if } a \leq x \leq b, \\ \dfrac{c-x}{c-b}, \text{ if } b \leq x \leq c, \\ 0, \text{ if } x \in \mathbb{R} \setminus [a,c]. \end{cases}$$

Definition 2. *We use* ***Tap*** *to denote the set of all trapezoidal fuzzy numbers. Tap* $:= \{(a,b,c,d) : [a,b,c,d] \text{ in } T_0\}$, *where, for any* $[a,b,c,d]$ *in* T_0, *the trapezoidal fuzzy number* (a,b,c,d) *is defined to be the fuzzy set u in* $F(\mathbb{R})$ *given by*

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, \text{ if } a \leq x \leq b, \\ 1, \text{ if } b \leq x \leq c, \\ \dfrac{d-x}{d-c}, \text{ if } c \leq x \leq d, \\ 0, \text{ if } x \in \mathbb{R} \setminus [a,d]. \end{cases}$$

Remark 1. *(i)* $u \in$ *Tag means that there is an* $[a,b,c] \in G_0$ *satisfying* $u = (a,b,c)$. *(ii)* $u \in$ *Tap means that there is an* $[a,b,c,d] \in T_0$ *satisfying* $u = (a,b,c,d)$. *The contents of this remark have already been given in [3].*

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We say that two fuzzy sets are *equal* if they have the same membership function.

3. Generalized Triangular Fuzzy Numbers and Generalized Trapezoidal Fuzzy Numbers

In this section, we recall and discuss the concepts and properties of the generalized triangular fuzzy numbers and the generalized trapezoidal fuzzy numbers introduced in [3]. The readers may also refer to the corresponding contents in [3] for details.

Definition 3. *We use **Trag** to denote the set of all generalized triangular fuzzy numbers.* $Trag := \{(a,b,c) : [a,b,c] \text{ in } G\}$, *where, for any* $[a,b,c]$ *in* G, *the generalized triangular fuzzy number* (a,b,c) *is defined to be the fuzzy set* u *in* $F(\mathbb{R})$ *in the following way:*

$$u \text{ is the triangular fuzzy number } (a,b,c) \text{ when } a<b<c;$$

$$u(x) = \begin{cases} \dfrac{c-x}{c-b}, \text{ if } b \le x \le c, \\ 0, \text{ if } x \in \mathbb{R} \setminus [b,c], \end{cases} \quad \text{when } a=b<c;$$

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, \text{ if } a \le x \le b, \\ 0, \text{ if } x \in \mathbb{R} \setminus [a,b], \end{cases} \quad \text{when } a<b=c;$$

$$u(x) = \begin{cases} 1, \text{ if } x = b, \\ 0, \text{ if } x \in \mathbb{R} \setminus \{b\}, \end{cases} \quad \text{when } a=b=c.$$

Clearly each (a,b,c) in Tag is the (a,b,c) in Trag. This means that the concept of generalized triangular fuzzy numbers is a kind of generalization of the concept of triangular fuzzy numbers. Hence Tag $\subseteq$ Trag.

Definition 4. *We use **Trap** to denote the set of all generalized trapezoidal fuzzy numbers.* $Trap := \{(a,b,c,d) : [a,b,c,d] \text{ in } T\}$, *where, for any* $[a,b,c,d]$ *in* T, *the generalized trapezoidal fuzzy number* (a,b,c,d) *is defined to be the fuzzy set* u *in* $F(\mathbb{R})$ *in the following way:*

$$u \text{ is the trapezoidal fuzzy number } (a,b,c,d) \text{ when } a<b\le c<d;$$

$$u(x) = \begin{cases} 1, \text{ if } b \le x \le c, \\ \dfrac{d-x}{d-c}, \text{ if } c \le x \le d, \\ 0, \text{ if } x \in \mathbb{R} \setminus [b,d], \end{cases} \quad \text{when } a=b\le c<d;$$

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, \text{ if } a \le x \le b, \\ 1, \text{ if } b \le x \le c, \\ 0, \text{ if } x \in \mathbb{R} \setminus [a,c], \end{cases} \quad \text{when } a<b\le c=d;$$

$$u(x) = \begin{cases} 1, \text{ if } b \le x \le c, \\ 0, \text{ if } x \in \mathbb{R} \setminus [b,c], \end{cases} \quad \text{when } a=b\le c=d.$$

Clearly each (a,b,c,d) in Tap is the (a,b,c,d) in Trap. This means that the concept of generalized trapezoidal fuzzy numbers is a kind of generalization of the concept of trapezoidal fuzzy numbers. Hence Tap $\subseteq$ Trap.

Remark 2. *(i)* $u \in Trag$ *means that there is an* $[a,b,c] \in G$ *satisfying* $u = (a,b,c)$. *(ii)* $u \in Trap$ *means that there is an* $[a,b,c,d] \in T$ *satisfying* $u = (a,b,c,d)$.

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Remark 3. *Each generalized triangular fuzzy number (a,b,c) is the generalized trapezoidal fuzzy number (a,b,b,c).*

The contents in Remarks 2 and 3 have already been given in [3].

The following Propositions 1 and 2 in [3] give some relationships between Trag and Trap, and between Tag and Tap, respectively.

Let A be a set. A mapping $f : A \to A$ is said to be the identity mapping on A if $f(x) = x$ for each $x \in A$. A mapping g is said to be an identity mapping if there is a set S and g is the identity mapping on S.

Define $\text{Trap}^1 := \{(a,b,c,d) : (a,b,c,d) \in Trap \text{ and } b = c\}$. Clearly $\text{Trap}^1 \subsetneqq \text{Trap}$ and $\text{Trap}^1 = \{(a,b,b,c) : (a,b,b,c) \in Trap\}$.

Proposition 1 ([3]). *(i) Trag = Trap^1. (ii) Define a mapping $K : Trag \to Trap^1$ as follows: for each $u \in Trag$, find an $[a,b,c] \in G$ satisfying $u = (a,b,c)$, and then define $K(u)$ to be (a,b,b,c). Then K is the identity mapping on Trag.*

We have that Trap^1= Trag $= \{(a,b,c) : [a,b,c] \in G\} = \{(a,b,b,c) : [a,b,c] \in G\} = \{(a,b,b,c) : [a,b,b,c] \in T\}$.

Clearly Remark 3 is an easy corollary of Proposition 1(ii).

Define $\text{Tap}^1 := \{(a,b,c,d) : (a,b,c,d) \in Tap \text{ and } b = c\}$. Clearly $\text{Tap}^1 \subsetneqq \text{Tap}$ and $\text{Tap}^1 = \{(a,b,b,c) : (a,b,b,c) \in Tap\}$.

Proposition 2 ([3]). *(i) Tag = Tap^1. (ii) Define a mapping $L : Tag \to Tap^1$ as follows: for each $u \in Tag$, find an $[a,b,c] \in G_0$ satisfying $u = (a,b,c)$, and then define $L(u)$ to be (a,b,b,c). Then L is the identity mapping on Tag.*

We have that Tap^1= Tag $= \{(a,b,c) : [a,b,c] \in G_0\} = \{(a,b,b,c) : [a,b,c] \in G_0\} = \{(a,b,b,c) : [a,b,b,c] \in T_0\}$.

For any $[a,b,c,d]$ and $[a_1,b_1,c_1,d_1]$ in $\mathbb{R}^4$, $[a,b,c,d] = [a_1,b_1,c_1,d_1]$ means that $a = a_1$, $b = b_1$, $c = c_1$ and $d = d_1$. For any $[a,b,c]$ and $[a_1,b_1,c_1]$ in $\mathbb{R}^3$, $[a,b,c] = [a_1,b_1,c_1]$ means that $a = a_1$, $b = b_1$ and $c = c_1$.

The following Theorem 1(ii) states the representation uniqueness of Trap.

Theorem 1 (Theorem 4.1(i)(ii) in [3]). *(i) Let $u = (a,b,c,d)$ be in Trap. Then $[u]_0 = [a,d]$ and $[u]_1 = [b,c]$. (ii) Let (a,b,c,d) and (a_1,b_1,c_1,d_1) be in Trap. Then $(a,b,c,d) = (a_1,b_1,c_1,d_1)$ if and only if $[a,b,c,d] = [a_1,b_1,c_1,d_1]$.*

In [3], we obtain the following relationship among Tag, Trag, Tap and Trap: Tag $\subsetneqq$ Trag, Tap $\subsetneqq$ Trap, Tag $\subsetneqq$ Tap, and Trag $\subsetneqq$ Trap.

4. Main Results

In this section, we give several properties of the generalized trapezoidal fuzzy numbers in terms of the cut sets. Also, as corollaries, we give corresponding conclusions of these properties for Tag, Tap and Trag, respectively.

We begin with the following representation theorem, which should be a known conclusion appearing earlier than [18]. A proof for this theorem is given in [18]. See Theorem 3.1 in [18] and its proof. In this paper we assume that $\mathbf{sup}\,\emptyset = \mathbf{0}$.

Theorem 2. *Let Y be a nonempty set. If $u \in F(Y)$, then for all $\alpha \in (0,1]$, $[u]_\alpha = \cap_{\beta<\alpha}[u]_\beta$.*

Conversely, suppose that $\{v_\alpha : \alpha \in (0,1]\}$ is a family of sets in Y with $v_\alpha = \cap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0,1]$. Define $u \in F(Y)$ by

$$u(x) := \sup\{\alpha \in (0,1] : x \in v_\alpha\}$$

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for each $x \in Y$. Then u is the unique element of $F(Y)$ which satisfies $[u]_\alpha = v_\alpha$ for all $\alpha \in (0,1]$; that is, u is the unique element of the set $\{w \in F(Y) : [w]_\alpha = v_\alpha$ for all $\alpha \in (0,1]\}$.

Remark 4. *Let Y be a nonempty set and let $\{v_\alpha : \alpha \in (0,1]\}$ be a family of sets in Y. Denote $S := \{w \in F(Y) : [w]_\alpha = v_\alpha$ for all $\alpha \in (0,1]\}$.*
(i) Suppose that the statement "$v_\alpha = \cap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0,1]$" does not hold. Then $S = \emptyset$.
(ii) Suppose that $v_\alpha = \cap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0,1]$. Then S is a singleton set.
(iii) S is an empty set or a singleton set.
(iv) Let $v \in F(Y)$. Then the set $S(v) := \{w \in F(Y) : [w]_\alpha = [v]_\alpha$ for all $\alpha \in (0,1]\}$ is a singleton set.
(v) Let $v \in F(Y)$. Then $S(v) = \{v\}$.

We show (i). By Theorem 2, for each $w \in F(Y)$, it holds that $[w]_\alpha = \cap_{\beta<\alpha}[w]_\beta$ for all $\alpha \in (0,1]$. Thus $S = \emptyset$. So (i) is proved.

By Theorem 2, (ii) holds. (iii) follows immediately from (i) and (ii).

We show (iv). As $v \in F(Y)$, by Theorem 2, for all $\alpha \in (0,1]$, $[v]_\alpha = \cap_{\beta<\alpha}[v]_\beta$. Then, by (ii), $S(v)$ is a singleton set. So (iv) is proved.

It is easy to see (a) for each $v \in F(Y)$, $v \in S(v)$; and (b) for each $v \in F(Y)$, $S(v) \neq \emptyset$. ((a)⇒(b).) Clearly (a) and (iv) hold if and only if (v) holds. So (v) holds.

Combining (b) and (iii) yields (iv).

The following Proposition 3 should be known. We cannot find the original reference which gave this conclusion, so we give a proof here for the self containing of this paper.

Let Y be a nonempty set and $u \in F(Y)$. Then $[u]_0$ is well-defined if and only if Y is a topological space.

Proposition 3. *Let Y be a nonempty set, $x \in Y$ and $u, v \in F(Y)$. (i) (i-1) $u(x) = \sup\{\alpha \in (0,1] : x \in [u]_\alpha\}$. (i-2) If $[u]_0$ is well-defined, then $u(x) = \sup\{\alpha \in [0,1] : x \in [u]_\alpha\}$. (ii) (ii-1) If for each $\alpha \in (0,1]$, $[u]_\alpha = [v]_\alpha$, then $u = v$. (ii-2) Assume that $u = v$. Then (ii-2a) for each $\alpha \in (0,1]$, $[u]_\alpha = [v]_\alpha$; (ii-2b) Y is a topological space if and only if $[u]_0 = [v]_0$. (ii-3) $u = v$ if and only if for each $\alpha \in (0,1]$, $[u]_\alpha = [v]_\alpha$. (ii-4) Assume that Y is a topological space. Then $u = v$ if and only if for each $\alpha \in [0,1]$, $[u]_\alpha = [v]_\alpha$.*

Proof. First we show (i). Put $u(x) = \xi$. If $\xi > 0$, then $\{\alpha \in (0,1] : x \in [u]_\alpha\} = (0,\xi]$, and so $\sup\{\alpha \in (0,1] : x \in [u]_\alpha\} = \sup(0,\xi] = \xi = u(x)$. If $\xi = 0$, then $\{\alpha \in (0,1] : x \in [u]_\alpha\} = \emptyset$, and so $\sup\{\alpha \in (0,1] : x \in [u]_\alpha\} = 0 = u(x)$. Thus (i-1) holds.

Below we show (ā) If $[u]_0$ is well-defined, then $\sup S_1 = \sup S_2$, where $S_1 := \{\alpha \in (0,1] : x \in [u]_\alpha\}$ and $S_2 := \{\alpha \in [0,1] : x \in [u]_\alpha\}$.

Case (I). Assume that $S_1 = \emptyset$. Then $\sup S_1 = 0$, and it holds that $S_2 = \emptyset$ or $S_2 = \{0\}$. Clearly $\sup S_2 = 0$ regardless of $S_2 = \emptyset$ or $S_2 = \{0\}$. So $\sup S_1 = \sup S_2$.

Case (II). Assume that $S_1 \neq \emptyset$. Then there is an $\alpha \in (0,1]$ with $x \in [u]_\alpha$. So $x \in [u]_0$ as $[u]_\alpha \subseteq [u]_0$. Thus we obtain (a) $S_2 = S_1 \cup \{0\}$. Clearly we have (b) $\sup S_1 > 0$. Thus $\sup S_2 =$ (by (a)) $\sup(S_1 \cup \{0\}) = \sup S_1 \vee \sup\{0\} = \sup S_1 \vee 0$ (i.e. $(\sup S_1) \vee 0$) $=$ (by (b)) $\sup S_1$. The proof of (ā) is completed.

Combining (i-1) and (ā) yields (i-2). Hence (i) is proved. (Obviously, combining (i-1) and (i-2) yields (ā).)

Now we show (ii-1). Notice that for each $y \in Y$,

$$v(y) = \text{(by (i-1))} \sup\{\alpha \in (0,1] : y \in [v]_\alpha\} = \sup\{\alpha \in (0,1] : y \in [u]_\alpha\} = \text{(by (i-1))}\ u(y).$$

So $u = v$ as $u, v \in F(Y)$. This proof of (ii-1) is essentially given in the proof of Theorem 3.1 in [18]. (see also (I) below)

Now we show (ii-2). (ii-2a) holds obviously. Assume that Y is a topological space. This means that $[u]_0$ and $[v]_0$ are well-defined. Then $u = v$ implies that $[u]_0 = [v]_0$. If $[u]_0 = [v]_0$ then $[u]_0$ and $[v]_0$

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are well-defined, which means that Y is a topological space. So (ii-2b) holds. (ii-3) follows immediately from (ii-1) and (ii-2a). (ii-4) follows immediately from (ii-1) and (ii-2).

(I) We can see that (ii-1) is equivalent to (ii-1)′ Given $v \in F(Y)$, if $u \in S(v)$ then $u = v$. It holds that (a) for each $v \in F(Y)$, $v \in S(v)$. Clearly (a) and (ii-1)′ hold if and only if Remark 4(v) holds (We use (b) to denote this statement.). Below (I-1) and (I-2) are two proofs of (ii-1).

(I-1) As Remark 4(v) is proved, by (b), (ii-1)′ holds; that is, (ii-1) holds.

(I-2) Suppose that u and v are in $F(Y)$ satisfying for all $\alpha \in (0, 1]$, $[u]_\alpha = [v]_\alpha$. Then it holds that (c) u and v are in $S(v)$. By Remark 4(iv), we have (d) $S(v)$ is a singleton set. By (c) and (d), we have that $u = v$. So (ii-1) is proved. ((d) also follows from (c) and Remark 4(iii).)

In some sense, all the proofs of (ii-1) given in this paper are essentially the same.

□

First, the corresponding author of this paper independently gave all contents of ChinaXiv:202507.00428, which include the contents from the sentence "In this paper we assume that $\mathbf{sup}\,\emptyset = \mathbf{0}$." at the second paragraph of this section to the "□" at the end of the proof of Proposition 3 (see ChinaXiv:202507.00428 at https://chinaxiv.org/abs/202507.00428). Then we gave the rest of this paper.

The corresponding author of this paper also independently gave at least all sentences that contain the expression "the unique element of" of this paper.

The following Proposition 4 presents a characterization of all cut sets of a fuzzy set in $\mathbb{R}$ when this fuzzy set in $\mathbb{R}$ is a given generalized trapezoidal fuzzy number.

Proposition 4. *Let $u \in F(\mathbb{R})$ and $(a, b, c, d) \in$ Trap. Then $u = (a, b, c, d)$ if and only if*

$$\text{for each } \xi \in [0, 1],\ [u]_\xi = [\xi(b - a) + a,\ c + (1 - \xi)(d - c)]. \tag{1}$$

Proof. We prove the "only if" part. Suppose that $u = (a, b, c, d)$. Then, by Definition 4 and easy calculations, (1) holds (see also (I) below).

We prove the "if" part. Suppose that (1) holds. By Proposition 3(i-2), $u(x) = \sup\{\alpha \in [0, 1] : x \in [u]_\alpha\}$ for each $x \in \mathbb{R}$. From this, by easy calculations, we can obtain that $u(x) = (a, b, c, d)(x)$ for each $x \in \mathbb{R}$ (see also (II) below). This means that $u = (a, b, c, d)$ as both u and (a, b, c, d) are in $F(\mathbb{R})$.

Another proof of the "if" part is as follows. Suppose that (1) holds. Note that the "only if" part says that for each $\xi \in [0, 1]$, $[(a, b, c, d)]_\xi = [\xi(b - a) + a,\ c + (1 - \xi)(d - c)]$. Thus (1) means that for each $\xi \in [0, 1]$, $[u]_\xi = [(a, b, c, d)]_\xi$. As both u and (a, b, c, d) are in $F(\mathbb{R})$, by Proposition 3(ii-1), $u = (a, b, c, d)$. This proof is based on the result of the "only if" part.

(I) One way to perform these calculations are to do it based on watching the graphs of the membership functions of (a, b, c, d). In this way, it is easy to calculate that for each $\xi \in [0, 1]$, $[(a, b, c, d)]_\xi = [\xi(b - a) + a,\ c + (1 - \xi)(d - c)]$ in all the four cases $a < b \le c < d$, $a = b \le c < d$, $a < b \le c = d$ and $a = b \le c = d$. So (1) holds as we suppose that $u = (a, b, c, d)$.

(II) One way to perform these calculations are to do it based on watching the graphs of the cut sets $[u]_\alpha$, $\alpha \in [0, 1]$. In this way, it is easy to calculate that $u(x) = (a, b, c, d)(x)$ for each $x \in \mathbb{R}$ in all the four cases $a < b \le c < d$, $a = b \le c < d$, $a < b \le c = d$ and $a = b \le c = d$.

□

For $u \in \mathbf{F}(\mathbb{R})$, we call u a 1-dimensional compact fuzzy number if u has the following properties:
(i) $[u]_1 \neq \emptyset$; and
(ii) for each $\alpha \in [0, 1]$, $[u]_\alpha$ is a compact interval of $\mathbb{R}$.
The set of all 1-dimensional compact fuzzy numbers is denoted by E.

Let $u \in$ Trap. Denote $u = (a, b, c, d)$. By Proposition 4, $[u]_1 = [b, c] \neq \emptyset$ and for each $\xi \in [0, 1]$, $[u]_\xi = [\xi(b - a) + a,\ c + (1 - \xi)(d - c)]$ is a compact interval of $\mathbb{R}$. Also $u \in F(\mathbb{R})$. Thus $u \in E$. So Trap $\subseteq E$. Below Example 1 shows that $E \setminus$ Trap $\neq \emptyset$. Hence Trap $\subsetneqq E$. So Tap $\subsetneqq$ Trap $\subsetneqq E$ and Tag $\subsetneqq$ Trag $\subsetneqq$ Trap $\subsetneqq E$, where the first $\subsetneqq$, the third $\subsetneqq$ and the fourth $\subsetneqq$ have already been given in [3] (see also Section 3 of this paper).

Example 1. *Define $u \in F(\mathbb{R})$ by*

$$u(x) = \begin{cases} e^{-x}, & \text{if } x \in [0,1], \\ 0, & \text{if } x \in \mathbb{R} \setminus [0,1]. \end{cases}$$

Then

$$[u]_\alpha = \begin{cases} [0, -\ln \alpha], & \text{if } \alpha \in [e^{-1}, 1], \\ [0,1], & \text{if } \alpha \in [0, e^{-1}]. \end{cases} \tag{2}$$

Thus $[u]_1 = \{0\} \neq \emptyset$, and for each $\alpha \in [0,1]$, $[u]_\alpha$ is a compact interval of $\mathbb{R}$. So $u \in E$. We claim that $u \notin$ Trap. Suppose that $u \in$ Trap. Denote $u = (a,b,c,d)$. Then $\{0\} = [u]_1 = [b,c]$ and $[0,1] = [u]_0 = [a,d]$, where the second = and fourth = follow from Proposition 4 or Theorem 1(i). So $a = b = c = 0$ and $d = 1$. Hence, by Proposition 4, $[u]_{e^{-1}} = [e^{-1}(0-0)+0,\ 0+(1-e^{-1})(1-0)] = [0, 1-e^{-1}]$. However, by (2), $[u]_{e^{-1}} = [0,1]$. This is a contradiction. Thus $u \notin$ Trap.

Corollary 1. *Let $u \in F(\mathbb{R})$ and $(a,b,c,d) \in$ Tap. Then $u = (a,b,c,d)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a,\ c+(1-\xi)(d-c)]$.*

Proof. Note that $(a,b,c,d) \in$ Tap implies that $(a,b,c,d) \in$ Trap (see also (I) below). So the desired result follows immediately from Proposition 4.

(I) Tap $\subsetneqq$ Trap. This means that if $w \in Tap$ then $w \in Trap$ but the converse is false.

□

From the above proof of Corollary 1, we can see that Corollary 1 is a corollary of Proposition 4.

Proposition 5. *Let $u \in F(\mathbb{R})$ and $(a,b,c) \in$ Trag. Then $u = (a,b,c)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a,\ b+(1-\xi)(c-b)]$.*

Proof. By Remark 3 or Proposition 1(ii), (a,b,c) is the (a,b,b,c) in Trap. And, by Proposition 4, $u = (a,b,b,c)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a,\ b+(1-\xi)(c-b)]$. So we obtain the desired result.

□

From the above proof of Proposition 5, we can see that Proposition 5 is a corollary of Proposition 4.

Corollary 2. *Let $u \in F(\mathbb{R})$ and $(a,b,c) \in$ Tag. Then $u = (a,b,c)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a,\ b+(1-\xi)(c-b)]$.*

Proof. $(a,b,c) \in$ Tag implies that $(a,b,c) \in$ Trag (see also (I) below). So the desired result follows immediately from Proposition 5.

(I) Tag $\subsetneqq$ Trag. This means that if $w \in$ Tag then $w \in$ Trag but the converse is false.

□

From the above proof of Corollary 2, we can see that Corollary 2 is a corollary of Proposition 5. So Corollary 2 is a corollary of Proposition 4.

The following Theorem 3 gives an equivalent condition for the equality of two generalized trapezoidal fuzzy numbers.

Theorem 3. *Let u and v be in Trap. Then $u = v$ if and only if there exist two distinct elements λ and τ in $[0,1]$ with $[u]_\lambda = [v]_\lambda$ and $[u]_\tau = [v]_\tau$.*

Proof. Clearly $u = v$ implies that $[u]_\xi = [v]_\xi$ for all $\xi \in [0,1]$. So the "only if" part is true.

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Conversely, assume that there exist two distinct elements λ and τ in $[0,1]$ with $[u]_\lambda = [v]_\lambda$ and $[u]_\tau = [v]_\tau$. Denote $u := (a,b,c,d)$ and $v := (a_1,b_1,c_1,d_1)$. Then

$$[\lambda(b-a)+a,\ c+(1-\lambda)(d-c)] = [u]_\lambda = [v]_\lambda = [\lambda(b_1-a_1)+a_1,\ c_1+(1-\lambda)(d_1-c_1)], \quad (3)$$
$$[\tau(b-a)+a,\ c+(1-\tau)(d-c)] = [u]_\tau = [v]_\tau = [\tau(b_1-a_1)+a_1,\ c_1+(1-\tau)(d_1-c_1)], \quad (4)$$

where the first = and the third = in (3) and the first = and the third = in (4) follow from Proposition 4. (3) implies (5) and (6) below. (4) implies (7) and (8) below.

$$\lambda(b-a)+a = \lambda(b_1-a_1)+a_1, \quad (5)$$
$$c+(1-\lambda)(d-c) = c_1+(1-\lambda)(d_1-c_1), \quad (6)$$
$$\tau(b-a)+a = \tau(b_1-a_1)+a_1, \quad (7)$$
$$c+(1-\tau)(d-c) = c_1+(1-\tau)(d_1-c_1). \quad (8)$$

We claim (a) (5) and (7) hold if and only if $a = a_1$ and $b = b_1$; and (b) (6) and (8) hold if and only if $c = c_1$ and $d = d_1$.

We show (a). Obviously $a = a_1$ and $b = b_1$ implies (5) and (7). Conversely, suppose that (5) and (7) hold. Computing (5)−(7), we obtain (c) $(\lambda-\tau)(b-a) = (\lambda-\tau)(b_1-a_1)$. (c) is equivalent to (d) $(b-a) = (b_1-a_1)$, as $\lambda \neq \tau$. Computing (5)−λ·(d), we obtain (e) $a = a_1$. Computing (d) + (e), we obtain $b = b_1$. Thus (a) is proved.

We show (b). Obviously $c = c_1$ and $d = d_1$ implies (6) and (8). Conversely, suppose that (6) and (8) hold. Computing (6)−(8), we obtain ($\bar{c}$) $(\tau-\lambda)(d-c) = (\tau-\lambda)(d_1-c_1)$. ($\bar{c}$) is equivalent to ($\bar{d}$) $(d-c) = (d_1-c_1)$, as $\lambda \neq \tau$. Computing (6)−$(1-\lambda)$·($\bar{d}$), we obtain ($\bar{e}$) $c = c_1$. Computing ($\bar{d}$) + ($\bar{e}$), we obtain $d = d_1$. Thus (b) is proved.

The above proofs of (a) and (b) are similar. See also (I) below.

By (5), (6), (7), (8), (a) and (b), we have that $a = a_1$, $b = b_1$, $c = c_1$ and $d = d_1$. Then obviously $u = v$ (see also (II) below). So the "if" part is true. The proof is completed.

(I) We can also show (a) and (b) as follows. Below two proofs of (a) and (b) are similar.

The following is a proof of (a). We see (5) and (7) as a system of linear equations in 2 unknowns a and b. We use (A) to denote this system of linear equations. Clearly (a) means that $a = a_1$ and $b = b_1$ is the unique solution of (A). Obviously, by (5) and (7), $a = a_1$ and $b = b_1$ is a solution of (A). So to show (a), we only need to show that (A) has a unique solution.

We can write (A) as

$$\begin{cases} \lambda b + (1-\lambda)a & = \lambda(b_1-a_1)+a_1, \\ \tau b + (1-\tau)a & = \tau(b_1-a_1)+a_1 \end{cases}$$

We can see that (A) is square. Computing the determinant of the coefficient matrix of (A), we obtain

$$\begin{vmatrix} \lambda, & (1-\lambda) \\ \tau, & (1-\tau) \end{vmatrix} = \lambda(1-\tau) - (1-\lambda)\tau = \lambda - \tau \neq 0.$$

Thus (A) has a unique solution. So (a) is proved.

The following is a proof of (b). We see (6) and (8) as a system of linear equations in 2 unknowns c and d. We use (B) to denote this system of linear equations. Clearly (b) means that $c = c_1$ and $d = d_1$ is the unique solution of (B). Obviously, by (6) and (8), $c = c_1$ and $d = d_1$ is a solution of (B). So to show (b), we only need to show that (B) has a unique solution.

We can write (B) as

$$\begin{cases} \lambda c + (1-\lambda)d & = c_1+(1-\lambda)(d_1-c_1), \\ \tau c + (1-\tau)d & = c_1+(1-\tau)(d_1-c_1) \end{cases}$$

We can see that (B) is square. Computing the determinant of the coefficient matrix of (B), we obtain

$$\begin{vmatrix} \lambda, & (1-\lambda) \\ \tau, & (1-\tau) \end{vmatrix} = \lambda(1-\tau) - (1-\lambda)\tau = \lambda - \tau \neq 0.$$

Thus (B) has a unique solution. So (b) is proved.

(II) In fact, by Theorem 1(ii), $u = v$ is equivalent to $[a, b, c, d] = [a_1, b_1, c_1, d_1]$, which means that $a = a_1$, $b = b_1$, $c = c_1$ and $d = d_1$. In other words, $u = v$ if and only if $a = a_1$, $b = b_1$, $c = c_1$ and $d = d_1$.

□

Corollary 3. *(i) Let u and v be in Tap. Then* $u = v$ *if and only if there exist two distinct elements* λ *and* τ *in* $[0, 1]$ *with* $[u]_\lambda = [v]_\lambda$ *and* $[u]_\tau = [v]_\tau$.
(ii) Let u and v be in Trag. Then $u = v$ *if and only if there exist two distinct elements* λ *and* τ *in* $[0, 1]$ *with* $[u]_\lambda = [v]_\lambda$ *and* $[u]_\tau = [v]_\tau$.
(iii) Let u and v be in Tag. Then $u = v$ *if and only if there exist two distinct elements* λ *and* τ *in* $[0, 1]$ *with* $[u]_\lambda = [v]_\lambda$ *and* $[u]_\tau = [v]_\tau$.

Proof. We show (i). u and v are in Tap implies that u and v are in Trap. So (i) follows immediately from Theorem 3. (This proof indicates that (i) is a corollary of Theorem 3.)

We show (ii). u and v are in Trag implies that u and v are in Trap (see also (I) below). So (ii) follows immediately from Theorem 3. (This proof indicates that (ii) is a corollary of Theorem 3.)

We show (iii). u and v are in Tag implies that u and v are in Trag. So (iii) follows immediately from (ii). (This proof indicates that (iii) is a corollary of (ii). So (iii) is a corollary of Theorem 3.)

(I) Trag $\subsetneqq$ Trap. This means that if $w \in$ Trag then $w \in$ Trap but the converse is false.

□

5. Conclusions

In this paper, we first present a characterization of all cut sets of a fuzzy set in $\mathbb{R}$ when this fuzzy set in $\mathbb{R}$ is a given generalized trapezoidal fuzzy number. Then we give an equivalent condition for the equality of two generalized trapezoidal fuzzy numbers. This condition is described in terms of the cut sets. As corollaries, we give corresponding conclusions of the above conclusions for Tag, Tap and Trag, respectively.

The results of this paper have potential effects on the research and applications of the generalized triangular fuzzy numbers and the generalized trapezoidal fuzzy numbers.

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