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Long Journey from Stevenson's Formally Complex Hypergeometric Polynomials to Real-by-Definition Romanovski-Routh Polynomials

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Abstract

The paper links Stevenson's formally complex hypergeometric polynomials to the real-by-definition Romanovski-Routh (R-Routh) polynomials. The spectral problem for Stevenson's second-order normal ordinary differential equation (NODE) was formulated in such a way that it could be re-used for the two SLPs associated with the 'trigonometric Rosen-Morse' (t-RM) potential on the finite interval and the implicit Milson potential on the line (both solvable by the R-Routh polynomials). Namely, the sought-for eigenfunction was required to represent the principal Frobenius solutions at both minus- and plus-infinity. We refer to these boundary conditions as the 'dual-PFS' problem. The exact solvability of the former SLP with the trigonometric Liouville potential was then proven by taking into account that the Romanovski-Routh polynomial of degree n must have exactly n real zeros as well as that the discrete energy spectrum in question had no upper bound. As the direct consequence of this proof, we then found that the mentioned d-PFS problem Stevenson's NODE and therefore the second SLP associated with the Milson potential on the line were exactly solvable via the quasi-rational solutions (q-RSs) composed of the R-Routh polynomials with degree-dependent indexes.

Keywords: normal ODE; Dirichlet problem; Sturm-Liouville problem; principal Frobenius solution; hypergeometric polynomials; Routh polynomials; Romanovski-Routh polynomials; trigonometric Rosen-Morse potential; Milson potential

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1. Introduction

While the classical Jacobi polynomials are the topic of numerous textbooks, there are two less known sequences of finite orthogonal hypergeometric polynomials discovered by Askey [1] and sensibly related by him to the (obscure at that time) paper by Romanovski [2]. Regretfully, his contribution to the subject was not properly appreciated either in the review article [3],which did cite Askey's breakthrough discovery , or in the following numerous studies on the potentials solvable by the Romanovski polynomials (which even did not mention his work).

Roughly speaking, Askey discovered a subset of complex Jacobi polynomials orthogonal with a complex weight and demonstrated that its real field realization results in the family of the Romanovsky polynomials associated with Pearson's [4] probability curve of type IV. Askey also re-introduced the reader to another family of the Romanovski polynomials associated with Pearson's probability curve of type VI. A few years after publication of Askey's paper [1], Lesky [5,6] termed these two families of the Romanovski polynomials as 'Romanovski/pseudo-Jacobi' and 'Romanovski/Jacobi' (R-Jacobi) polynomials and proved that they exhaust all the possible finite orthogonal real-field realization of the complex Jacobi polynomials.

Before proceeding with these introductory comments, let us clarify the terminology adopted in this series of the publications. As it has illuminated in great details in [7], the complex Jacobi

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polynomials admit two real-field reductions: the renowned real Jacobi polynomials with arbitrarily chosen indexes and the real-by-definition Routh polynomials [8] overlooked by mathematicians for a century [7]. The manifold of the real Jacobi polynomials has the orthogonal subsets of two types: the Jacobi orthogonal polynomial systems (OPS) formed by classical Jacobi polynomials and finite orthogonal sequence composed of the R-Jacobi polynomials. On the contrary, the manifold of the Routh polynomials has the finite orthogonal subset also discovered by Romanovski [2] and termed by us [9] as 'Romanovski-Routh' (R-Routh) polynomials. We use this term as the precise replacement for Lesky's more messy term 'Romanovski/pseudo-Jacobi polynomials'. From our point of view, Lesky [5,6] invented the latter simply because he was unaware of Routh' paper [8].

The alternative term 'Routh-Romanovski polynomials' often appearing in the literature is based on the erroneous assertion that Romanovski [2] rediscovered the Routh polynomials. Though it is true that Romanovski, like most mathematician of that time, was unaware of Routh' work [8], he should be credited for the discovery of the finite orthogonal subsets of both Jacobi and Routh polynomials. The fact that the Routh polynomials may not compose any OPS was possibly the main reason for them to have so little attention in mathematical literature.

As explained in Appendix A, the two hypergeometric expressions (9.9.1) in the monograph [10] for the so-called 'pseudo-Jacobi' polynomials are valid for arbitrary real values of the parameter N and therefore constitute the alternative hypergeometric representations of the Routh polynomials. For this reason, we use the two terms 'Routh' and 'pseudo-Jacobi' as the precise equivalents, making our terminology fully consistent with Lesky's [5,6] names for two other families of the Romanovski polynomials.

While the real monic polynomials specified by the first formula (9.9.1) in [10] represent the properly scaled complex Jacobi polynomials in the renowned special case of the complex conjugated Jacobi indexes and imaginary argument [1], the second formula (as demonstrated below) goes back to Stevenson's [11] sketchy note on Schrödinger's "Kepler problem" in a spherical space [12]. The cited formulas (9.9.1) thus present the little-known bridge between the two families of formally complex hypergeometric polynomials. In particular, Askey's assertion that the first family is composed of the real polynomials in the imaginary argument allowed us to confirm Stevenson's speculation that the introduced-by-him hypergeometric polynomials in the complex variable ξ (${}^{\xi}S$ in our notation) are necessarily real.

The Routh (pseudo-Jacobi) polynomials (9.9.1) turn into the R-Jacobi polynomials iff the polynomial degree does not exceed $N-\frac{1}{2}$. Again, the cited formulas under this restriction

provide the sought-for connection between the R-Jacobi polynomials defined via the Rodriguez formula [1,3] and their polynomial components of the quasi-rational eigenfunctions in [11]. There are two separate groups of papers assembled in Appendix B, which refer only to one hypergeometric representation, while very rarely even mentioning the other. To our knowledge, the interrelationship between the two representations has been never discussed in literature, except the formulas (9.9.1) which are listed in [10] without any proof or references.

The mentioned hypergeometric realizations of the R-Routh polynomials allowed the author to prove the exact solvability of the two Sturm-Liouville problems (SLPs) converted by the Liouville transformations [13] to the one-dimensional Schrödinger equations with the so-called [14,15] 'trigonometric Rosen-Morse' (t-RM) potential and with the four-parameter Milson potential on the line [16]. The present analysis corrects our earlier argumentation [9], in support of the exact solvability of the second SLP via the R-Routh polynomials. A more thorough analysis of [11] revealed some ambiguities, which made questionable our original attempt [9] to prove the exact solvability of the Milson potential [16], as well as its translationally shape invariant (TSI) limit represented by the renowned 'Gendenshtein' [17] potential (the 'Scarf II' potential in the terminology of [18]), by simply referring the reader to Stevenson's Note [11].

One of the major achievements of this paper was to formulate the spectral problem for Stevenson's second-order 'normal' [19] ordinary differential equation (*N*ODE) in such a way that it

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could be re-used for the two SLPs under consideration. It is crucial that the principal Frobenius solution (PFS) near the given singular endpoint is uniquely determined for the NODE and its reductions represented by the Sturm-Liouville equations (SLEs) of our interest. We introduce the so-called 'dual PFS (*d*-PFS) problem via the requirement that the sought-for solution represents the PFSs near both singular endpoints, assuring that the same boundary criterion is applied to the solutions of the given NODE and associated SLEs.

Since the PFS remains the solution with a larger ChExp under any linear fractional transformation (LFT) of its argument, we can easily extend the *d*-PFS boundary condition to the complex NODE expressed in terms of Stevenson's complex variable ξ_S (see Appendix C for details).

Examination of the ChExps of the NODE in question reveals that they may be both positive and therefore any solution obeys the DBC at the given singular endpoint. As a result, the Dirichlet boundary conditions (DBCs) cannot be used to distinguish the PFS from its Frobenius counterpart. The DBC unambiguously specifies the PFS iff the NODE is converted to its 'prime' form [20] such the ChExps has the same absolute value and opposite signs. Note that the latter Dirichlet problem does not have any special spectral parameter, i.e., all the three parameters are varied independently.

If λ is chosen as the spectral parameter, we come to the rational SLP (RSLP) with the trigonometric ('*t*-RM') Liouville potential. It is usually taken for granted that the n-th eigenfunction of the given singular SLP must have exactly n nodes. The author found the rigorous proof of this assertion in the treatise by Gestesy et al. [21]. It was demonstrated that the Wronskian of any two PFSs of the RCSLE in question does vanish at ±∞, which is the sufficient prerequisite for Theorem 2.1 in [21]. This proves that that the nth solution of the d-PFS problem must have exactly n-1 nodes, assuming that the eigenvalues are arranged in the monotonically increasing order.

Since the energy spectrum of the *t*-RM potential is unbounded from above, we then asserted that the given RSLP was exactly solvable by the quasi-rational solutions (q-RSs) formed by the R-Routh polynomials with degree-dependent indexes [14,15]. We then came back to the formulated d-PFS problem for Stevenson's NODE and confirmed his statement that it was exact solvable by the formally complex hypergeometric polynomials with the degree-dependent parameters.

After proving the exact solvability of the d-PFS problem for Stevenson's NODE, we chose the spectral parameter to be $l(l+1)$, instead of λ, which brought us to the SLP initiated in [16]. We proved that any eigenfunction of the Dirichlet problem formulated for the corresponding 'prime' [20] SLE (*p*-SLE) necessarily constituted a q-RS of the given d-PFS problem, with the polynomial component represented by a R-Routh polynomial. Since the latter belongs a finite sequence of orthogonal polynomials, its degree unambiguously determines the number of eigenfunction nodes.

The algebraic energy spectrum and eigenfunctions for the limiting case of the Gendenshtein potential used in Appendix D, as the simple illustration of the presented mathematical algorithm.

While the SLP initiated in [16] was reformulated as the *p*-SLE solved under the DBCs, the latter conditions were proven to be precisely equivalent to the requirement for the eigenfunctions of the given CSLE to be squarely integrable with the density function in question. This proof confirmed that any squarely integrable solution of the Schrödinger equation with the Milson potential on the line [16] has the quasi-rational form [9], with the polynomial component composed of a R-Routh polynomial. As clarified in Section 3, this is not true for the *t*-RM potential, which has the continuous spectrum (in the physical sense) at certain values of its parameters.

2. Search for d-PFSs Using the DBCs

The main purpose of this Section is to formulate the spectral problem for Stevenson's NODE (2) in [11], which we re-write as

$$\left[\frac{d^2}{dy^2} + I[y;\lambda,\mu,l]\right]\Phi^{\pm}[y;\lambda,\mu,l] = 0 \quad , \tag{1}$$

where the rational 'Stevenson' invariant

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$$\mathrm{I}[\mathrm{y};\lambda,\mu,l] := \frac{\lambda + 2\mu \mathrm{y}}{(1+\mathrm{y}^2)^2} - \frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{1+\mathrm{y}^2} \tag{2}$$

is nothing but the 'normal' [19] form of the Riemann's P-equation [22], with the poles at $\pm i$ and ∞. The characteristic exponents (ChExps) for the pole at infinity are equal to $-l$ and l+1, with the positive exponent difference (ExpDiff) of 2l+1 for $l > -\frac{1}{2}$. The subscripts ± indicate that $\Phi^{\pm}[\mathrm{y};\lambda,\mu,l]$ are the PFSs of the given NODE at ±∞ accordingly, i.e., the FSs with the common ChExp of $-l$ satisfying the boundary conditions

$$\Big| \lim_{\mathrm{y}\to\pm\infty} |\mathrm{y}|^{l} \Phi^{\pm}[\mathrm{y};\lambda,\mu,l] \Big| < \infty \, . \tag{3}$$

Note that the PFSs do not obey the DBCs at infinity if l is nonpositive, including the physically important case l=0.

The main prerequisite for our choice of the boundary conditions imposed on solutions of the NODE (1) is that the chosen boundary conditions can be easily extended to both SLPs mentioned in Introduction. We formulate these SLPs in Sections 3 and 4 by requiring the parameters λ and respectively l to specify the energy at fixed values of two other parameters, taking advantage of the fact the corresponding coefficient function is positive on the real axis.

We term any real solution $\phi_\tau[\mathrm{y};\lambda_\tau,\mu_\tau,l_\tau]$ of the NODE (1) as 'd-PDF' if it constitutes a PFS near both poles at -∞ and +∞. One of the major results of this paper is Theorem 2 in the end of Section 3, which assures that any solution of the d-PDF problem sketched in subsection 2.1 has the quasi-rational form, with the hypergeometric polynomial components discovered by Stevenson [11]. (The interconnection between the Dirichlet problem mentioned in Theorem 2 and d-PDF problem is outlined in subsection 2.2.) It will be proven in subsection 2.5 that there is an infinite sequence of the solutions $\phi_n[\mathrm{y};\lambda_n,\mu,l]$ for any value of μ and $l > -\frac{1}{2}$. These 'd-PDF solutions are then re-treated in Section 3 as being the eigenfunctions of the RSLP with the trigonometric Liouville potential. On other hand, the d-PDF solutions of the RSLP with the Liouville potential on the line (if exist at the given values of the parameters λ and μ) form a finite sequence $\phi_n[\mathrm{y};\lambda,\mu,l_n]$ ($0 \le n \le n_{max}$), as the direct consequence of the fact that the Milson potential [16] has the exponential asymptotics at both quantization ends.

Though this paper mainly focuses on the eigenfunctions of the two RSLPs, which are both composed of the R-Routh polynomials, subsection 2.4 brings reader's attention to the more general issue of the q-RSs formed by the Routh polynomials. As first noticed by Quesne [23] and then then refined by the author [9] in a more general context, these solutions can be used as the transformation functions (TFs) for the 'rational' Darboux transformations (RDTs), or, to be more precise, for the Darboux transformation (DTs) with quasi-rational TFs.

2.1. The d-PFS Problem for Second-Order ODE

As explained above, we are looking for all the solutions $\phi_\tau[\mathrm{y};\lambda_\tau,\mu_\tau,l_\tau]$ of the NODE (1) satisfying the boundary conditions

$$\Big| \lim_{\mathrm{y}\to\pm\infty} |\mathrm{y}|^{l_\tau} \phi_\tau[\mathrm{y};\lambda_\tau,\mu_\tau,l_\tau] \Big| < \infty \, , \tag{4}$$

where τ identifies the parameter values enumerating all the possible d-PFSs. It should be stressed the d-PFSs form an uncountable manifold, having the countable eigenfunctions of the SLPs ($\tau \equiv n$) as its

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either infinite or finite subsets. It will be proven in subsection 2.5 that there is an infinite sequence of the d-PFSs for an arbitrary value of μ and any positive value of the ExpDiff 2*l*+1, so (4) takes the form

$$\Big|\lim_{y\to\pm\infty} |y|^{l}\phi_n[y;\lambda_n(l,\mu),\mu,l]\Big|<\infty \qquad (5)$$

It is demonstrated in Section 3 that the boundary conditions (5) unambiguously determine the eigenfunctions of the Dirichlet problem for the *p*-SLE (66) or, which is equivalent, all the possible d-PFSs of the given family of the SLEs sharing the same canonical form (55),

The fundamentally important result of Stevenson's paper [11] not properly acknowledged in the literature is his proof that the condition (5) holds for the q-RSs composed of the hypergeometric polynomials in the complex variable:

$$\xi_S := 2/(1+i\,y) \qquad (6)$$

It is explicitly confirmed in Appendix A that the latter polynomials multiplied by the power function $(y-i)^n$ become real after been expressed in y (as anticipated by Stevenson) and, moreover, that the polynomial components of the resultant q-RSs $\phi_n[y;\lambda_n,\mu,l]$ form the infinite sequence of the R-Routh polynomials with degree-dependent indexes, which was more recently introduced by Compean and Kirchbach [14,15] to produce the quasi-rational eigenfunctions of the Schrödinger equation with the *t*-RM potential.

2.2. Reformulating the d-PFS Problem as Dirichlet Problem for p-ODE

If $-\tfrac{1}{2} < l < 0$, then the ChExps of both Frobenius solutions near the poles of the NODE (1) at infinity are positive and therefore the PFS cannot be selected anymore via the requirement that it obeys the DBC at the corresponding singular endpoint. The specified range of the parameter *l* has no significance for the Kepler problem on the sphere because it is allowed to take only nonnegative values. However, one can be more cautious, when analyzing the Darboux transforms (DTs) of the *t*-RM potential [24,25], keeping in mind that the RDT may simply eliminate the discrete energy spectrum in the quantum-mechanical sense (in violation of the conventional rules of the SUSY quantum mechanics [18]).

To be able to re-formulate the search for the PFSs as the Dirichlet problem, we converted the NODE to its 'prime' form [20] such the ChExps has the same absolute value and opposite signs. Namely, the (self-adjoint by definition) 'algebraic' [20] *p*-ODE:

$$\left\{\frac{d}{dy}\sqrt{y^2+1}\,\frac{d}{dy}-\not{q}[y;\lambda,\mu,l]\right\}\Psi_{\pm}[y;\lambda,\mu,l]=0 \qquad (7)$$

is obtained from the NODE (1) by the gauge transformation

$$\Psi^{\pm}[y;\lambda,\mu,l] = (y^2+1)^{-1/4}\,\Phi^{\pm}[y;\lambda,\mu,l], \qquad (8)$$

chosen in such a way [20] that the ChExps for the pole at infinity differ only by their signs and as a result the DBC unambiguously pinpoints the PFS:

$$\lim_{y\to\pm\infty}\Psi^{\pm}[y;\lambda,\mu,l]=0 \qquad (9)$$

In many cases it is much easier to verify that the given solution of the *p*-ODE (7) vanishes at infinity rather than to prove that it is the *d*-PFS.

It was shown in [20] that

$$\not{q}[y;\lambda,\mu,l] := -\sqrt{y^2+1}\left(\mathrm{I}[y;\lambda,\mu,l]+\mathcal{I}\{\sqrt{y^2+1}\}\right) \qquad (10)$$

where

$$\mathfrak{I}\{\rho[y]\} := \tfrac{1}{4}\dot{\rho}^{2}[y]/\rho[y] - \tfrac{1}{2}\ddot{\rho}[y], \tag{11}$$

with dot standing for the derivative with respect to y. One can easily verify that [7]

$$\sqrt{y^2+1}\,\mathfrak{I}\{\sqrt{y^2+1}\} = \frac{3}{4(y^2+1)^{3/2}} - \frac{1}{4(y^2+1)^{1/2}}, \tag{12}$$

which, coupled with (2), gives

$$\not{q}[y;\lambda,\mu,l] = -\frac{\lambda + 3/4 + 2\mu y}{(1+y^2)^{3/2}} + \frac{(l+1/2)^2}{(1+y^2)^{1/2}}. \tag{13}$$

2.3. PFSs of NODE (1) Near Poles at Infinity

The purpose of this subsection is to use Stevenson's complex change of variable (6) to obtain the hypergeometric representation for the generic PFS of the NODE (1) near each pole at -∞ and at +∞. This problem (while lying out of scope of this paper) deserves a separate attention, keeping in mind that the PFSs below the lowest eigenvalue are necessarily nodeless and therefore can be used (at least in theory) as the TFs for the DTs of both *t*-RM and Milson potentials. We refer the reader to the papers of Ferandez et al. [24,25], where this issue is discussed in great details for the *t*-RM potential.

As mentioned in Appendix C, we are only interested in the root of the quadratic equation (A47),

$$\alpha(\lambda,\mu) \equiv \alpha_R(\lambda,\mu) + \alpha_I(\lambda,\mu)\, i := -\tfrac{1}{2}\sqrt{\lambda+1+2\mu\, i} + \tfrac{1}{2}, \tag{14}$$

such that

$$\alpha_R(\lambda,\mu) < \tfrac{1}{2}. \tag{15}$$

Such a choice of the root assures that the hypergeometric series in the right-hand side of (A52) converges on the unit circle, making possible the analytical continuation of the given solution beyond the circle.

Keeping in mind that

$$2\alpha_I(\lambda,\mu) = \mu / [1 - 2\alpha_R(\lambda,\mu)], \tag{16}$$

one finds

$$\lambda = [\alpha_R(\lambda,\mu) - \tfrac{1}{2}]^2 - \mu^2 / [\alpha_R(\lambda,\mu) - \tfrac{1}{2}]^2. \tag{17}$$

If $\mu \neq 0,$ the second summand in the sum (17) monotonically increases from -∞ to 0 as α_R varies from 0 to -∞. Consequently, the sum is a monotonically decreasing function of α_R, changing from +∞ to -∞ as α_R varies from -∞ to 0. In the 'edge' case $\mu = 0$ the function (26) changes from -1 to +∞, as α_R varies from 0 to -∞, and as a result, the solutions of our interest do not exist for negative values of λ.

Expressing the PFS (A52) in terms of y, we come to the following solutions of the NODE (1)

$$\Phi^{\pm}[y;\lambda,\mu,l] \propto \sqrt{\frac{dy}{d\xi}}\, \underset{\sim}{\Phi}[2/(1+i\,y);\alpha(\lambda,\mu),l] \quad \text{for } \pm y \geq \sqrt{3}, \tag{18}$$

where we took advantage of the fact that the hypergeometric series in the right-hand-side of (A52) converges within the unit circle $|\xi_S| \leq 1$. As the direct consequence of (C5), with ξ_S given by (6), one finds

$$\sqrt{\frac{dy}{d\xi_S}} = e^{\frac{1}{4}\pi i}(1+i\,y)/\sqrt{2}. \tag{19}$$

Substituting (6) and

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$$\xi_S - 1 = \frac{1 - i\,y}{1 + i\,y} \tag{20}$$

into the power functions of ξ_S and $\xi_S - 1$ in the right-hand side of (A52), coupled with (19), one finds

$$\Phi^{\pm}[y;\lambda,\mu,l] = N^{\pm}(\lambda,\mu,l)\,(1+iy)^{-l-\alpha(\lambda,\mu)}\,(1-i\,y)^{\alpha(\lambda,\mu)}\times$$
$$F\big(2\alpha_R(\lambda,\mu)+l,\ l+1-2\alpha_I(\lambda,\mu)i, 2l+2; 2/(1+i\,y)\big) \quad \text{for } \pm y \geq \sqrt{3}, \quad (21)$$

where the multiplier $N^{\pm}(\lambda,\mu,l)$, as clarified below, is introduced to make the solution (21) real in the limits $y\to\pm\infty$. The derived expression is reminiscent of (3) in [11], with x standing for y here, except that we replaced the complex powers of $y-i$ and $y+\imath$ for the well-defined complex powers of $1+ i\,y$ and $1- iy$ accordingly. Indeed, bearing in mind that

$$\frac{1-i\,y}{1+i\,y} = exp(-2i\,arctan\,y)\,, \tag{22}$$

we can rewrite (21) as

$$\Phi^{\pm}[y;\lambda,\mu,l] = N^{\pm}(\lambda,\mu,l)(1+y^2)^{\alpha_R(\lambda,\mu)}(1+i\,y)^{-l-2\alpha_R(\lambda,\mu)}\,exp[2\alpha_I(\lambda,\mu)\,arctan\,y]$$
$$\times F\big(2\alpha_R(\lambda,\mu)+l,\ l+1+i\,2\alpha_I(\lambda,\mu), 2l+2; 2/(1+i\,y)\big) \quad \text{for } \pm y \geq \sqrt{3}, \ (23)$$

Next, taking into account that

$$(1+i\,y)^{\beta_R} := (1+y^2)^{\frac{1}{2}\beta_R}\,exp[i\,\beta_R\,arctan\,y] \tag{24}$$

for any real β_R, we confirm that the formally complex pretendents for the real-by-definition PFSs near ±

$$\Phi^{\pm}[y;\lambda,\mu,l] = N^{\pm}(\lambda,\mu,l)(1+y^2)^{-\frac{1}{2}l}\,exp[2\alpha_I(\lambda,\mu)\,arctan\,y - \tfrac{1}{2}l - \alpha_R(\lambda,\mu)]$$
$$\times F\big(2\alpha_R(\lambda,\mu)+l,\ l+1+i\,2\alpha_I(\lambda,\mu), 2l+2; 2/(1+i\,y)\big) \quad \text{for } \pm y \geq \sqrt{3} \ \ (25)$$

vanishes at infinity as $|y|^{-l}$. We then set

$$N^{\pm}(\lambda,\mu,l) := exp\Big[\pm i\,\tfrac{1}{2}\pi\big(l+2\alpha_R(\lambda,\mu)\big)\Big], \tag{26}$$

which assures that

$$\lim_{y\to\pm\infty}\Big(|y|^{l}\,\Phi^{\pm}[y;\lambda,\mu,l]\Big) = \pm 1. \tag{27}$$

As anticipated the solutions (8) of the p-ODE (7),

$$\Psi^{\pm}[y;\lambda,\mu,l] = N^{\pm}(\lambda,\mu,l)(1+y^2)^{-\frac{1}{2}(l+\frac{1}{2})}\,exp[2\alpha_I(\lambda,\mu)\,arctan\,y - \tfrac{1}{2}l - \alpha_R(\lambda,\mu)]$$
$$\times F\big(2\alpha_R(\lambda,\mu)+l,\ l+1+i\,2\alpha_I(\lambda,\mu), 2l+2; 2/(1+i\,y)\big) \quad (\pm y \geq \sqrt{3}), \ (28)$$

vanish at ±∞ accordingly for any $l > -\tfrac{1}{2}$.

Proposition 1: *The functions (25) and consequently (28) are real and therefore constitute the (real by definition) PFSs of the NODE (1) and respectively p-ODE (7) at ±∞ for any real y.*

This assertion was recently made in [24,25], but I could not independently confirm this statement (see Appendix B below for a detailed discussion of this issue). For this reason, I decided to formulate this statement as the proposition (at least for now), hoping that a more thorough analysis would confirm it.

On other hand, as demonstrated below, Proposition 1 certainly holds for the hypergeometric polynomials obtained by the truncation of the given series. However, its proof is based on the very specific feature of the Jacobi polynomials with complex conjugate indexes, restricted to the imaginary argument [1,26] (see Appendix A below for details) and cannot be easily extended to the formally complex functions (25) or (28).

2.4. Pairs of Real q-RSs Formed by Routh Polynomials

As pointed out by Stevenson [11], the hypergeometric series in the right-hand side of (25) terminate if

$$\tilde{\alpha}_{n;R}(l,\mu) := \alpha_R\left(\lambda_n(l,\mu),\mu\right) = -\tfrac{1}{2}(l+n). \tag{29}$$

Setting

$$\lambda_n(l,\mu) := (l+n+1)^2 - \frac{\mu^2}{(l+n+1)^2} - 1, \tag{30}$$

$$N^{\pm}\left(\lambda_n(\mu,l),\mu,l\right) = (\pm i)^{-n}, \tag{31}$$

and

$$\tilde{\alpha}_{n;I}(l,\mu) := \alpha_I\left(\lambda_n(l,\mu),\mu\right) = \frac{\mu}{2(l+n+1)}, \tag{32}$$

we can represent both PFSs in the common quasi-rational form:

$$\pm\Phi^{\pm}[y;\lambda_n(l,\mu),\mu,l] \equiv \tilde{\phi}_n[y;\tilde{\alpha}_n(l,\mu)], \tag{33}$$

where

$$\tilde{\phi}_n[y;\tilde{\alpha}] := (y^2+1)^{\tilde{\alpha}_R}\, exp(2\tilde{\alpha}_I\, arctan\ y) P_n\left(y; 2\tilde{\alpha}_I, -2\tilde{\alpha}_R\right) = \tilde{\phi}^*_n[y;\tilde{\alpha}] \tag{34}$$

are the quasi-rational functions with the polynomial components defined via (A3) in Appendix A below.

We approached the important milestone of this paper: the proof that the monic polynomials (A3) in y are real. While directing the reader to Appendix A for details, let us only mention that our proof (for any real values of the parameter N) is essentially based on the equivalence of the formally complex hypergeometric expressions (9.9.1), listed in [10] for nonnegative integers N, without any proof or supporting references. On other hand, the author is unaware of any study discussing the link between these two formally complex hypergeometrical formulas for the Routh polynomials (in our terms), converted to the monic form.

As pointed to in Introduction and further clarified in Appendix A, we use this term as the precise equivalent for the term 'pseudo-Jacobi polynomials', initiated in [10], with no limitations of the real parameter N. The monic Routh polynomials in question turn into the monic R-Routh polynomials introduced via (A14) in Appendix A and detailed in the following subsection with q-RSs of the Dirichlet problem formulated in subsection 2.2.

It is important that each eigenfunction is accompanied by the real q-RS

$$\tilde{\phi}_n[y;\tilde{\alpha}_n(-l-1,\mu)] := (y^2+1)^{\frac{1}{2}(l+1)}\, exp[2\,\tilde{\alpha}_{n;I}(-l-1,\mu)\, arctan\ y] \times$$
$$(y^2+1)^{-\frac{1}{2}n} P_n\left(y; 2\tilde{\alpha}_{I;n}(-1-l,\mu), n-1-l\right). \tag{35}$$

It was proven in [7,9] that, as originally speculated by Quesne [23], there exits the q-RSs (35) composed the Routh polynomials with no real zeros , which can be then used as the TFs for constructing new exactly solvable rational CSLEs. A brief outline of these solutions is presented in Section 5 below, but this is supposed to be the subject of a separate publication.

2.5. Infinite Sequences of Quasi-Rational d-PFSs Expressible in Terms of R-Routh Polynomials

By applying the gauge transformation (8) to the q-RSs (33) with the real part of the parameter $\tilde{\alpha}_n(l,\mu)$ given by (25), we find that the solutions of the *p*-SLE (7):

$$\psi_n[y;\mu,l]:=(y^2+1)^{-1/4}\tilde{\phi}_n[y;\tilde{\alpha}_n(l,\mu)]$$
$$=(y^2+1)^{-\frac{1}{2}(l+\frac{1}{2})}exp[2\tilde{\alpha}_{n;I}(\mu,l)\arctan y$$
$$\times(y^2+1)^{-\frac{1}{2}n}P_n\left(y;2\tilde{\alpha}_{n;I}(\mu,l),n+l\right)\quad\text{for}\quad l>-\tfrac{1}{2}. \qquad (36)$$

satisfy the DBCs at the both limits y→-∞ and y→+∞ iff $l > -\frac{1}{2}$. This implies that the q-RSs $\tilde{\phi}_n[y;\tilde{\alpha}_n(l,\mu)]$ represent the d-PFSs $\phi_n[y;\lambda_n(l,\mu),l,\mu]$ introduced in subsection 2.1.

As originally pointed to by Cryer [26] and later exploited in great details by Askey [1] (though with no mentioning of Cryer's contribution to this subject), the polynomials

$$\mathfrak{R}_n^{(\boldsymbol{\lambda})}(y):=(-i)^n\,P_n^{(\boldsymbol{\lambda},\boldsymbol{\lambda}^*)}(iy) \qquad (37)$$

represent one of the three families of the Romanovsky polynomials [4] if the polynomial degree is restricted from above by the constraint

$$0\le n<-\boldsymbol{\lambda}_R-\tfrac{1}{2}, \qquad (38)$$

with $\boldsymbol{\lambda}$ standing for $a + bi$ in [1].

In following [3,14,15,23,27], we [7,20] reserve the letter R solely for the R-Routh polynomials defining them via (3.5) and (3.6) in [23]:

$$R_n^{(2\boldsymbol{\lambda}_I,\boldsymbol{\lambda}_R+1)}(y):=\mathfrak{R}_n^{(\boldsymbol{\lambda}_R+\boldsymbol{\lambda}_I i)}(y)\quad\text{for n=0, …,}\left\lfloor-\boldsymbol{\lambda}_R-\tfrac{1}{2}\right\rfloor. \qquad (39)$$

(It is worth mentioning that Quesne's notation [23] for the R-Routh polynomials differs from those in [3,14,15,27].) On other hand, Askey [1] uses the notation $R_n^{(\alpha,\beta)}(x)$ for the R-Jacobi, not R-Routh polynomials. As mentioned in the end of the previous subsection and proven in Appendix A, the monic R-Jacobi polynomials can be represented in the hypergeometric form

(A14), assuming that their degrees are smaller than $N+\frac{1}{2}\equiv-\boldsymbol{\lambda}_R-\frac{1}{2}$.

Examination of the inequality

$$-\tilde{\alpha}_{n;R}(l,\mu)-\tfrac{1}{2}-n=l+\tfrac{1}{2}>0\quad\left(n=0,1,\,…,\left\lfloor-\boldsymbol{\lambda}_R-\tfrac{1}{2}\right\rfloor\right) \qquad (40)$$

reveals that the polynomials (39) form the infinite sequence of the R-Routh polynomials with the degree-dependent indexes — the astonishing fact disclosed in Compean and Kirchbach's enlightening study [14] on the eigenfunctions of the '*t*-RM' potential. The cited authors also presented the overview of the literature on the mathematical grounds of Romanovski's [2] finite sequences of the orthogonal polynomials mostly unknown to physicists. This list was then largely expanded in [3].

Confusingly, the term 'pseudo-Jacobi polynomials' was initiated in [10] via (9.9.1) only for nonnegative integers N. Later Koornwinder [28], in his additions to the monograph [10], extended the range of definition for this parameter to any real values larger then $-\frac{1}{2}$, while misleadingly referring to these polynomials as 'pseudo-Jacobi' or 'Romanovski-Routh' polynomials.

As asserted by Stevenson, the solution of his Eq. (3) in [11] is discontinuous at y = 0 along the real axis, unless the hypergeometric series is truncated into a polynomial. He stated that this result was proven by him, using the analytical continuation of the given solution beyond the unit circle $|\xi_S|=1$, with reference to Whittaker and Watson's treatise [23]. In Appendix C we analyzed the restrictions on the complex variable ξ_S for the four possible analytical continuations given by (5.3.6)-

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(5.3.9) in [29] and found that none of them (at least according to our understanding of this issue) can be used to justify Stevenson's assertion.

As mentioned in Introduction, we will prove the exact solvability of the *d*-PFS problem under consideration, after re-formulating the Dirichlet problem for the *p*-ODE (7) as the SLP, with λ standing for the spectral parameter. We then make advantage of Gesztesy et al.'s [21] results to prove that the constructed quasi-rational eigenfunctions exhaust all the possible solutions of the formulated SLP.

As for now, let us simply notice that truncating (C14) via (29) brings us to the new hypergeometric realization for the monic pseudo-Jacobi polynomials:

$$P_n(y;2\alpha_I,-2\alpha_{R;n})=\frac{\langle 1-2\alpha-n\rangle_n}{\langle 2-2n-4\alpha_R\rangle_n}(y-i)^n\,F(-n,-n-\alpha,\ \alpha^*+1;(y+i)/(y-i)) \tag{41}$$

with

$$\langle\nu\rangle_n=\frac{\Gamma(\nu+n)}{\Gamma(\nu)} \tag{42}$$

standing for the rising factorial [30].

3. Exact Solvability of the t-RM Potential

Keeping in mind that the coefficient function of λ in the NODE (1) is positive for any real y, let us rewrite (1) in the form of the CSLE

$$\left[\frac{d^2}{dy^2}+I^o[y;\mu,l]+\lambda\,\rho_S[y]\right]\Phi[y;\lambda,\mu,l]=0\quad (l>-\tfrac{1}{2}) \tag{43}$$

with the reference polynomial fraction

$$I^o[y;\mu,l]:=I[y;0,\mu,l] \tag{44}$$

and the density function

$$\rho_S[y]:=(1+y^2)^{-2}\ , \tag{45}$$

such that the change of variable (A13) in Appendix B converts it into the Schrödinger equation with the t-RM potential [31]. Here we use 'circle' as the subscript to indicate that we deal with the reference polynomial fraction (RefPF) but not with 'Bose invariant' [16].

$$I[y;\lambda,\mu,l]=I^o[y;\mu,l]+\lambda\rho_S[y]\,.$$

It will be proven that the q-RSs

$$\tilde{\phi}_n[y;\tilde{\alpha}_n(l,\mu)]=N_n(l,\mu)(y^2+1)^{-\frac{1}{2}l}\,exp[2\tilde{\alpha}_{n;I}(\mu,l)\,arctan\ y]$$
$$\times(y^2+1)^{-\frac{1}{2}n}P_n\left(y;2\tilde{\alpha}_{n;I}(l,\mu),-2\tilde{\alpha}_{n;R}(l,\mu)\right) \tag{46}$$

for n=0,…, $\left\lfloor -2\tilde{\alpha}_{n;R}(l,\mu)+\tfrac{1}{2}\right\rfloor$,

with the eigenvalues (30) constitute all the possible eigenfunctions $\phi_\tau[y;l,\mu]$ of the CSLE

$$\left[\frac{d^2}{dy^2}+I^o[y;0,\mu,l]+\lambda_\tau(l,\mu)\rho_S[y]\right]\tilde{\phi}_\tau[y;\tilde{\alpha}_n(l,\mu)]=0\qquad (l>-\tfrac{1}{2}) \tag{47}$$

selected via the boundary conditions (5) re-written as

$$\left|\lim_{y\to\pm\infty}|y|^l\,\phi_n[y;\tilde{\alpha}_n(l,\mu)]\right|<\infty\qquad (l>-\tfrac{1}{2}) \tag{48}$$

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The crucial point for our analysis is that the Wronskian of two eigenfunctions defined in such a way vanishes at both endpoints:

$$0<\lim_{y\to\pm\infty}\left|\,|y|^{2l+1}\,W\left\{\tilde{\phi}_{n''}[y;\tilde{\alpha}_{n''}(l,\mu)],\tilde{\phi}_{n'}[y;\tilde{\alpha}_{n'}(l,\mu)]\right\}\right|<\infty\quad(n''\neq n'),\tag{49}$$

iff $l>-\frac{1}{2}$, which turns out to be the sufficient prerequisite for the intertwining theorem for zeros of two sequential eigenfunction (see Corollary 2.3 in [21] for the more general formulation of this criterion). As the direct consequence of the intertwining theorem, we assert that the (n+1)-th eigenfunction of the CSLE must have exactly n nodes.

Moreover, based on Theorem 2.1 in [21], we assert that the eigenfunctions (46) are orthogonal with the weight (45):

$$\int_{-\infty}^{+\infty}\tilde{\phi}_{n''}[y;\tilde{\alpha}_{n''}(l,\mu)],\tilde{\phi}_{n'}[y;\tilde{\alpha}_{n'}(l,\mu)]\rho_S[y]dy=0\qquad(n'\neq n).\tag{50}$$

But here is a catch: the indexes of the R-Routh polynomials forming the eigenfunctions (46) depend on the polynomial degrees. As a result, the normalization condition (A12) listed in Appendix A (with n′=n″) can not be used to compute the eigenfunction normalization factors.

As clarified in Appendix B, to be useful for constructing the normalized (in the physical sense) eigenfunctions of the Schrödinger equation with the t-RM potential, the normalization of the eigenfunctions (46) must be performed with the same weight (45):

$$\int_{-\infty}^{+\infty}\left|\tilde{\phi}_n[y;\tilde{\alpha}_n(l,\mu)]\right|^2\rho_S[y]dy=1.\tag{51}$$

In other words,

$$N_n^{-2}(l,\mu)=\int_{-\infty}^{+\infty}(y^2+1)^{2\tilde{\alpha}_{n;R}(\mu,l)-2}\,exp[4\tilde{\alpha}_{n;I}(\mu,l)\arctan y]\tag{52}$$

$$P_n^2\left(y;2\tilde{\alpha}_{n;I}(l,\mu),-2\tilde{\alpha}_{n;R}(l,\mu)\right)dy\quad\text{for } n=0,\ldots,\left\lfloor-2\tilde{\alpha}_{R;n}(l,\mu)+\tfrac{1}{2}\right\rfloor,$$

which differs from the normalization of the monic R-Routh polynomials via (A12) in Appendix A.

Theorem 1. *The formulated SLP is exactly solvable* via *the q-RSs (46).*

Proof of Theorem 1. Since the monic R-Routh polynomial of degree n in the right-hand side of (46) has exactly n-1 real zeros, the SLP in question may not have any solution between the eigenvalues $\lambda_{n-1}(l,\mu)$ and $\lambda_n(l,\mu)$ for $l>-\frac{1}{2}$. Keeping in mind that the given discrete energy spectrum is unbounded from above, we assert that the q-RSs (46) exhaust all the possible *d*-PFSs of the CSLE (43). □

By making the gauge transformations

$$\psi_n[y;l,\mu]=\rho^{-\frac{1}{2}}[y]\tilde{\phi}_n[y;\tilde{\alpha}_n(l,\mu)]\tag{53}$$

with

$$\rho[y]=\sqrt{y^2+1},\tag{54}$$

we come to the *p*-SLE

$$\left\{\frac{d}{dy}\sqrt{y^2+1}\frac{d}{dy}-q_S[y;l,\mu]+\lambda_n(l,\mu)w[y]\right\}\psi_n[y;l,\mu]=0\tag{55}$$

with the free-term and weight functions

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$$\not\phi_S[y;l,\mu] := \not\phi[y;0,\mu,l] = -\frac{3/4 + 2\mu y}{(1+y^2)^{3/2}} + \frac{(l+1/2)^2}{(1+y^2)^{1/2}} \tag{56}$$

and

$$\psi[y] := \frac{1}{(1+y^2)^{3/2}} > 0 \tag{57}$$

respectively. By definition the ChExps for its poles at ±∞ have the same absolute value $l + 1/2 > 0$ and the opposite signs, i.e., the PFSs are unambiguously determined by the DBCs

$$\lim_{y\to\pm\infty} \psi_n[y;l,\mu] = 0. \tag{58}$$

Corollary 1. *The Dirichlet problem for the p-SLE (55) is exactly solvable* via *the q-RSs (53).*

We are finally ready to prove one of the major results of this paper.

Theorem 2. *The Dirichlet problem for the p-ODE (7) is exactly solvable* via *the q-RSs (36).*

Proof of Theorem 2. Suppose that the Dirichlet problem formulated in Section 2 has a solution $\phi_\tau[y;\lambda_\tau,\mu_\tau,l_\tau]$ with $\lambda_\tau \neq \lambda_n(l_\tau,\mu_\tau)$ for $l_\tau \in (-1/2, \infty)$ and any nonnegative integer n. Then it must be an eigenfunction of the *p*-SLE (55) for $l = l_\tau$, $\mu = \mu_\tau$, which implies (as the direct consequence of Theorem 1) that λ_τ must coincide with one of the eigenvalues (30) at the specified values of the parameters l and μ. However, this conclusion contradicts the assumption that $\lambda_\tau \neq \lambda_n(l_\tau,\mu_\tau)$ for any $n \geq 0$. □

We reached one of the major milestones of this paper:

Corollary 2. *The d-PFS problem for the NODE (1) is exactly solvable* via *the q-RSs (53).*

In next Section we make use of Theorem 2 to prove that the Milson potential on the line [16] is exactly solvable in terms of the d-PFSs of the NODE (1).

4. A Family of SLPs with a Finite Number of Eigenfunctions Expressible via R-Routh Polynomials with Degree-Dependent Indexes

Keeping in mind that the coefficient function of $l(l+1)$ in the NODE (1) is positive for any real y, one can, alternatively to (43), rewrite (1) as the Routh-reference (RRef) CSLE [16]

$$\left[\frac{d^2}{dy^2} + {}_i\mathrm{I}[y;h_o;\kappa;E]\right]\Phi^{\pm}_{M}[y;h_o;\kappa;E] = 0 \tag{59}$$

where the Bose invariant

$${}_i\mathrm{I}[y;h_o;\kappa;E] := {}_i\mathrm{I}^o[y;h_o] + E\rho_M[y;\kappa] \tag{60}$$

is formed by the Routh RefPF (RRefPF)

$${}_i\mathrm{I}^o[y;h_o] := -\frac{h_o}{4(y-i)^2} - \frac{h_o^*}{4(y+i)^2} + \frac{2h_{o;R}+1}{4(y^2+1)} \tag{61}$$

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$$=\frac{h_{o;R}+h_{o;I}y}{(y^2+1)^2}+\frac{1/4}{y^2+1} \tag{62}$$

$$=I[y;h_{o;R},\tfrac{1}{2}h_{o;I},-\tfrac{1}{2}] \tag{63}$$

and the density function

$$\rho_M[y;\kappa]:=\frac{{}_iT_2[y;1,\kappa]}{(1+y^2)^2}. \tag{64}$$

The complex parameter h_o and its complex conjugate h_o^* used to parametrize the RefPF (61) specify the zero-energy ExpDiffs ${}_i\lambda_o$ and ${}_i\lambda_o^*$ for the poles of the CSLE (59) at $-i$ and $+i$ respectively [9], namely,

$$h_o+1={}_i\lambda_o^2. \tag{65}$$

The second-degree tangent polynomial (TP)

$${}_iT_2[y;a_2,\kappa_R+i\kappa_I]\equiv a_2(y^2+1)+\kappa_I\,y+\kappa_R-1 \tag{66}$$

with a_2 set to 1 is assumed to stay positive on the real axis. The reference energy point dictated by the last term in (61) is chosen in such a way that the ExpDiff for the pole of the CSLE at infinity vanishes at E=0.

In his pioneering work [16] Milson examined in depth the limiting case of the even TP (*e*-TP) for real (positive) $\kappa=\kappa_+\ (\kappa_I=0)$. The *e*-TP version of the Milson potential was later rediscovered by Lévai [32], who showed that its eigenfunctions are expressible in terms of the Jacobi polynomials with complex-conjugated indexes in the imaginary argument *iy*. while the energy spectrum is determined by some real roots of the quartic equation.

This was the essential step forward, compared with [16], since Milson was able only to derive the implicit spectral equation for the eigenvalues of the given SLP by truncating the formally complex solutions. He also provided the explicit formula for the number of the bound energy levels. However, no attempt was made to prove that the resultant q-RSs are real. It turned out that the truncated hypergeometric series used in [16] to derive the spectral equation was nothing but the hypergeometric expression suggested by Askey [1] for the Jacobi polynomials (37) in the imaginary argument *iy*.

For the *e*-TP version of the Milson potential [16] this issue was scrutinized by the author [9], though (regeretfully0 with no mentioning of Lévai's contribution [32] to this subject (see [33] for the detailed analysis of the relationship between the cited approaches). The thorough analysis of the general case: $\kappa=\kappa_+ + i\kappa_I\ (\kappa_I\neq 0)$ was presented in [7].

Introducing the energy-dependent quantities

$$l(E):=-\tfrac{1}{2}+\sqrt{-E}>-\tfrac{1}{2}, \tag{67}$$

and

$$h(h_o;\kappa;E)\equiv h_R(h_o;\kappa;E)+i\,h_I(h_o;\kappa;E):=h_o+(\kappa-1)E, \tag{68}$$

we represent the Bose invariant (60) as

$${}_iI[y;h(h_o;\kappa;E);E]=\frac{h_R(h_o;\kappa;E)+h_I(h_o;\kappa;E)y}{(y^2+1)^2}-\frac{[l(E)+\tfrac{1}{4}]^2-\tfrac{1}{4}}{y^2+1}, \tag{69}$$

which brings us to the Stevenson invariant (2) with

$$\lambda=h_R\left(h_o;\kappa;E\right),\ \mu=\tfrac{1}{2}h_I\left(h_o;\kappa;E\right), \tag{70}$$

and

$$l \equiv l(E) \tag{71}$$

Again. we choose the square root of the complex e-dependent quantity (68):

$$\underset{\sim}{\alpha}(h_o;\kappa;l) \equiv \underset{\sim}{\alpha}_R(h_o;\kappa;l) + \underset{\sim}{\alpha}_I(h_o;\kappa;l)i := {}^1\!/_2\sqrt{h(h_o;\kappa;E(l))+1} + {}^1\!/_2 \tag{72}$$

in such a way that

$$\underset{\sim}{\alpha}_R(h_o;\kappa;l) < {}^1\!/_2 . \tag{73}$$

If Proposition 1 holds, then the solutions (25) with λ, μ, and l defined via (70) as functions of E represent the PFSs of the CSLE (59) near its poles at ±∞ at the energy E.

Let $E_n(h_o;\kappa)$ be the (n+1)-th eigenvalue of the CSLE (59) and

$$l_n(h_o;\kappa) := -{}^1\!/_2 + \sqrt{-E_n(h_o;\kappa)} > -{}^1\!/_2 , \tag{74}$$

Regardless of the validity of this Proposition, the q-RS (33),

$$\tilde{\phi}_n[y;\underset{\sim}{\alpha}_n(h_o;\kappa)] = (y^2+1)^{-{}^1\!/_2 l_n(h_o;\kappa)} exp(2\underset{\sim}{\alpha}_{n;I}(h_o;\kappa)\, arctan\ y) ,$$

$$\times(y^2+1)^{-{}^1\!/_2 n} P_n\left(y; 2\underset{\sim}{\alpha}_{n;I}(h_o;\kappa), -2\underset{\sim}{\alpha}_{n;R}(h_o;\kappa)\right) , \tag{75}$$

where

$$\underset{\sim}{\alpha}_n(h_o;\kappa) \equiv \underset{\sim}{\alpha}_{n;R}(h_o;\kappa) + \underset{\sim}{\alpha}_{n;I}(h_o;\kappa)i := \underset{\sim}{\alpha}\left(h_o;\kappa;l_n(h_o;\kappa)\right) \tag{76}$$

constitutes the (real by definition) d-PFS of the CSLE (59) with the exactly n nodes, assuming that the real part of (72) obeys the constraint (29). i.e.,

$$\underset{\sim}{\alpha}_{n;R}\left(h_o;\kappa\right) = -{}^1\!/_2[l_n(h_o;\kappa)+n] < {}^1\!/_2 . \tag{77}$$

Substituting

$$\underset{\sim}{\alpha}_{n;I}\left(h_o;\kappa\right) = \frac{h\left(h_o;\kappa;E_n l_n(h_o;\kappa)\right)}{2[l_n(h_o;\kappa)+n+1]} \tag{78}$$

into the real part of the E-dependent quantity (68),

$$h_R\left(h_o;\kappa;E_n(h_o;\kappa)\right) = [l_n(h_o;\kappa)+n+1]^2 - \underset{\sim}{\alpha}^2_{n;I}(h_o;\kappa) - 1 , \tag{79}$$

brings us to the following quartic equation

$$[l_n(h_o;\kappa)+n+1]^2\left\{{}^1\!/_4[l_n(h_o;\kappa)+n+1]^2 - h_{o;R} - (\kappa_R-1)[l_n(h_o;\kappa)+{}^1\!/_2]^2\right\}$$

$$= {}^1\!/_4\{h_{o;I} - \kappa_I[l_n(h_o;\kappa)+{}^1\!/_2]^2\} \tag{80}$$

with respect to with respect to $l_n(h_o;\kappa)$, or, after replacing $l_n(h_o;\kappa)+{}^1\!/_2$ for Λ_n,

$$(\kappa_R - {}^1\!/_4\kappa_I^2)(\Lambda_n+n+{}^1\!/_2)^4 - {}^1\!/_4(h_{o;I}-\kappa_I\Lambda_n^2)^2 - [h_{o;R}+1-\Lambda_n^2(\kappa_R-1)](\Lambda_n+n+{}^1\!/_2)^2 = 0 . \tag{81}$$

Since the TP (66) does not real roots, a positive root Λ_n necessarily exists if the given quartic equation has the negative free term:

$$(n+{}^1\!/_2)^4 - (h_{o;R}+1)(n+{}^1\!/_2)^2 - {}^1\!/_4 h^2_{o;I} < 0 \quad (n = 0,1,\ldots,n_{max}) . \tag{82}$$

If $h_{o;I} \neq 0$, the quadratic polynomial

$$X^2 - (h_{o;R}+1)X - {}^1\!/_4 h^2_{o;I} \tag{83}$$

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has the positive leading coefficient and the negative free terms, its roots must have opposite signs and therefore it is negative iff X lies between the roots. Since X is allowed to take only positive values, this implies that the polynomial (83) is negative as far as X is smaller than the positive root, i.e., iff

$$0 \le n \le n_{max} < \tfrac{1}{2}\sqrt{2(h_{o;R}+1)+2\,|\,h_o+1\,|} - \tfrac{1}{2}. \tag{84}$$

Note that the derived upper bound for the number of the bound energy levels is independent of κ and therefore matches the one obtained in [16] for $\kappa = \kappa_R \;(\kappa_I = 0)$, with

$$h_o + 1 = -b + ic \tag{85}$$

If $h_{o;I} = 0$ then the inequality (82) takes the form:

$$(n+\tfrac{1}{2})^2 < h_{o;R} + 1 \;\; (n = 0, 1, \ldots, n_{max}), \tag{86}$$

with the right-hand side positive due to (65), i.e.,

$$0 \le n \le n_{max} = \left\lfloor \sqrt{h_o + 1} - \tfrac{1}{2} \right\rfloor \;\; (h_{o;I} = 0) \tag{87}$$

The algebraic energy spectrum for the translationally form-invariant (TFI) limit [34] of the CSLE (59), converted by the Liouville transformation [13] to the Schrodinger equation with the Gendenshtein (Scarf II) potential [17,27] is discussed in Appendix D.

Lemma 1. *The CSLE (59) has no d-PFSs below the energy* $E_{n_{max}}(h_o;\kappa)$.

Proof of Lemma 1.

Again we take advantage of Corollary 2.3 in [21], bearing in mind that the Wronskian of two eigenfunctions

$$
\begin{aligned}
{}_i\phi_n[y;h_o;\kappa] = {}_i N_n(h_o;\kappa)(y^2+1)^{-\frac{1}{2}l_n(h_o;\kappa)}\, exp[2\,\underset{\sim}{\alpha}_{I;n}(h_o;\kappa)\, arctan\; y] \times \\
(y^2+1)^{-\frac{1}{2}n}\, P_n\left(y; 2\,\underset{\sim}{\alpha}_{I,n}(h_o;\kappa), n + l_n(h_o;\kappa)\right)
\end{aligned} \tag{88}
$$

vanishes at both endpoints:

$$0 < \lim_{y\to\pm\infty} \left| \,|\,y\,|^{l_{n'}(h_o;\kappa)+l_{n''}(h_o;\kappa)+1}\, W\{{}_i\phi_{n''}[y;h_o;\kappa], {}_i\phi_{n'}[y;h_o;\kappa]\} \right| < \infty$$

$$(n' \ne n''). \tag{89}$$

In particular, we conclude that the eigenfunctions (85) must be orthogonal with the weight (64):

$$\int_{-\infty}^{+\infty} {}_i\phi_{n''}[y;h_o;\kappa], {}_i\phi_{n'}[y;h_o;\kappa]\rho_M[y;\kappa]dy = \delta_{n''n'} \tag{90}$$

Furthermore, we assert that the (n+1)-th eigenfunction of the CSLE must have exactly n nodes, and therefore may not be any eigenvalue, other than ${}_iE_n(h_o;\kappa)$, below the energy

$${}_iE_{n_{max}}(h_o;\kappa) = -\left[l_{n_{max}}(h_o;\kappa) + \tfrac{1}{2}\right]^2, \tag{91}$$

which completes the proof. □

Our final step is to prove that there may not exist any negative eigenvalue above the energy (91). To do it, we convert the RRef CSLE (59) to its prime form

$$\left\{\frac{d}{dy}\sqrt{y^2+1}\frac{d}{dy} - {}_i\mathcal{q}[y;h_o] + {}_iE_n(h_o;\kappa)\, {}_i w[y;\kappa]\right\} {}_i\psi_n[y;h_o;\kappa] = 0, \tag{92}$$

where

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$$_i\phi[y;h_o] = -\sqrt{y^2+1}\left(I^o[y;h_o] + \mathcal{G}\{\sqrt{y^2+1}\}\right), \quad (93)$$

with the second summand in the parentheses defined via (12), which gives

$$_i\phi[y;h_o] = -\frac{h_{o;R} + \tfrac{1}{2}h_{o;I}\,y + \tfrac{3}{4}}{(y^2+1)^{3/2}} \quad (94)$$

As for the weight function of the p-SLE (92), it is related to the density function (64) in the conventional way:

$$_iw[y;\kappa] := \sqrt{y^2+1}\,\rho_M[y;\kappa] > 0, \quad (95)$$

which gives

$$_iw[y;\kappa] := \frac{_iT_2[y;1,\kappa]}{(y^2+1)^{3/2}}. \quad (96)$$

One can directly verify that each q-RSs

$$_i\psi_n[y;h_o;\kappa] = (y^2+1)^{-\frac{1}{2}[l_n(h_o;\kappa)+\frac{1}{2}]} exp[-2\underset{\sim}{\alpha}_{I;n}(h_o;\kappa)\,arctan\ y]\times$$
$$(y^2+1)^{-\frac{1}{2}n} P_n\left(-y; -2\underset{\sim}{\alpha}_{I,n}(h_o;\kappa), n + l_n(h_o;\kappa)\right) \quad (97)$$

satisfies the DBCs

$$\lim_{y\to\pm\infty} {_i\psi_n}[y;h_o;\kappa] = 0, \quad (98)$$

as expected.

Theorem 3. *The formulated Dirichlet problem is exactly solvable* via *the q-RSs (97).*

Proof of Theorem 3. Since the (n+1)-th eigenfunction (108) has exactly n nodes, the Dirichlet problem in question may not have any solution between two sequential eigenvalues

$_iE_{n-1}(h_o;\kappa)$ and $_iE_n(h_o;\kappa)$. The most challenging issue is to prove that no eigenvalue $\underset{\sim}{E}$ exists inside the interval

$$_iE_{n_{max}}(h_o;\kappa) < \underset{\sim}{E} < 0. \quad (99)$$

Suppose that the p-SLE (92) solved under the DBCs has a solution at an energy $\underset{\sim}{E}$ within the range (110) which has the number of nodes N larger than n_{max}. This eigenfunction must be a solution of the p-ODE (7) with

$$\lambda = h_R\left(h_o;\kappa;\tilde{E}\right),\ \mu = \tfrac{1}{2}h_I\left(h_o;\kappa;\tilde{E}\right),\ l \equiv l(\tilde{E}). \quad (100)$$

However, according to Theorem 2, any solution of this problem must have the form (97), which may not be true if $N > n_{max}$. □

One can directly verify that the eigenfunctions (97) are squarely integrable with the weight (96):

$$\int_{-\infty}^{+\infty} {_i\psi_n^2}[y;h_o;\kappa]\,{_iw}[y;\kappa]dy < \infty \quad (101)$$

for n bounded by (87). Also, any solution of the p-SLE (92) is squarely integrable with the weight (106), iff it satisfies the DBC at both endpoints. This implies that any solution of the RRef CSLE (59), squarely integrable with the density function (64), necessarily coincides with one of the q-RSs (75).

5. Discussion

Theorem 3 represents the sought-for end point of our analysis, presenting the rigorous proof that the formulated SLP for the RRef CSLE (59) is exactly solvable via the R-Routh polynomials with degree-dependent indexes. The presented proof brings the Milson potential [16] to the same level as the two discovered-by-the-author families of the Jacobi-reference (JRef) and Laguerre-reference (LRef) potentials [35] with the eigenfunctions expressible via the *classical* Jacobi and accordingly *classical* Laguerre polynomials with the degree-dependent indexes.

In this connection let us remind the reader that our argumentation was vitally rested on the observation that the Wronskian of any two *d*-PFSs of the RCSLE in question vanishes at ±∞. Another common element of our proofs was the assertion that the nth quasi-rational *d*-PFS has exactly n-1 nodes. Combining these two statements allowed the author to invoke Theorem 2.1 in [21], which dismissed the existence of any other eigenvalue either below or between the eigenvalues of the constructed quasi-rational eigenfunctions.

The stated conclusion covered both (t-RM and Gendenshtein) TSI Liouville potentials of the CSLEs (43) and (59) accordingly. The reader might object that these potentials must be exactly solvable simply because they are TSI, with reference to Gendenshtein's celebrated paper [17]. However, from author's point of view [9], Gendenshtein (contrary to the widely spread opinion) never proved the exact solvability of an arbitrary TSI potential. While deleting the bound eigenstates by the sequential DTs, he did not bother to verify the number of nodes for the eigenfunction with the lowest eigenvalue in the given sequence or, in other words, it was taken for granted that the $\mathrm{D}\mathfrak{T}$ of the first-excited state, using the nodeless eigenfunction as the transformation function (TF), does not have any nodes. Though the cited notion is possibly correct (at least no exception has been found so far), its accurate proof would require imposing numerous restrictions on the behavior of the eigenfunctions near the quantization endpoints (like the invoked-here disappearance of the Wronskian of any two *d*-PFSs at ±∞). The imposed limitations would significantly degrade the power of Gendenstein's original speculations [17], which launched the new direction in the theory of solvable quantum-mechanical potentials.

The proofs presented in Sections 3 and 4 form the foundations for the theorems that the rational Rudjak-Zakhariev transformations (RRZTs) [36] of the CSLEs (43) and (59) result in exactly solvable SLEs with all the eigenfunctions representable in the quasi-rational form. The detailed analysis of this problem for the RRef CSLE (59), including its TFI limit associated with the Gendenshtein potential, has been presented in [7]. In our earlier studies (see [20] and the references therein) we termed the mentioned transformations as the canonical Liouville-Darboux transformations (CLDTs) to stress that any Rudjak-Zakhariev transformation (RZT) of the given CSLE results in the DT of the corresponding Liouville potential and otherwise. It should be however stressed that the RZT is defined with no relation to the possible Liouville transformation of the given CSLE to the Schrödinger equation and the conventional DT is nothing but the particular case of the RZT for the canonical eigenequation.

It is remarkable that the rational Rudjak-Zakhariev transforms ($\mathrm{RRZ}\mathfrak{T}\mathrm{s}$) of the RRef CSLE with the simple-pole density function defined by (D1) in Appendix D are quantized via finite sequences of exceptional orthogonal polynomial (EOPs), as the direct consequence of the observation by Odake and Sasaki [37,38] that the Gendenshtein potential belongs to group A of the TSI potentials. On the contrary, the TFI CSLE (43), associated with the TSI *t*-RM potential belongs to Group B and, as a result, its eigenfunctions are formed by the so-called [7] 'Routh-seed' (RS) Heine polynomials' with degree-dependent indexes. (We introduced this term in [20] to emphasize that the polynomials in questions satisfy Heine-type differential equations [39] with the exponential parameters dependent on the polynomial degrees.)

Taking into account that each eigenvalue is determined by a positive root of the quartic equation (81) with the positive leading coefficient and negative free term, the author [9] concluded that the nth quasi-rational eigenfunction must be accompanied by another q-RS composed of a Routh polynomial

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of degree n-1. (In [9] this conclusion was limited solely to the *e*-TP, but it was then extended in [7] to an arbitrary TP without real roots.)

It was Quesne [23], who stimulated author's interest to this subject, by speculating that the Schrödinger equation with the Gendenshtein potential had q-RSs composed of polynomials without real zeros. The straightforward proof of the existence of Routh polynomials with no real zeros was presented by us in [7] and sketched below for purely illustrative purposes..

The crucial element of the cited proof is the (derived in [7]) representation of the monic Routh polynomials in the form of the Wronskian of the sequential monic R-Routh polynomials with the same indexes:

$$P_m(x;b,-a-1) = W\{\hat{R}_1^{(-2b,a+1)}(y),\ldots,\hat{R}_m^{(-2b,a+m)}(y)\} / \prod_{k=1}^{m-1} k!\,, \tag{102}$$

starting from the first-degree polynomial. Since the R-Routh polynomials form an orthogonal set with the measure defined on the real axis, the Wronskian in the right-hand side, according the renowned Theorem 1 in [40], may not have real zeros for any even m.

Our next step is to use the q-RSs (35) formed by the Routh polynomials of an even degree:

$$\phi_{2j}[y;\alpha_n(-l-1,\mu)] := (y^2+1)^{\frac{1}{2}(l+1)} exp[2\alpha_{I;2j}(-l-1,\mu)\, arctan\ y] \times$$
$$(y^2+1)^{-j} P_{2j}\left(y;2\alpha_{I;2j}(-1-l,\mu),2j-1-l\right) \tag{103}$$

as the quasi-rational TFs for the RRZTs. Without going into the details, it would be sufficient for the purpose of this paper only to mention that each RRZT is equivalent to the RDT of the Schrödinger equation with the corresponding Liouville potential.

Below we outline this universal scheme of constructing new exactly solvable RCSLEs for the TSI t-RM potential, while addressing the reader to [7] for any details concerning rational extensions of both Milson potential [16] and its TSI limit [17].

According to Theorem 1, the Schrödinger equation (A18) with the potential (A23) is exactly solvable via R-Routh polynomials. Our aim is to prove that this is also true for the rational DT (RDT) of this potential with the TF

$$R_{2j}(\chi;-a-1,b) = sin^{-a}\chi\, exp[\alpha_{I;2j}(-a-1,b)(\pi-2\chi)] \times$$
$$sin^{2j}\chi\, P_{2j}\left(cot\,\chi;2\alpha_{I;2j}(-a,b),2j-a\right), \tag{104}$$

The fast-track examination of the TF (104) reveals that the RDT in question reduces by 1 the ExpDiff for the poles of the Schrödinger equation at both singular endpoints χ = 0 and χ = π, as it has been already observed in [24].

The RDT in question thus generates the new bound state described by the nodeless eigenfunction

$$R_0(\chi;a,b\,|\,-,2j) = R_{2j}^{-1}(\chi;-a-1,b) \tag{105}$$

at the energy

$$E_m(-a-1,b) := (m-a)^2 - \frac{b^2}{(m-a)^2} - 1\,, \tag{106}$$

lying below the discrete energy spectrum $E_n(a,b)$ of the potential (A23):

$$E_m(-a-1,b) - E_0(a,b) = (m+1)(m-2a-1)\left[1+\frac{b^2}{(a+1)^2(m-a)^2}\right] < 0 \tag{107}$$

for $m < 2a+1$.

If $a > 0$ then the solution (106) of the transformed Schrödinger equation vanishes at both singular endpoints χ = 0 and χ = π and therefore represents the nodeless eigenfunction. assuming that this

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Schrödinger equation is solved under the DBCs. Note that the poles of the Schrödinger equation (A14) are limit-circle (LC) singularities if $a < 1/2$ and, as a result, both solutions (44) and (45) in [45] turned out to be squarely integrable, while the DBCs unambiguously selects the first of the two.

The eigenfunctions of the excited eigenstates in the transformed potential are described by the conventional formula [18]

$$\mathrm{R}_{n+1}(\chi; a,b\,|\,-,2j) = \frac{\mathrm{W}\{\mathrm{R}_{2j}(\chi;-a-1,b),\mathrm{R}_{n}(\chi;a,b)\}}{\mathrm{R}_{2j}(\chi;-a-1,b)}. \tag{108}$$

As proven by the author in [20] for an arbitrary Gauss-reference (GRef) potential (i.e., for any JRef, LRef, or RRef potential), the eigenfunctions (108) have the quasi-rational form, with the polynomial components composed of the so-called 'polynomial determinants' (PDs), which obey the aforementioned Heine-type ODEs with the degree-dependent exponent parameters and for this reason are termed by us 'Gauss-seed' (GS) in general or more specifically 'RS' Heine polynomials.

Representing (108) as

$$\mathrm{R}_{n+1}(\chi; a,b\,|\,-,2j) = \dot{\mathrm{R}}_{n}(\chi;a,b) - ld\,\mathrm{R}_{2j}(\chi;-a-1,b)\mathrm{R}_{n}(\chi;a,b), \tag{109}$$

with dot and the symbolic expression ld standing for the first derivative with respect to χ and the logarithmic derivative accordingly, one finds that the q-RSs (108) vanish at both singular endpoints $\chi = 0$ and $\chi = \pi$ for $a > 0$ and therefore constitute the eigenfunctions of the transformed Schrödinger equation, as stated above. Note that the conventional rules of the SUSY quantum mechanics [18] utilized in [24,25] take this result for granted.

Our final step is to prove that the transformed Schrödinger equation may have no excited bound states other than the ones specified the eigenfunctions (108). This proof in the more general case of the Milson potential [7] or the t-RM potential here constitutes the main stimulus for this research.

Theorem 4. *The eigenfunction for any excited eigenstate of the transformed Schrödinger equation can be represented in the quasi-rational form (108).*

Proof of Theorem 4. Suppose that the transformed Schrödinger equation has an eigenfunction $\mathrm{R}_{\tau}(\chi; a,b\,|\,-,2j)$ with an eigenvalue

$$\mathrm{E}_{\tau}(a,b\,|\,-,2j) \neq \mathrm{E}_{n}(a,b). \tag{110}$$

By applying to this equation the RDT with the TF (105), we come back to the Schrödinger equation (A18). Let us show that the corresponding RDT of the eigenfunction $\mathrm{R}_{\tau}(\chi; a,b\,|\,-,2j)$,

$$\mathrm{R}_{2j}(\chi;-a-1,b)\mathrm{W}\{\mathrm{R}_{2j}^{-1}(\chi;-a-1,b),\mathrm{R}_{\tau}(\chi;a,b\,|\,-,2j)\}, \tag{111}$$

vanishes at both singular endpoints $\chi = 0$ and $\chi = \pi$ iff $a > 1$. Indeed, taking into account that

$$\dot{\mathrm{W}}\{\mathrm{R}_{2j}^{-1}(\chi;-a-1,b),\mathrm{R}_{\tau}(\chi;a,b\,|\,-,2j)\} = \left[\mathrm{E}_{2j}(a,b) - \mathrm{E}_{\tau}(a,b)\right]\times \mathrm{R}_{\tau}(\chi;a,b\,|\,-,2j)/\mathrm{R}_{2j}(\chi;-a-1,b), \tag{112}$$

one finds

$$0 < \lim_{\sin\chi\to 0} \left|\sin^{-2a}\chi\,\dot{\mathrm{W}}\{\mathrm{R}_{2j}^{-1}(\chi;-a-1,b),\mathrm{R}_{\tau}(\chi;a,b\,|\,-,2j)\}\right| < \infty \tag{113}$$

and therefore

$$0 < \lim_{\sin\chi\to 0} \left|\sin^{-2a-1}\chi\,\mathrm{W}\{\mathrm{R}_{2j}^{-1}(\chi;-a,b),\mathrm{R}_{\tau}(\chi;a,b\,|\,-,2j)\}\right| < \infty \quad \text{iff } a > 1. \tag{114}$$

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We thus proved that the q-RSs (111) vanishes as $sin^{a+1}\chi$ near each singular endpoint and therefore constitutes the eigenfunction of the Schrödinger equation (A14) with the eigenvalue (110), which contradicts Theorem 1. □

Winding up this discussion, leave us mentioned that the Wronskian representation (102) of the Routh polynomial also indicates that the RDT with the TF (103) is equivalent to the rational Darboux-Crum transformation (RDCT) using the even number of the sequential eigenfunctions, which represents the simplest illustration of the general theorem proven by Odake and Sasaki [38]. In particular, use of the TF composed of the second-degree Routh polynomial results in the potential (14) in [41], which was generated by the second-order RDCT, using the eigenfunctions of the first and second excited states as seed functions.

6. Conclusions

One of the major results of this paper is the notion of the *d*-PFS problem relating the SLPs for the SLE (43) relating the SLPs for the SLEs (43) and (59). Another important innovation common for both SLPs is the proof that the Wronskian of any two d-PFSs of each SLE vanishes at both singular endpoints, which assures [21] that the nth eigenfunction of the given SLE has exactly n-1 nodes. Keeping in mind that the SLE (43) with the trigonometric ('*t*-RM') Liouville potential has the infinite discrete energy spectrum (14,15), we then assert in Theorem 1 that the SLP in question is exactly solvable via the R-Routh polynomials – the result taken for granted in [14,15]. (As stressed in Discussion, the fact that this potential is TSI, does not automatically rule out the possibility for it to be quasi-exactly solvable).

Next, we take advantage of the exact solvability of the SLP for the SLE (43) to prove in the end of Section 3 (see Corollary 2) that the *d*-PFS problem for the NODE (1) is exactly solvable in terms of the *q*-RSs discovered by Stevenson [11].

Finally, Theorem 3 formulated in Section 3 allows the author to complete the sought-for-a-long time [9] proof that the Milson potential on line [16] is exactly solvable by the R-Routh polynomials. Originally [9], the author made this assertion, wrongly assuming that the reader can be simply referred to Stevenson's Note [11] for the supporting arguments. However, a more thorough analysis of his arguments revealed that they include some unproved statements, which I found it difficult to confirm, based on the conventional analytical extensions [29] of the complex hypergeometric series in his formula (3) for the eigenfunctions (see Appendix C for details).

The important innovative element allowing the author to prove this result is the concept of the 'prime' [20] SLEs, which can be solved under the DBCs to find all the possible *d*-PFSs.

This concept is broadly used in the cited papers [7,9,42,44] to prove the exact solvability of the RRZTs of the exactly solvable RCSLEs.

As stressed at the beginning of Section 5, Theorem 3 brings the Milson potential [16] to the same level as the two families of the rational potentials [35] with the eigenfunctions expressible via the *classical* Jacobi and accordingly *classical* Laguerre polynomials with the degree-dependent indexes. Our many years efforts [9] to present the rigorous argumentation in support of this theorem was stimulated by our studies on the RDTs of the Milson potential and its TSI limit represented by the Gendenshtein (Scarf II) potential. All the possible rational Darboux-Crum transforms (RDCTs) of the latter potential solvable via EOPs have been constructed by us in [42] under the assumption that the potential itself is exactly solvable via the R-Routh polynomials, with the reassuring reference to Stevenson's note. However, as mentioned above, the more thorough analysis of Stevenson's argumentation revealed that it has some hidden gaps, which were successfully filled in this paper.

As already stressed in Discussion the crucial difference between the Gendenshtein and t-RM potentials is that they belong to groups A and B in Odake and Sasaki's [35,37] classification scheme of the TSI potentials. Since the RCSLE with the simple-pole density function (A56) has the energy-independent ExpDiffs for its poles at ±∞, the indexes of the R-Routh polynomials forming the polynomial components of the quasi-rational eigenfunctions (A58) do not depend on the polynomial

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degrees. This remarkable feature of the rational TSI potentials of group A allowed the author to make use of the orthonormalization condition (9.9.2) in [10] for the analytical computation of the normalizing factors for the eigenfunctions of the Schrödinger equation with the Gendenshtein potential.

On the contrary, we were unable to find the normalizing factors for the eigenfunctions of the Schrödinger equation with the *t*-RM potential. In this respect, it is worth to mention again the formula for the given normalizing factors first listed in [53] (without any hints on the underlying derivation) and then simply reintroduced in several publications [55,56,62] without any additional comments. The author was unable to either confirm the validity of the normalizing condition (21) in [53] or to prove its erroneousness.

As a by-side novel result of this paper, I can also mention the link between the two formally complex hypergeometric expressions (9.9.1) in [10], which, to our knowledge, were never discussed in this context in the quantum-mechanical applications of the R-Routh polynomials. The proof of their equivalence presented in Theorem 6 in Appendix A created the link between Stevenson's formally complex hypergeometric polynomials (A3) and the family of the Romanovski polynomials [2] little known by mathematicians, until Askey's breakthrough work [1] stimulated a broader interest in this subject.

As discussed in some details in Appendix B, there are two groups of studies, utilizing one of the hypergeometric expressions (A1) or (A2), while even not mentioning the other in many cases (or, in the best scenario, simply making an implicit reference, notwithstanding any outline of how these two analytical approaches are related).

In the following studies we will discuss various RDCTs of the Milson potential exactly solvable via RS Heine polynomials. The generic scenario is the sequence of the rational extensions obtained by eliminating the lowest energy eigenstate one by one. Another promising direction is to formulate the 'enhanced' Adler theorem [43] similar to that proven in [44] for the JRef potential on the line. The exactly solvable RDCT of the Milson potential can be then constructed [44] using 'juxtaposed' [45–47] pairs of the eigenfunctions (83).

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Appendix A. Hypergeometric Expressions for Monic Routh Polynomials

The main purpose of this Appendix is to reexamine the hypergeometric expressions (9.9.1) in [10}:

$$P_n(y;\nu,N)=(-2i)^n\frac{\langle -N+i\nu\rangle_n}{\langle n-2N-1\rangle_n}F[-n,n-2N-1,-N+i\nu;\tfrac{1}{2}-\tfrac{1}{2}iy] \tag{A1}$$

and

$$P_n(y;\nu,N)=(y+i)^n F(-n,1-n+N-i\nu,2N+2-2n;2i/(y+i)) \tag{A2}$$

which were originally introduced in the cited monograph only for nonnegative integers N. Stevenson's polynomials (34) are nothing but the formal complex conjugates of (A2):

$$P_n(y;\nu,N)=(y-i)^n F(-n,1-n+N+i\nu,2N+2-2n;-2i/(y-i)). \tag{A3}$$

By paraphrasing, to a larger extent, the arguments in [3], Theorem 5 below presents the rigorous proof of Askey's assertion that the hypergeometric polynomials (A1) in y are real. The author found

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it necessary to reproduce the underlying formulas for the complex Jacobi polynomials in *i*y since the aforementioned assertion presents the basis for the by-no-means trivial proof that the monic polynomials (A2) and therefore their formal conjugates (A3) become real after being expressed in terms of y.

While Askey's paper [1] does appears in the comprehensive reference list compiled by Raposo et al. [3], where the author originally found the citation to his paper, the cited authors [3] did not properly understand the significance of his results. In particular, their statement that 'the orthogonality properties of the complex Jacobi polynomials are not well known.' Is certainly misleading. They were discovered by Askey [1] and then used to reintroduce two families of Romanovski polynomials, including the family of the R-Routh polynomials thoroughly illuminated in the cited (otherwise quite remarkable) review [3].

Before proceeding to the mentioned proof, let us remind the reader that the notion of the real-by-definition [8] 'Routh polynomials', defined by us [7,9] via (37) with

$$\boldsymbol{\lambda} \equiv -\mathrm{N}-1+\nu i\,, \tag{A4}$$

is used here in exactly the same sense as Jordaan and Toókos [48] employ the term 'pseudo-Jacobi polynomials'. As a result, Lesky's [5,6] epithet 'Romanovski/pseudo-Jacobi polynomials' becomes identical to the term 'R-Routh polynomials' which is used by us iff the polynomial degree does not exceed $\mathrm{N}-\frac{1}{2}$. On other hand, both expressions (A1) and (A2) remain valid for any Routh polynomial converted to its monic form.

It is worth stressing once again that the commonly cited formula (37) for the Routh polynomials was first introduced by Cryer [26] in his analysis of the infinite *real* polynomial sequences with the coefficients satisfying the three-term recurrence relations. It was Compean and Kirchbach [14], who brought author's attention to this archaic study rediscovering some of Routh' results [8]. The fact that the polynomials (37) in y are real was stated in [26] (and later in [1]), based on their definitions via the real Rodriguez formula.

The author personally prefers the direct deviation presented in the already cited paper by Raposo et al. [3], twenty years after publication of Askey's paper [1]. Regretfully, the cited authors used the non-standard definition of the Jacobi polynomials, which differed from the conventional one [31] by the factor $(-1)^n$. For this reason, we prefer to reproduce their arguments here, though in a slightly modified form.

Theorem 5. *The polynomials (A1) have real coefficients, when expressed in terms of y.*

Proof of Theorem 5. Let us start from the general expression [49]

$$\mathrm{P}_n^{(\alpha,\beta)}(x)=\frac{1}{2^n}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\langle\alpha+1\rangle_{n-k}\langle\beta+1\rangle_k(x-1)^k(x+1)^{n-k} \tag{A5}$$

for the Jacobi polynomials with arbitrarily chosen complex indexes α and β, which (as stated above) differs from (92) in [3] by the factor $(-1)^n$. (This is also true for the Rodriguez formula (93) in [3], compared with (1.1) in [49].) Rewriting

$$\mathrm{P}_n^{(\boldsymbol{\lambda},\boldsymbol{\lambda}^*)}(iy)=\frac{1}{2^n}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\langle\boldsymbol{\lambda}+1\rangle_{n-k}\langle\boldsymbol{\lambda}^*+1\rangle_k(1-iy)^k(1+iy)^{n-k} \tag{A6}$$

as

$$\begin{aligned}(-i)^n\mathrm{P}_n^{(\boldsymbol{\lambda},\boldsymbol{\lambda}^*)}(iy)&=\frac{1}{2^{n+1}}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\\&\times\Big[(-i)^n\langle\boldsymbol{\lambda}+1\rangle_{n-k}\langle\boldsymbol{\lambda}^*+1\rangle_k(1-iy)^k(1+iy)^{n-k}\end{aligned} \tag{A7}$$

$$+i^{n}\langle \boldsymbol{\lambda}^{*}+1\rangle_{n-k}\langle \boldsymbol{\lambda}+1\rangle_{k}(1+iy)^{k}(1-iy)^{n-k}\Big]$$

then explicitly confirms that the polynomials (40) are real functions of y. □

It was Askey [1], who first represented the cited polynomials in the hypergeometric form:

$$\mathfrak{R}_{n}^{(\boldsymbol{\lambda})}(y)=i^{n}\frac{\langle \boldsymbol{\lambda}+1\rangle_{n}}{n!}F\big(-n,n+2\boldsymbol{\lambda}_{R}+1;1+\boldsymbol{\lambda};(1-iy)/2\big) \tag{A8}$$

(see (1.17) in [1]). Expressing the complex index of the Routh polynomial (A5) via (A4) and comparing the resultant expression with (A1), we confirm that the latter polynomials have the form

$$\mathfrak{R}_{n}^{(-N-1+\nu i)}(y)=\frac{\langle n-2N-1\rangle_{n}}{2^{n}n!}P_{n}(y;\nu,N) \tag{A9}$$

Theorem 6. *The formally complex hypergeometric polynomials (A2), with no restriction on the real parameter N, represent the alternative representation for the Routh polynomials (A8) converted to their monic form.*

Proof of Theorem 6. To prove the equivalence of the two hypergeometric expressions (A1) and (A2), let us first rewrite the polynomials (A1) in the slightly different form:

$$P_{n}(y;\nu,N)=(-2i)^{n}\frac{\langle 1-n+N-i\nu\rangle_{n}}{\langle 2N+2-2n\rangle_{n}}F[-n,n-2N-1,-N+i\nu;\tfrac{1}{2}-\tfrac{1}{2}iy], \tag{A10}$$

bearing in mind that

$$\langle \boldsymbol{\nu}\rangle_{n}=\sum_{k=0}^{n-1}(\boldsymbol{\nu}+k)=(-1)^{n}\sum_{k=0}^{n-1}(-\boldsymbol{\nu}+1-n+k)=(-1)^{n}\langle -\boldsymbol{\nu}+1-n\rangle_{n}$$
$$\equiv(-1)^{n}\frac{\Gamma(1-\boldsymbol{\nu})}{\Gamma(1-n-\boldsymbol{\nu})}. \tag{A11}$$

Next, taking into account that the second summand in the analytical continuation of the hypergeometric function via (15.3.7) in [29],

$$F(\alpha,\beta;\gamma;\xi_{S}^{*})=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-\xi_{S}^{*})^{-\alpha}F(\alpha,1-\gamma+\alpha,1-\beta+\alpha;1/\xi_{S}^{*})$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-\xi_{S}^{*})^{-\beta}F[\beta,1-\gamma+\beta,1-\alpha+\beta;1/\xi_{S}^{*}] \tag{A12}$$

for $|\arg\left(-\xi_{S}^{*}\right)|<\pi$

vanishes for $\alpha=-n$, one finds

$$(y+i)^{n}F(-n,1-n+N-i\nu,2N+2-2n;2i/(y+i))=$$
$$\frac{\langle 1-n+N-i\nu\rangle_{n}}{\langle 2N+2-2n\rangle_{n}}(-2i)^{n}F(-n,n-2N-1,-N+i\nu;(1-iy)/2), \tag{A13}$$

which brings us to (9.9.1) in [10]. We thus confirmed that the monic polynomials (A2) must also have real coefficients, when expressed in terms of y. □

The latter assertion is automatically extended to Stevenson's hypergeometric polynomials (A3), since they are nothing but the complex conjugates of the proven-to-be real polynomials (A2).

Corollary 3. *The formally complex hypergeometric polynomials (A3) discovered by Stevenson [11] represent the alternative representation for the Routh polynomials (A8), after the latter were converted to their monic form.*

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We are ready to switch our discussion to finite orthogonal sequences of the Routh polynomials discovered by Romanovski [2]. In following Quesne [23], the R-Routh polynomials are defined via the relation:

$$R_n^{(2\nu,N)}(y) := \Re_n^{(-N-1+\nu i)}(y) \quad \text{for } n=0,\ldots, \lfloor N+\tfrac{1}{2}\rfloor . \tag{A14}$$

According to the orthogonalization conditions (9.9.2) in [10], the monic R-Routh polynomials

$$\hat{R}_n^{(2\nu,N)}(y) = P_n(y;\nu,N) = \frac{(-2)^n n!\Gamma(2N+2-2n)}{\Gamma(2N+2-n)} R_n^{(2\nu,N)}(y) \tag{A15}$$

are mutually orthogonal with the weigh specified by the integral:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{R}_{n''}^{(2\nu,-N)}(y)\hat{R}_{n'}^{(2\nu,-N)}(y)(y^2+1)^{-N-1} exp(2\nu\, arctan\, y)\,dy$$
$$= \frac{2^{2n-2N-1} n!\Gamma(2N+1-2n)\Gamma(2N+2-2n)}{\Gamma(2N+2-n)\left|\Gamma(N+1-n+\nu i)\right|^2}\delta_{n''n'} . \tag{A16}$$

(It seems that the factor $2^{\alpha+\beta+1}$ is absent in Askey's orthogonalization conditions (1.8) in [1].)

Appendix B. Some Remarks on the t-RM Potential

The Liouville transformation of the CSLE (43), using the change of variable

$$y(\chi) = cot\,\chi , \tag{A17}$$

results in the one-dimensional Schrödinger equation

$$\left\{-\frac{d^2}{d\chi^2} + V_L(\chi;l,\mu) - E\right\} R(\chi;l,\mu;E) = 0 , \tag{A18}$$

where

$$R(\chi;l,\mu;E) = sin\,\chi\, \Phi^+[cot\,\chi;E,\mu,l] . \tag{A19}$$

Taking into account that the Schwarzian derivative of the function (A17) vanishes:

$$\{y(\chi),\chi\} \equiv \{cot\,\chi,\chi\} = 0 \tag{A20}$$

and

$$1 + y^2(\chi) = sin^{-2}\chi , \tag{A21}$$

one finds

$$V_L(\chi;l,\mu) := -sin^{-4}\chi\, I^o[cot\,\chi;0,\mu,l] \tag{A22}$$

and therefore

$$V_L(\chi;l,\mu) := \frac{l(l+1)}{sin^2\chi} - 2\mu\, cot\,\chi \quad (0<\chi<\pi), \tag{A23}$$

with the parameters l and μ standing for a and b in [14,15,23,24,41]. As pointed to in [15], the parameter l in (A23) may take any real value larger than $-\tfrac{1}{2}$.

It was Infeld and Hull [31], who first pointed to the fact that the Kepler problem on a hypersphere [11]

$$\left[-sin^{-2}\chi\frac{d}{d\chi}\left(sin^2\chi\frac{d}{d\chi}\right) + l(l+1)sin^{-2}\chi - 2\mu\, cot\,\chi - \lambda\right] S(\chi;\mu,l;\lambda) = 0 \tag{A24}$$

can be reduced to the one-dimensional Schrödinger equation with the potential (A23) by the transformation [31]

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$$\mathrm{S}(\chi;\mu,l;\lambda)=sin^{-1}\chi\,\mathrm{R}(\chi;l,\mu;\lambda)\equiv\Phi^{+}[cot\,\chi;\lambda,\mu,l]. \qquad \text{(A25)}$$

As far as the potential in question is written in the cited form, it would be more appropriate to refer to it as the trigonometric modification of the Manning-Rosen (MR) potential [50}. As a matter of fact, Companion and Kirchbach [14] came to (A23), starting from the MR potential (misleadingly referred to as the 'Eckart' potential, in following De et al. [51]).

After re-writing (A23) in the form of the trigonometric modification of the Rosen-Morse (RM) potential:

$$\mathrm{V_L}(\pi/2-\mathrm{x};a,b)\equiv\mathrm{V}_{t\text{-RM}}(\mathrm{x};a,b)=\frac{a(a+1)}{cos^2\mathrm{x}}-2b\,tan\,\mathrm{x}\quad(-\pi/2<\mathrm{x}<\pi), \qquad \text{(A26)}$$

De et al. [51], in their Table 2, termed the potential (A19) as 'Rosen-Morse I', while referring to the its hyperbolic version (initiated by Rosen and Morse [52]) as 'Rosen-Morse II'. The latter terminology was adopted by Cooper et al., in their celebrated review [18] and then broadly accepted in the quantum-mechanical literature.

Representing the eigenfunction of the Schrödinger equation (A18) as

$$\mathrm{R_n}(\chi;a,b):=\mathrm{R}(\chi;a,b;E_\mathrm{n})=sin\,\chi\;\phi_\mathrm{n}[cot\,\chi;\alpha_\mathrm{n}(a,b)], \qquad \text{(A27)}$$

choosing the angular variable χ via the relation

$$\chi:=\tfrac{1}{2}\pi-arctan\,\mathrm{y}, \qquad \text{(A28)}$$

(such that it increases from 0 to π as y varies from -∞ to +∞), and expressing the q-RSs (34) in terms of χ, coupled with (A21), one finds

$$exp[-\pi\,\alpha_{\mathrm{I;n}}(a,b)]\,\mathrm{R_n}(\chi;a,b)=$$

$$sin^{a+\mathrm{n}+1}\chi\,exp[-2\alpha_{\mathrm{I,n}}(a,b)\chi]P_\mathrm{n}\left(cot\,\chi;2\alpha_{\mathrm{I;n}}(a,b),\mathrm{n}+a\right), \qquad \text{(A29)}$$

which brings us to (10) in [15], with

$$\alpha_\mathrm{m}:=2b/(a+\mathrm{m}+1)=4\,\alpha_{\mathrm{I,m}}(a,b),\;\;\beta_\mathrm{m}:=-a-\mathrm{m}, \qquad \text{(A30)}$$

and m standing for n in (A24) or n-1 in both [14] and [15]. With this in mind, the reader (if necessary) can easily link Compean and Kirchbach's [14,15] results to our notation (also remembering that, in following Quesne [23], we use the slightly different notation (A15) for the R-Routh polynomials.)

As it has been pointed to in Introduction, there are two separate groups of the studies, which express the eigenfunctions of either Schrödinger equation (A18) or the 'Kepler problem on the hypersphere' (A24) in terms of one and only one of the hypergeometric expressions (9.9.1) in [10] (i.e., either to (A1) or (A2) here) without even mentioning the alternative. It was Kirchbach et al. [3,14,15], who introduced the notion of the R-Routh polynomials in quantum-mechanical applications, starting from the real Rodriguez formula. This is precisely the way how Cryer [26] (and later Askey [1]) came up to the real Jacobi polynomials in the imaginary argument, and the right-hand side of (A13) is nothing but the conventional hypergeometric expression [1] for these polynomials.

On other hand, the formally complex quasi-rational eigenfunctions of the trigonometric eigenequation (A24), with the polynomial components of the form (A2), appeared (most probably under influence of Stevenson's Note [11]) in the paper of Bogush et al. [53] in the early eighties. The intriguing feature of this paper written in Russian (see [54] for the online access) is that the authors reported the normalizing factors for the eigenfunctions, though with no mention of any background formulas for their derivation. As stressed in Section 4, this could not be done, using the orthonormalization conditions (A16) for the R-Routh polynomials, since the normalization of the eigenfunctions in question requires the different weight. I am not aware of any other analytical expression for the integral of the squared hypergeometric polynomials under consideration.

To clarify the relationship of the cited expression to (A3), let us first come back to the parameters $l=a$ and $\mu=b$ and evaluate the q-RSs (A25) at the energies $\lambda_\mathrm{n}(l,\mu)$. Defining the R-Routh polynomials in (34) via (A2), with the argument expressed in terms of χ:

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$$2/(1-i\cot\chi)=2ie^{-i\chi}\sin\chi\equiv 1-e^{-2i\chi}=\xi_S^*, \tag{A31}$$

we come to the trigonometric formula

$$\begin{aligned} S_n(\chi;l,\mu)=\tilde{\phi}_n[\cot\chi;\tilde{\alpha}_n(l,\mu)]=\sin^l\chi\, \exp\{[ni-2\alpha_{I;n}(l,\mu)]\chi\} \\ \times F\left(-n,\, l+1-i\, 2\alpha_{I;n}(l,\mu), 2l+2; 1-e^{-2i\chi}\right), \end{aligned} \tag{A32}$$

for the eigenfunctions in question, with $\alpha_{I;n}(l,\mu)$ given by (32). Disregarding the normalizing factor, the derived expression precisely matches that in [53], with $\alpha R\equiv 2\mu$ and n standing for $l+1+n$ here. (For reader's convenience we can point to the hypergeometric expression (7) in the more recent paper [55] of these authors, with $e\equiv 2\mu$ and n again standing for $l+1+n$ here.) The trigonometric expression (A36) must be a real function of χ, as it is nothing but the real quasi-rational truncation (44) of the PFS (A26). This by-no-means trivial issue was simply ignored in the cited papers. As mentioned above, the reported analytical expression for the normalizing factor requires a more thorough analysis.

Originally the citation to [53] was found by us in the more recent study by Gritsev and Kurochkin [56], citing the aforementioned expressions for the normalized eigenfunctions, again without any additional comments on their derivation. The author luckily came across the latter paper, while examining the reference list of the preprint by Compean and Kirchbach [57]. In the following related studies Kirchbach et al. [58–61] simply referred the reader to the definition of the R-Routh polynomials in [3] (similar to (A14), except that we use Quesne's convention for the indexes).

On other hand, the cited hypergeometric expressions (including the normalizing factors originally reported in [53]) were more recently reintroduced by Redkov and Otchik [62] without any clarifying comments or any references to the solutions of the trigonometric eigenequation (A24) in [14,58,59,63] via the R-Routh polynomials.

It is interesting that the monograph [64] published by the aforementioned authors the same year did present the apparently original derivation of the hypergeometric expression for the polynomial components of the (formally complex) quasi-rational eigenfunctions, though with no mentioning of the corresponding normalizing factors. It was taken for granted that the constructed solutions are real. Again, there was no reference to the alternative solution via the (real-by-definition) R-Routh polynomials or to the monograph [9], which might provide the more solid grounds for their derivation

The R-Routh polynomials in the form (A2) reappeared more recently in the works of Fernández et al. [24,25], who were apparently also unaware of the hypergeometric expressions (9.9.1) in [10], which would help to relate their results to the R-Routh polynomials introduced in [14] via the Rodriguez formula.

. Since the authors cited above [24,25] stated by these authors that Proposition 1 suggested in subsection 2.3 above can be readily proven, it seems useful to readdress this issue, using their terminology. First let us substitute the complex conjugate of (A31) into the formally complex solution (25) vanishing at -∞, which brings us to the following trigonometric

$$\begin{aligned} \underset{\sim}{\Phi}(\chi;a,b;E):=\Phi^+[\cot\chi;E,b,a]=\sin^a\chi\,\exp\{i\left(a+2\alpha_R(E,b)\right)-2\alpha_I(E,b)]\chi\} \\ \times F\left(2\alpha_R(E,b)+a,\, a+1+i\, 2\alpha_I(E,b), 2a+2; -2ie^{i\chi}\sin\chi\right), \end{aligned} \tag{A33}$$

with awareness that the listed hypergeometric series converges only for $0<\chi\le\pi/6$. Next, note that the change of variable

$$\xi:=\tfrac{1}{2}(1-i\cot\chi), \tag{A34}$$

leading to the hypergeometric equation (40) in [24], is equivalent to the LFT

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$$\xi = \frac{\xi_S - 1}{\xi_S} \qquad \text{(A35)}$$

in our notation, On other hand, the complex conjugation of (A33) brings us to Fernández and Reyes' [25] regular solution (35), expressed in terms of the complex conjugated variable

$$\zeta = \xi_S^* . \qquad \text{(A36)}$$

Comparing the wavefunction

$$\mathrm{R}^*(\chi; a,b;E) = \sin^{a+1}\chi\, exp\{-[2\alpha_I(E,b) + i(2\alpha_R(E,b)+a)]\chi\}$$
$$\times \mathrm{F}(2\alpha_R(E,b)+a,\ a+1-i\ 2\alpha_I(E,b), 2a+2; 2ie^{-i\chi} sin\chi) \qquad \text{(A37)}$$
$$(0 < \chi \le \pi/6)$$

with (44) in [24] and (35) in [25], one finds

$$\beta \equiv \nu = 2\alpha_R(E,b),\quad \alpha \equiv \mu = 4\alpha_I(E,b) . \qquad \text{(A38)}$$

As pointed to in [25], the hypergeometric series in the right-hand side of (A32) diverges for $\pi/6 < \chi < 5\pi/6$, in agreement with our results. To bypass this problem, Fernández and Reyes [25] made use of the analytical continuation (15.3.8) in [29], introducing the hypergeometric series in the new variable

$$(1-\zeta)^{-1} = e^{2i\chi} . \qquad \text{(A39)}$$

However, this analytical continuation is valid only if

$$|arg(1-\zeta)| = 2\chi < \pi , \qquad \text{(A40)}$$

i.e., for $0 < \chi < \pi/2$.

The cited authors [25] also declared that 'it is straightforward to show that their solution (36) is real; if E is real'. However, I was unable to figure out how this could be done and, for this reason, preferred to formulate this result as Proposition 1. No proof in this paper uses this sole speculation as its prerequisite. However, the validity of this assertion would require a more rigorous justification if one tries to use these solutions as the TFs for the DTs, as it has been done in [25].

Appendix C. Canonical Form of the Hypergeometric Equation in Complex Variable $\xi_S(y)$

let us first note that it represents the LFT. As a result, the Schwarzian derivative $\{y, \xi_S\}$ vanishes [13] and the invariant of the complex rational CSLE (RCSLE)

$$\left\{\frac{d^2}{d\xi_S^2} + \underset{\sim}{I}[\xi_S;\lambda,\mu,l]\right\}\Phi[\xi_S;\lambda,\mu,l] = 0 \qquad \text{(A41)}$$

is related to the Stevenson invariant (2) via the elementary formula

$$\underset{\sim}{I}[\xi_S;\lambda,\mu,l] = \left(\frac{dy}{d\xi_S}\right)^2 I[y;\lambda,\mu,l] , \qquad \text{(A42)}$$

$$= 1/4\left(\frac{dy}{d\xi_S}\right)^2\left[\frac{\lambda+2i\mu(1-2/\xi_S)}{4(\xi_S-1)} - \frac{l(l+1)}{\xi_S^2}\right]\frac{\xi_S^4}{\xi_S-1} , \qquad \text{(A43)}$$

bearing in mind that $-iy = 1-2/\xi_S$ and

$$1+y^2 = 4(\xi_S - 1)/\xi_S^2 . \qquad \text{(A44)}$$

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Taking into account that

$$\frac{dy}{d\xi_S} = 2i/\xi_S^2, \tag{A45}$$

we can then re-write (A43) as

$$\underset{\sim}{I}[\xi_S;\lambda,\mu,l] = \frac{1-(2l+1)^2}{4\xi_S^2} - \frac{\lambda+2\mu i}{4(1-\xi_S)^2} + \frac{1-(2l+1)^2-4\mu i}{4\xi_S(1-\xi_S)} \,. \tag{A46}$$

Setting

$$[2\alpha(\lambda,\mu)-1]^2 := \lambda + 2\mu i + 1, \tag{A47}$$

$$[2\alpha^*(\lambda,\mu)-1]^2 = \lambda - 2\mu i + 1, \tag{A48}$$

$$[2\alpha(\lambda,\mu)-1]^2 - [2\alpha^*(\lambda,\mu)-1]^2 = 4\mu i \tag{A49}$$

gives

$$\underset{\sim}{I}[\xi_S;\lambda,\mu,l] = \frac{1-(2l+1)^2}{4\xi_S^2} - \frac{[2\alpha(\lambda,\mu)-1]^2}{4(1-\xi_S)^2} + \frac{1-(2l+1)^2+[2\alpha^*(\lambda,\mu)-1]^2-[2\alpha^*(\lambda,\mu)-1]^2}{4\xi_S(1-\xi_S)} \,. \tag{A50}$$

Comparing (A410) with the invariant (9) in § 2.7.2 in [13], we can represent the hypergeometric parameters *a*, *b*. *c* defined via (7) in the cited subsection as follows

$$c-1 = 2l+1,\; c-a-b = 1-2\alpha(\lambda,\mu),\; b-a = 1-2\alpha^*(\lambda,\mu) \tag{A51}$$

By choosing the third hypergeometric parameter c to be larger than 1, we assure that the solution

$$\underset{\sim}{\Phi}[\xi_S;\alpha(\lambda,\mu),l] := \xi_S^{l+1}(\xi_S-1)^{\alpha(\lambda,\mu)}\, F(2\alpha_R(\lambda,\mu)+l,\; l+1+2i\,\alpha_I(\lambda,\mu), 2l+2;\xi_S) \quad \text{for} \quad |\xi_S| \le 1$$

(A52)

has the larger of the two *real* ChExp for the pole of the NODE (A41) at ξ_S=0 and therefore represents the PFS near the pole in question. The choice of the complex ChExp for the pole of the NODE (A41) at ξ_S=1 was motivated by the requirement that the given hypergeometric series converges on the circle of the radius 1 as far as the constraint (15) holds. Finally, the square root of the complex parameter $(b-a)^2$ was chosen in such a way that the first hypergeometric parameter *a* becomes real, making possible the truncation of the series into a hypergeometric polynomial.

To verify Stevenson' assertion that he proved the exact solvability of the problem in terms of the q-RSs, using the analytical continuation of the solution (A52) beyond the unit circle $|\xi_S| = 1$, let us analyze the restrictions on the complex variable ξ_S for the four possible analytical continuations given by (5.3.6)-(5.3.9) in [29].

Let us start from the analytical continuation (5.3.7) in the variable ξ_S^{-1}, which was detailed in the monograph [22] cited by Stevenson as the source for the background mathematical material. The author unable to proceed with the necessary argumentation because the analytical continuation in question has the cut at y=0 due to the constraint

$$|\arg(-1+i\,y)| < \pi \tag{A53}$$

and, therefore, it is applicable only to either positive or negative values of y. The cited analytical extension cannot be thus used for constructing the sought-for solutions with the continuous logarithmic derivatives at y=0.

We came to the similar problem, while trying to use the analytical continuation (15.3.6) in [29]:

$$F(l+1+2\alpha_R,\; l+1+2i\alpha_I, 2l+2;\xi_S) = \frac{\Gamma(2l+2)\Gamma(-2\alpha)}{\Gamma(l+1-2\alpha_R)\Gamma(l+1-2i\alpha_I)}$$

$$\times F(l+1+2\alpha_R,\; l+1+2i\alpha_I, 1-\, 2\alpha; 1-\xi_S)$$

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$$+\frac{\Gamma(2l+2)\Gamma(2\alpha^*)}{\Gamma(l+1+2\alpha_R)\Gamma(l+1-2i\alpha_I)}(1-\xi_S)^{-\alpha}\,F(l+1-2i\alpha_I, l+1-2\alpha_R; 1-2\alpha; 1-\xi_S)$$

$$(|\,arg(1-\xi_S)\,|<\pi), \qquad \text{(A54)}$$

with

$$1-\xi_S = exp(2i\chi) \quad \text{for} \quad 0<\chi<\pi. \qquad \text{(A55)}$$

It seems that the representation (A54) is applicable only to the segment $0<\chi<\pi/2$, which is also true for the analytical continuations (15.3.8) and (15.3.9).

Appendix D. Gendenshtein Potential as the TSI Limit

In the particular case of the single-pole density function ($\kappa = \kappa_R = 1$)

$$\rho_\Diamond[y] := \frac{1}{1+y^2}, \qquad \text{(A56)}$$

associated with the Gendenshtein potential [17], the complex quantity (68) and real quantities (70) become energy-independent, he complex quantity (72) becomes independent of l:

$$\underset{\sim}{\alpha}_n(h_o) := \underset{\sim}{\alpha}(h_o;1;l) = \tfrac{1}{2}\sqrt{h_o^*+1}+\tfrac{1}{2}, \qquad \text{(A57)}$$

and. as a result, the indexes of the R-Jacobi polynomials become independent of the polynomials degree:

$$_i\phi_n[y;h_o;1] = {}_iN_n(h_o;1)(y^2+1)^{\underset{\sim}{\alpha}_R(h_o)}\,exp[2\underset{\sim}{\alpha}_I(h_o)\,arctan\ y]\times$$
$$P_n\left(y; 2\underset{\sim}{\alpha}_I(h_o), -2\underset{\sim}{\alpha}_R(h_o)\right). \qquad \text{(A58)}$$

As pointed by the author [65], this is the direct consequence of the fact that the ChExps for the poles at the singular endpoints turn out to be energy independent.

The same year Odake and Sasaki [37,38] came up with the classification scheme of the translationally shape-invariant (TSI) potentials dividing them into two groups A and B. As clarified by us more recently [34], it is really the question of the energy-dependence of the ExpDiffs for the poles of the RCSLE used to express the eigenfunction of the given Schrödinger equation in the quasi-rational form. (As revealed o in [34], the potential may belong to different groups, depending on the change of variable used to convert the given Schrödinger equation to the RCSLE). The common remarkable feature of the TFI SLEs is that the Wronskians of the eigenfunctions can be represented in the form of the weighted Wronskians of orthogonal polynomials. The fundamentally important consequence of this observation is that each polynomial Wronskian represents an exceptional orthogonal polynomial (EOP) if the degrees of the given polynomial set are specified by a partition with no more than one odd-length segment [37,38]. The same year Gòmez-Ullate et al. [66] put forward a similar concept for the exceptional orthogonal polynomial systems (X-OPSs) formed by infinitely many polynomials. In contrast, Odake and Sasaki's analysis covered both finite and infinite EOP sequences, making their treatise especially useful for our purposes.

In the limit $\kappa = 1$ the quartic equation (88) turns into the quadratic equation:

$$(\Lambda_n+n+\tfrac{1}{2})^4-(h_{o;R}+1)(\Lambda_n+n+\tfrac{1}{2})^2-\tfrac{1}{4}h_{o;I}^2=0 \qquad \text{(A59)}$$

in $(\Lambda_n+n+\tfrac{1}{2})^2$, with the positive root given by the radical formula

$$(\Lambda_n+n+\tfrac{1}{2})^2=\tfrac{1}{2}(h_{o;R}+1)+\tfrac{1}{4}\sqrt{|\,h_o+1\,|^2}. \qquad \text{(A60)}$$

As the direct consequence of the fact that the RRef CSLE (59) with the density function (A56) belong to Group A of the TFI CSLEs, the orthonormalization relations (90) turn into the orthonormalization relations (A16) for the R-Routh polynomials, which gives

$$ {}_i N_n^2(h_o;1) = \frac{\Gamma(2-4\alpha_R(h_o)-n)}{2^{1-n-2\alpha_R(h_o)} n!\Gamma(1-4\alpha_R(h_o)-2n)\Gamma(2-4\alpha_R(h_o)-2n)} \times |\Gamma(1-2\alpha^*(h_o)-n)|^2 . \tag{A61} $$

(Remember that we were unable to analytically compute the normalization factors for the eigenfunctions of the TFI RCSLE (43), which belongs to Group B.)

Abbreviations

The following abbreviations are used in this manuscript:

ChExp	characteristic exponent
CLDT	canonical Liouville-Darboux transformation
CSLE	canonical Sturm-Liouville equation
DBC	Dirichlet boundary condition
d-PFS	dual principal Frobenius solution
EOP	exceptional orthogonal polynomial
ExpDiff	exponent difference
D$\mathfrak{T}$	Darboux transform
DT	Darboux transformation
FS	Frobenius solution
JRef	Jacobi-reference
GRef	Gauss-reference
GS	Gauss-seed
LC	limit-circle
LFT	linear fractional transformation
LRef	Laguerre-reference
*N*ODE	normal ordinary differential equation
ODE	ordinary differential equation
OPS	orthogonal polynomial system
PD	polynomial determinant
PF	polynomial fraction
PFS	principal Frobenius solution
p-ODE	prime ordinary differential equation
p-SLE	prime Sturm-Liouville equation
RCSLE	rational Sturm-Liouville equation
RDC$\mathfrak{T}$	rational Darboux-Crum transform
RDCT	rational Darboux-Crum transformation
RDT	rational Darboux transformation
RD$\mathfrak{T}$	rational Darboux transform
RDT	rational Darboux transformation
RefPF	reference polynomial fraction
RRef	Routh-reference
RS	Routh-seed
RRZT	rational Rudjak-Zakhariev transformation
RRZ$\mathfrak{T}$	rational Rudjak-Zakhariev transform
RZT	Rudjak-Zakhariev transformation
SLE	Sturm-Liouville equation
SLP	Sturm-Liouville problem
TF	transformation function
TP	tangent polynomial
t-RM	trigonometric Rosen-Morse
TFI	translationally form-invariant
TSI	translationally shape-invariant

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