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Article

Operator- and Matrix-Valued Strip-Analytic Abu-Ghuwaleh Transforms: Wiener–Mellin Inversion, System Symbols, and Stable Recovery

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Abstract

We develop a unified operator- and matrix-valued strip-analytic extension of the Abu-Ghuwaleh transform program. The central object is a strongly measurable operator-valued orbit density whose boundary representation induces a continuous dilation-convolution operator acting on the Fourier transform of a weighted Hilbert-space-valued signal. In this setting the transform admits two complementary inversion mechanisms: Mellin contour inversion and contour-free Wiener–Mellin inversion on the logarithmic scale. We prove exact factorization formulas on named weighted signal spaces, derive branchwise Mellin diagonalization formulas with operator-valued system symbols, obtain inversion theorems under bounded invertibility assumptions, and formulate a log-scale Fourier multiplier representation suitable for FFT-based recovery. We then prove Young-type boundedness on the logarithmic side and stability estimates on frequency windows away from singularities of the multiplier. The finite-dimensional matrix case is obtained as a direct specialization of the Hilbert-space theory, and in that setting the Wiener inverse is derived from a standard matrix Wiener criterion. Finally, we isolate an explicit Gamma-type kernel family for which the system symbol is computable in closed form and yields concrete injectivity and stability constants. The paper is intended as the natural operator-theoretic successor to the scalar strip-analytic stage of the master-integral-transform program.

Keywords: Abu-Ghuwaleh transform; strip-analytic kernels; operator-valued integral transforms; matrix-valued systems; Mellin inversion; Wiener–Mellin inversion; system symbols; stable recovery

1. Introduction

The master integral transform with entire kernels established a broad scalar framework in which an entire kernel generates a family of oscillatory integral transforms together with completeness, injectivity, and Mellin–Fourier inversion under density hypotheses on the nonzero Taylor indices [1]. The subsequent finite-Laurent sequel shows that the first meromorphic extension enlarges the discrete spectrum from one-sided to two-sided. The strip-analytic sequel then replaces the discrete spectral picture by a continuous one and identifies the correct scalar Wiener–Mellin inversion mechanism.

The present paper develops the next structural layer in that sequence. We move from scalar strip-analytic kernels to *operator-valued* and *matrix-valued* strip-analytic kernels. The point of this step is not decorative generality. Once the scalar strip-analytic theory is understood as a continuous dilation-convolution operator on the Fourier side, the natural next question is whether the same architecture survives for coupled systems. The answer is yes: the forward transform becomes a system-valued dilation-convolution operator, the scalar Mellin symbol is replaced by an operator-valued system symbol, and the two inversion mechanisms — Mellin contour inversion and Wiener–Mellin inversion — persist provided one imposes bounded invertibility along the relevant spectral sets.

This operator-valued sequel is mathematically natural for several reasons. First, it provides a rigorous system version of strip-analytic AGT rather than leaving the theory scalar forever. Second, it isolates the correct object controlling invertibility: not a scalar multiplier, but an operator-valued

Mellin/log-Fourier symbol. Third, it interfaces naturally with adjacent literatures on vector-valued Hardy and de Branges type spaces, operator-valued Wiener theory, and functional models for structured systems. Those neighboring theories suggest that the strip-analytic system extension is not merely possible, but structurally well aligned with modern harmonic and functional analysis.

The paper is deliberately written as a first theorem package rather than as the most general imaginable operator-valued theory. The kernel class is chosen so that the orbit representation is explicit, Bochner integration is justified on named spaces, and the inversion mechanisms are exact rather than heuristic. This restraint is part of the design: it keeps the theory honest while still producing a genuinely new stage in the program.

Main Results

The main results may be summarized as follows.

- (1) We define an operator-valued strip-admissible orbit class and prove that the associated transform is a continuous dilation-convolution operator acting on the Fourier transform of a weighted Hilbert-space-valued signal.
- (2) We derive branchwise Mellin diagonalization formulas with operator-valued system symbol

$$\mathfrak{M}_K(s) = \int_0^\infty K(\xi) \xi^{-s} \frac{d\xi}{\xi},$$

and prove Mellin inversion under bounded invertibility on a vertical line.

- (3) We pass to logarithmic coordinates and show that the transform becomes an additive convolution equation with operator-valued kernel. This yields a contour-free Wiener–Mellin inversion theorem under existence of an operator-valued Wiener inverse.
- (4) We prove Young-type boundedness and practical stability estimates on frequency windows away from singularities of the multiplier.
- (5) We show that the matrix-valued case is a finite-dimensional specialization of the general Hilbert-space theory and prove an explicit injectivity-and-stability proposition for a Gamma-type kernel family.

Roadmap of the paper

Section 2 fixes the weighted Hilbert-space-valued signal classes and the basic AGT notation. Section 3 introduces strip-admissible orbit densities and proves the continuous factorization theorem together with a logarithmic Young estimate. Section 4 establishes the operator-valued Mellin diagonalization and contour inversion formulas. Section 5 develops the Wiener–Mellin inversion theory on the logarithmic scale. Section 6 proves the basic stability estimate away from multiplier singularities. Section 7 specializes the theory to the matrix-valued case and derives a practical Wiener criterion. Section 8 introduces an explicit Gamma-type kernel family and proves a sharper injectivity-and-stability proposition. Section 9 records a compact numerical blueprint for system recovery. The final sections discuss the place of the paper in the broader post-MIT program.

2. Hilbert-Valued Signals and the Base Transform

Let $\mathcal{H}$ be a complex separable Hilbert space. Fix $p \in (-1, 1)$ and parameters $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$. For a strongly measurable function $f : \mathbb{R} \rightarrow \mathcal{H}$, define the weighted signal

$$\Phi(x) := |x|^p e^{-i\frac{\pi p}{2} \operatorname{sgn} x} f(x). \quad (1)$$

Its Fourier transform is the Bochner integral

$$F(\omega) := \hat{\Phi}(\omega) = \int_{\mathbb{R}} \Phi(x) e^{i\omega x} dx,$$

whenever the integral is defined.

Definition 1 (Signal spaces). We write

$$X_p(\mathcal{H}) := \left\{ f : \mathbb{R} \rightarrow \mathcal{H} : \Phi \in L^1(\mathbb{R}; \mathcal{H}) \cap L^2(\mathbb{R}; \mathcal{H}) \right\}.$$

For Mellin inversion we use the subclass $X_p^M(\mathcal{H}) \subset X_p(\mathcal{H})$ consisting of signals whose weighted even and odd parts satisfy

$$\Phi_e, \Phi_o \in L^1((0, \infty), x^{-c} dx; \mathcal{H}) \cap L^1((0, \infty), x^{-c'} dx; \mathcal{H})$$

for some $c < c'$. For logarithmic inversion we use the class $X_p^{\log}(\mathcal{H}) \subset X_p(\mathcal{H})$ for which

$$g_0(t) := F(e^t)$$

belongs to $L^1(\mathbb{R}; \mathcal{H}) \cap L^2(\mathbb{R}; \mathcal{H})$.

Remark 1. These are working spaces designed to justify the exact inversion arguments below. They are not intended as optimal hypotheses.

Definition 2 (Operator-valued strip-analytic AGT). Let $\mathcal{K} : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H})$ be a strongly measurable orbit kernel. The associated operator-valued strip-analytic Abu-Ghuwaleh transform is defined by

$$\mathcal{G}_{\mathcal{K}}^{(p)}[f](\theta) := \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{K}(\theta x) \Phi(x) dx, \quad \theta > 0, \quad (2)$$

whenever the integral converges in the Bochner sense.

3. Strip-Admissible Orbit Densities and Continuous Factorization

The operator-valued theory is most transparent when the orbit kernel is itself given by a boundary-frequency representation.

Definition 3 (Strip-admissible orbit density). Let $\sigma > 0$. A strongly measurable function

$$K : (0, \infty) \rightarrow \mathcal{B}(\mathcal{H})$$

is called strip-admissible if

$$\int_0^\infty \|K(\xi)\|_{\text{op}} e^{\sigma \xi} \frac{d\xi}{\xi} < \infty. \quad (3)$$

Its associated orbit kernel is

$$\mathcal{K}(\tau) := \int_0^\infty K(\xi) e^{i\xi\tau} \frac{d\xi}{\xi}, \quad \tau \in \mathbb{R}, \quad (4)$$

where the integral converges absolutely in operator norm. The exponential moment in (3) implies that $\mathcal{K}$ extends holomorphically to the strip

$$\Sigma_\sigma := \{\tau \in \mathbb{C} : |\Im \tau| < \sigma\}.$$

Remark 2. The explicit orbit representation is the operator-valued analogue of the scalar strip-admissible framework. It is chosen precisely so that continuous dilation-convolution factorization and Bochner-integral inversion are exact.

Theorem 1 (Continuous operator factorization). Let $f \in X_p(\mathcal{H})$ and let K be strip-admissible. Then for every $\theta > 0$,

$$\mathcal{G}(\theta) = \frac{1}{2\pi} \int_0^\infty K(\xi) F(\xi\theta) \frac{d\xi}{\xi}, \quad (5)$$

where $\mathcal{G} := \mathcal{G}_{\mathcal{K}}^{(p)}[f]$. Equivalently,

$$\mathcal{G} = \mathcal{T}_{\mathcal{K}}F, \quad (\mathcal{T}_{\mathcal{K}}F)(\theta) := \frac{1}{2\pi} \int_0^\infty K(\xi)F(\xi\theta) \frac{d\xi}{\xi}. \quad (6)$$

The integral converges absolutely and locally uniformly in $\theta > 0$.

Proof. For fixed $\theta > 0$ and $x \in \mathbb{R}$,

$$\|\mathcal{K}(\theta x)\Phi(x)\|_{\mathcal{H}} \leq \left(\int_0^\infty \|K(\xi)\|_{\text{op}} \frac{d\xi}{\xi} \right) \|\Phi(x)\|_{\mathcal{H}}.$$

By (3), the operator-norm integral is finite, and since $\Phi \in L^1(\mathbb{R}; \mathcal{H})$, the majorant is integrable. Tonelli's theorem therefore permits interchange of the x - and ξ -integrals in (2):

$$\mathcal{G}(\theta) = \frac{1}{2\pi} \int_0^\infty K(\xi) \left(\int_{\mathbb{R}} \Phi(x) e^{i\xi\theta x} dx \right) \frac{d\xi}{\xi}.$$

The inner integral is exactly $F(\xi\theta)$, proving (5). The bound

$$\|\mathcal{G}(\theta)\|_{\mathcal{H}} \leq \frac{\|\Phi\|_{L^1(\mathbb{R}; \mathcal{H})}}{2\pi} \int_0^\infty \|K(\xi)\|_{\text{op}} \frac{d\xi}{\xi}$$

yields local uniform convergence. $\square$

Passing to logarithmic variables yields a standard convolution equation and therefore a harmonic-analysis estimate.

Proposition 1 (Young-type boundedness on the logarithmic side). *Assume the hypotheses of Theorem 1 and let $f \in X_p^{\log}(\mathcal{H})$. Set*

$$\theta = e^t, \quad h(t) := \mathcal{G}(e^t), \quad g_0(t) := F(e^t), \quad q(u) := K(e^{-u}). \quad (7)$$

Then

$$h(t) = \frac{1}{2\pi} \int_{\mathbb{R}} q(u)g_0(t-u) du, \quad (8)$$

and

$$\|h\|_{L^2(\mathbb{R}; \mathcal{H})} \leq \frac{1}{2\pi} \|q\|_{L^1(\mathbb{R}; \mathcal{B}(\mathcal{H}))} \|g_0\|_{L^2(\mathbb{R}; \mathcal{H})}. \quad (9)$$

The same estimate holds with L^2 replaced by L^1 .

Proof. From (5), the change of variables $\xi = e^{-u}$ gives

$$h(t) = \frac{1}{2\pi} \int_{\mathbb{R}} K(e^{-u})g_0(t-u) du,$$

which is (8). Since $q \in L^1(\mathbb{R}; \mathcal{B}(\mathcal{H}))$ and $g_0 \in L^2(\mathbb{R}; \mathcal{H})$, the vector-valued Young inequality for Bochner convolution yields (9). $\square$

4. Operator-Valued Mellin Diagonalization and Contour Inversion

As in the scalar theory, parity separation clarifies the inversion formulas. Write

$$\Phi = \Phi_e + \Phi_o, \quad \Phi_e(x) = \frac{\Phi(x) + \Phi(-x)}{2}, \quad \Phi_o(x) = \frac{\Phi(x) - \Phi(-x)}{2}.$$

Then

$$F(\omega) = 2 \int_0^\infty \Phi_e(x) \cos(\omega x) dx + 2i \int_0^\infty \Phi_o(x) \sin(\omega x) dx.$$

Definition 4 (System Mellin symbol). For a strip-admissible density K , define the operator-valued Mellin symbol

$$\mathfrak{M}_K(s) := \int_0^\infty K(\xi) \xi^{-s} \frac{d\xi}{\xi}, \quad (10)$$

whenever the integral converges absolutely in operator norm.

Theorem 2 (Operator-valued Mellin diagonalization). Assume the hypotheses of Theorem 1. Let $f \in X_p^M(\mathcal{H})$, and suppose there exists a strip $a < \Re s < b$ such that (10) converges absolutely there. Then, for every s with $a < \Re s < b$,

$$\mathcal{M}_\theta\{\mathcal{G}^c\}(s) = \frac{\Gamma(s) \cos(\pi s/2)}{\pi} \mathfrak{M}_K(s) \mathcal{M}\{\Phi_e\}(1-s), \quad (11)$$

$$\mathcal{M}_\theta\{\mathcal{G}^s\}(s) = \frac{\Gamma(s) \sin(\pi s/2)}{i\pi} \mathfrak{M}_K(s) \mathcal{M}\{\Phi_o\}(1-s). \quad (12)$$

Proof. The proof is the operator-valued version of the scalar argument. By the strip assumptions on K and the Mellin-strip assumptions on f , all exchanges of integral order are justified by Fubini–Tonelli for Bochner integrals. The change of variables $u = \xi\theta x$ and the classical Mellin formulas for cosine and sine produce the identities, with the operator-valued factor (10) pulled out on the left. $\square$

Theorem 3 (Operator-valued Mellin contour inversion). Assume the hypotheses of Theorem 2. Suppose there exists $c \in (a, b)$ such that $\mathfrak{M}_K(1-u)$ is boundedly invertible for every u on the line $\Re u = c$ and

$$\sup_{\Re u=c} \|\mathfrak{M}_K(1-u)^{-1}\|_{\text{op}} < \infty.$$

Assume also that the branch quotients below are Bochner- L^1 along that line. Then

$$\Phi_e(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} |x|^{-u} \frac{\pi}{\Gamma(1-u) \sin(\pi u/2)} \mathfrak{M}_K(1-u)^{-1} \mathcal{M}_\theta\{\mathcal{G}^c\}(1-u) du, \quad (13)$$

$$\Phi_o(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} |x|^{-u} \frac{i\pi}{\Gamma(1-u) \cos(\pi u/2)} \mathfrak{M}_K(1-u)^{-1} \mathcal{M}_\theta\{\mathcal{G}^s\}(1-u) du. \quad (14)$$

Consequently,

$$f(x) = e^{i\frac{\pi p}{2} \operatorname{sgn} x} |x|^{-p} (\Phi_e(x) + \Phi_o(x)) \quad (15)$$

for almost every $x \in \mathbb{R} \setminus \{0\}$.

Proof. Solve (11) and (12) for the Mellin transforms of Φ_e and Φ_o , then apply Mellin inversion in the Bochner sense. Uniform boundedness of the inverse system symbol on the contour guarantees that the integrals are well-defined. $\square$

5. Operator-Valued Wiener–Mellin Inversion

We now pass to logarithmic coordinates. With the notation in (7), the convolution equation (8) becomes the basic additive model.

Theorem 4 (Log-scale Fourier multiplier representation). Assume the hypotheses of Theorem 1 and let $f \in X_p^{\log}(\mathcal{H})$. Then $h, g_0 \in L^1(\mathbb{R}; \mathcal{H}) \cap L^2(\mathbb{R}; \mathcal{H})$, (8) holds in $L^1 \cap L^2$, and Fourier transformation in t yields

$$\hat{h}(\eta) = A(\eta) \hat{g}_0(\eta), \quad A(\eta) := \frac{1}{2\pi} \int_0^\infty K(\xi) \xi^{-i\eta} \frac{d\xi}{\xi} = \frac{1}{2\pi} \mathfrak{M}_K(i\eta). \quad (16)$$

If $A(\eta)$ is boundedly invertible for almost every η and

$$\sup_{\eta \in \mathbb{R}} \|A(\eta)^{-1}\|_{\text{op}} < \infty,$$

then

$$g_0(t) = \mathcal{F}_t^{-1}[A(\eta)^{-1}\widehat{h}(\eta)](t), \quad F(e^t) = g_0(t). \quad (17)$$

Proof. Since (3) implies $q \in L^1(\mathbb{R}; \mathcal{B}(\mathcal{H}))$ and $g_0 \in L^1 \cap L^2$, the convolution in (8) is well-defined. Termwise Fourier transformation gives

$$\widehat{h}(\eta) = \frac{1}{2\pi} \widehat{q}(\eta) \widehat{g}_0(\eta),$$

where the Fourier transform of q is an operator-valued multiplier. A change of variables $\xi = e^{-u}$ identifies this multiplier with the integral in (16). The inverse formula follows by applying the bounded inverse pointwise in frequency and then inverse Fourier transforming. $\square$

Definition 5 (Operator-valued Wiener inverse). Let $q \in L^1(\mathbb{R}; \mathcal{B}(\mathcal{H}))$. We say that q admits a Wiener inverse if there exists $\ell \in L^1(\mathbb{R}; \mathcal{B}(\mathcal{H}))$ such that

$$\widehat{\ell}(\eta) \widehat{q}(\eta) = 2\pi \text{Id}_{\mathcal{H}} \quad \text{for every } \eta \in \mathbb{R}.$$

Theorem 5 (Operator-valued Wiener–Mellin inversion). Assume the hypotheses of Theorem 4 and suppose that the logarithmic kernel q admits a Wiener inverse $\ell \in L^1(\mathbb{R}; \mathcal{B}(\mathcal{H}))$. Define

$$\nu(\xi) := \ell(-\log \xi), \quad \xi > 0.$$

Then the inverse of the system AGT is given by

$$F(\theta) = \int_0^\infty \nu(\xi) \mathcal{G}(\theta/\xi) \frac{d\xi}{\xi}, \quad \theta > 0. \quad (18)$$

Consequently,

$$\Phi = \mathcal{F}^{-1}[F], \quad f(x) = e^{i\frac{\pi p}{2} \text{sgn } x} |x|^{-p} \Phi(x)$$

for almost every $x \neq 0$.

Proof. From the Wiener inverse identity one obtains

$$g_0(t) = \int_{\mathbb{R}} \ell(u) h(t-u) \, du$$

on the additive side. Returning to multiplicative variables via $\xi = e^{-u}$ gives (18). $\square$

6. Stability away from singularities of the multiplier

The multiplier representation makes numerical conditioning completely transparent.

Proposition 2 (Practical stability on frequency windows). Assume the hypotheses of Theorem 4. Let $I \subset \mathbb{R}$ be measurable and suppose there exists $\eta_0 > 0$ such that

$$\|A(\eta)^{-1}\|_{\text{op}} \leq \eta_0^{-1} \quad \text{for a.e. } \eta \in I.$$

For exact data $\widehat{h}$ and noisy data $\widehat{h}_\delta$, define

$$g_{0,I} := \mathcal{F}_t^{-1}[\mathbf{1}_I A^{-1} \widehat{h}], \quad g_{0,I}^\delta := \mathcal{F}_t^{-1}[\mathbf{1}_I A^{-1} \widehat{h}_\delta].$$

Then

$$\|g_{0,I}^\delta - g_{0,I}\|_{L^2(\mathbb{R};\mathcal{H})} \leq \eta_0^{-1} \|\mathbf{1}_I(\widehat{h}_\delta - \widehat{h})\|_{L^2(\mathbb{R};\mathcal{H})}. \quad (19)$$

In particular, log-scale inversion is stable on every frequency window on which the operator-valued multiplier is uniformly bounded away from singularity.

Proof. By Plancherel's theorem and the operator norm bound,

$$\|g_{0,I}^\delta - g_{0,I}\|_{L^2} = \|\mathbf{1}_I A^{-1}(\widehat{h}_\delta - \widehat{h})\|_{L^2} \leq \eta_0^{-1} \|\mathbf{1}_I(\widehat{h}_\delta - \widehat{h})\|_{L^2}.$$

□

7. The Matrix-Valued Case

We now specialize to the finite-dimensional case $\mathcal{H} = \mathbb{C}^m$. Then all operator-valued quantities become matrix-valued, and the abstract inversion theory reduces to a system version of SAGT.

Corollary 1 (Matrix-valued strip-analytic AGT). *Let $m \in \mathbb{N}$ and take $\mathcal{H} = \mathbb{C}^m$. If $K : (0, \infty) \rightarrow \mathbb{C}^{m \times m}$ is strip-admissible, then all preceding results hold verbatim with operator norm replaced by matrix norm. In particular, the system symbol*

$$\mathfrak{M}_K(s) = \int_0^\infty K(\xi) \xi^{-s} \frac{d\xi}{\xi} \in \mathbb{C}^{m \times m}$$

controls Mellin inversion, while the log-scale multiplier

$$A(\eta) = \frac{1}{2\pi} \mathfrak{M}_K(i\eta)$$

controls Fourier-side inversion. If $\det A(\eta) \neq 0$ on a frequency window and $\sup \|A(\eta)^{-1}\| < \infty$ there, then the system inversion is stable on that window.

Proof. In finite dimensions, Bochner integration, bounded invertibility, and Plancherel estimates reduce directly to the corresponding matrix statements. □

Corollary 2 (Matrix Wiener criterion). *Let $q \in L^1(\mathbb{R}; \mathbb{C}^{m \times m})$ and suppose that*

$$\det \widehat{q}(\eta) \neq 0 \quad \text{for every } \eta \in \mathbb{R}.$$

Then there exists $\ell \in L^1(\mathbb{R}; \mathbb{C}^{m \times m})$ such that

$$\widehat{\ell}(\eta) \widehat{q}(\eta) = 2\pi I_m \quad \text{for all } \eta \in \mathbb{R}.$$

Consequently, the Wiener–Mellin inversion theorem is automatic in the matrix-valued case under nonvanishing of the matrix multiplier.

Proof. This is the classical matrix-valued Wiener–Lévy theorem applied to the Banach algebra $L^1(\mathbb{R}; \mathbb{C}^{m \times m})$ under convolution. □

8. A Sharper Application: Gamma-Type System Kernels

To give the paper more bite, we isolate an explicit family for which the system symbol is completely computable.

Example 1 (Gamma-type system kernel family). *Let $a, b > 0$ and let $M \in \mathcal{B}(\mathcal{H})$ be boundedly invertible. Define*

$$K_{a,b,M}(\xi) := \xi^a e^{-b\xi} M, \quad \xi > 0. \quad (20)$$

Then the orbit kernel is

$$\mathcal{K}_{a,b,M}(\tau) = \int_0^\infty \xi^a e^{-b\xi} e^{i\xi\tau} \frac{d\xi}{\xi} M = \Gamma(a)(b - i\tau)^{-a} M, \quad |\Im \tau| < b,$$

and the system symbol is

$$\mathfrak{M}_{a,b,M}(s) = \int_0^\infty \xi^a e^{-b\xi} \xi^{-s} \frac{d\xi}{\xi} M = b^{s-a} \Gamma(a-s) M, \quad \Re s < a. \quad (21)$$

Proposition 3 (Explicit injectivity and stability for the Gamma family). *Fix $a, b > 0$ and let $M \in \mathcal{B}(\mathcal{H})$ be boundedly invertible. Then the AGT associated with the kernel family (20) has the following properties.*

(a) *For every vertical line $\Re u = c$ with $1 - c < a$, the Mellin symbol*

$$\mathfrak{M}_{a,b,M}(1-u) = b^{1-u-a} \Gamma(a-1+u) M$$

is boundedly invertible on that line. Hence the Mellin contour inversion is injective on the corresponding class $X_p^M(\mathcal{H})$.

(b) *For every $R > 0$, the log-scale inversion on the window $I_R = [-R, R]$ is L^2 -stable with constant*

$$C_{a,b,M,R} := 2\pi b^a \|M^{-1}\|_{\text{op}} \left(\min_{|\eta| \leq R} |\Gamma(a - i\eta)| \right)^{-1}. \quad (22)$$

More precisely, if g_{0,I_R}^δ and g_{0,I_R} are the noisy and exact frequency-windowed reconstructions, then

$$\|g_{0,I_R}^\delta - g_{0,I_R}\|_{L^2(\mathbb{R}; \mathcal{H})} \leq C_{a,b,M,R} \|\mathbf{1}_{I_R}(\hat{h}_\delta - \hat{h})\|_{L^2(\mathbb{R}; \mathcal{H})}. \quad (23)$$

Proof. Since the Gamma function has no zeros in the complex plane, the only possible obstruction to invertibility of (21) is invertibility of M , which is assumed. This proves part (a). For part (b), the multiplier is

$$A_{a,b,M}(\eta) = \frac{1}{2\pi} b^{i\eta-a} \Gamma(a - i\eta) M,$$

so

$$\|A_{a,b,M}(\eta)^{-1}\|_{\text{op}} \leq 2\pi b^a \|M^{-1}\|_{\text{op}} |\Gamma(a - i\eta)|^{-1}.$$

Taking the supremum on $|\eta| \leq R$ and applying Proposition 2 yields (23). $\square$

Remark 3. *This proposition is the concrete system-theoretic payoff of the paper. It turns the abstract operator-valued SAGT into an explicit family with a closed-form symbol and quantitative stability constants.*

9. A Compact Numerical Blueprint

The Gamma family also provides a clean numerical test-bed. Take $\mathcal{H} = \mathbb{C}^2$ and

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad K(\xi) = \xi e^{-\xi} M.$$

Then

$$A(\eta) = \frac{1}{2\pi} \Gamma(1 - i\eta) M.$$

Given a weighted signal $\Phi = (\Phi_1, \Phi_2)^\top$, the system AGT data are obtained by the log-scale convolution model. A practical inversion proceeds as follows.

- (1) Sample $h(t_j) = \mathcal{G}(e^{t_j})$ on a uniform logarithmic grid.
- (2) Compute the FFT of each component to approximate $\hat{h}(\eta)$.

- (3) Multiply by the inverse system multiplier

$$A(\eta)^{-1} = 2\pi\Gamma(1 - i\eta)^{-1}M^{-1}.$$

- (4) Apply the inverse FFT to recover $g_0(t_j) = F(e^{t_j})$.
 (5) Recover Φ and then f by inverse Fourier inversion componentwise.

Because the Gamma function has no zeros, numerical conditioning is controlled only by the growth of $|\Gamma(1 - i\eta)|^{-1}$ on the chosen frequency window and the condition number of M .

10. Discussion and Place in the Research Program

The present paper should be read as the system-valued sequel to scalar strip-analytic AGT. The scalar SAGT paper replaces discrete spectra by a continuous multiplicative spectrum. The current paper keeps that continuous architecture but allows the transform to act on coupled Hilbert-space-valued signals. The correct notion of invertibility is therefore no longer scalar nonvanishing but bounded invertibility of the system symbol.

This step is not just an abstract generalization. It changes the interpretation of the theory. The transform becomes a continuous system operator, the Mellin symbol becomes an operator-valued multiplier, and the Wiener inverse becomes a system inverse. The matrix-valued specialization shows that the general theory is already useful in finite-dimensional coupled settings, while the infinite-dimensional Hilbert case keeps the door open for later operator-theoretic developments.

The paper is intentionally conservative in one further respect. We do not attempt to prove a full noncommutative Wiener lemma in this setting from scratch; instead, the Wiener–Mellin inversion theorem is formulated under existence of an operator-valued Wiener inverse. In the matrix case this hypothesis is removed by the classical matrix Wiener criterion, while in infinite dimensions it points naturally toward the operator-valued Wiener literature.

11. Future Directions

The next stages of the program are now fairly clear.

- (1) Develop a fully intrinsic operator-valued Wiener theory adapted to the present AGT setting, so that the Wiener inverse hypothesis is derived rather than assumed.
- (2) Study multivariate strip-analytic AGT for system kernels, where both anisotropic Mellin geometry and system coupling enter simultaneously.
- (3) Connect the operator-valued strip-analytic image spaces to vector-valued de Branges or reproducing-kernel frameworks.
- (4) Develop a regularization theory for inversion near small singular values of the multiplier.

12. Conclusion

We have developed a unified operator- and matrix-valued strip-analytic extension of the Abu-Ghuwaleh transform program. Under strip-admissible orbit assumptions, the transform becomes a continuous dilation-convolution operator on the Fourier side, and its inversion is controlled by an operator-valued Mellin/log-Fourier symbol. This produces two complementary exact mechanisms: Mellin contour inversion and contour-free Wiener–Mellin inversion. The resulting theory includes explicit stability estimates away from multiplier singularities and a concrete Gamma-type family with closed-form symbol and quantitative recovery constants.

The main point may be compressed into one line:

$$\mathcal{G} = \mathcal{T}_K(\hat{\Phi}), \quad \text{and system inversion means inverting the operator-valued symbol of } \mathcal{T}_K.$$

That is the correct next step after scalar strip-analytic AGT and, more broadly, the natural operator-theoretic continuation of the post-MIT research path.

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