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Set of Orthogonal Conditional Probability Distributions and Quantum Oscillator States

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Abstract: The notion of an infinite set of orthogonal conditional probability distributions describing quantum system states is introduced. The example of such orthogonal probability distributions is considered being associated with quantum oscillator states. The connection with separable and entangled probability distributions is discussed in the case of the probability distributions describing the oscillator energy states.

Keywords: probability representation; Hilbert space; quantum states; quantizer; dequantizer; density operator.

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1. Introduction

The nature of physical system states is described by quantum mechanics developed during the last century. Schrödinger equation [1,2] for the complex wave function of a system provides the possibility to find the system states and their evolution. In quantum mechanics, classical observables, like the position and momentum, are described by operators $\hat{q}$ and $\hat{p}$ acting onto wave functions. This picture differs from the human intuition of usual classical mechanics, but the development of quantum technologies should be described in the picture unusual from the classical point of view. Also, due to the existence of uncertainty relations, it is impossible to simultaneously measure the particle's position and velocity (momentum) in the experiment [3–5]; see also [6–8]. L.D. Landau elaborated more general picture, where states of quantum systems (called mixed states) were described by density matrices or density operators [9]; mixed states, describe, for example, molecules at given temperatures.

In classical thermodynamics, the molecular states are described by probability distributions. Since the initial attempts to find an analogous description of quantum system states failed, quasiprobability distributions, such as the Wigner function [10], the Husimi function [11], and the Glauber–Sudarshan function [12,13] were constructed; they are not probability distributions but have all information on the density operators of quantum system states. L.D. Landau described the properties of charge in magnetic field – Landau levels – in quantum mechanics [9]. Finally and fortunately, the probability representation of quantum states was shown to be constructed [14–33]. The correlated oscillator states were introduced and studied in [34–37]. Also, a charge moving in magnetic field was considered in [14] and various properties of quantum behavior of charges in magnetic fields were studied in [38–41].

The aim of our work is to introduce and discuss some properties of the probability representation of quantum oscillator states; namely, to consider a concrete construction of the invertible map of pure state vectors $|\psi\rangle$ belonging to the Hilbert space $\mathcal{H}$ of the oscillator states along with the map of the density operator $\hat{\rho}$ onto the function containing complete information on mixed states of the quantum oscillator. For such construction of the map, we introduce the notion of two sets of operators $\hat{U}(\vec{X})$ and $\hat{D}(\vec{X})$, where $\vec{X} = (X_1, \ldots, X_N)$ and $X_1, X_2, \ldots, X_N$ are either continuous or discrete parameters. After constructing the map, we study properties of functions called symbols of the operators.

A particular case of such a construction is the case where the obtained function symbols of density operators of quantum states have the properties of probability distribution functions; we call this map the probability representation of quantum oscillator states. The introduced map describes the oscillator states by the functions with specific properties, some of which have not been discussed in the literature.

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For example, the functions associated with the density operators of the oscillator energy levels obey new nonlinear relations of the Hermite polynomials describing the oscillator energy levels.

An analogous construction can be applied for the Wigner functions describing the oscillator energy levels, taking into account the fact that the Wigner functions describe the properties of quasidistribution functions of coherent states or the energy level states of quantum oscillators.

The technique of constructing an invertible map of operators onto functions is valid for many different cases. For this, one should find the pairs of quantizer $\hat{D}(X)$ and dequantizer $\hat{U}(X)$ operators depending on specific parameters and acting in a Hilbert space. There are many such possibilities, and there exists a connection between the selected pairs of the quantizer–dequantizer choice.

2. Quantizer and Dequantizer Operators

To formulate the approach for obtaining new properties of probability distributions used to describe pure and mixed quantum system states, first we recall the notion of quantizer $\hat{D}(X)$ and dequantizer $\hat{U}(X)$ operators for the one-dimensional oscillator [42,43]. These operators are used to construct an invertible map of the operators acting in a Hilbert space $\mathcal{H}$ of quantum states with arbitrary operators $\hat{A}$ acting onto the vectors $\mid \psi\rangle$, including the state density operators $\hat{\rho}$ of both pure and mixed states, onto the function $f_A(X)$ called symbol of the operator $\hat{A}$. The invertible map is given by the following relations:

$$f_A(X) = \mathrm{Tr}\big[\hat{A}\hat{U}(X)\big], \tag{1}$$

$$\hat{A} = \int f_A(X)\hat{D}(X)\, dX, \tag{2}$$

with the argument $X = \{X_1, X_2, \ldots, X_N\}$, where some parameters X_k can be continuous ones and some parameters can be discrete ones; in the latter case, the integration in (2) is considered as summation over the discrete parameters. For density operators of quantum states $\hat{\rho}$, we have symbols of the operators $f_{\hat{\rho}}(X)$ with extra properties, corresponding to the conditions $\hat{\rho}^\dagger = \hat{\rho}$ and $\mathrm{Tr}\,\hat{\rho} = 1$.

Pure state vectors of multimode oscillator satisfy the Schrödinger equation with Hamiltonian

$$\hat{H} \mid \psi_{\vec{n}}\rangle = E_{\vec{n}} \mid \psi_{\vec{n}}\rangle; \quad \vec{n} = (n_1, n_2, \ldots, n_N), \tag{3}$$

and the density operator $\hat{\rho}_{\vec{n}}(X)$ satisfies the corresponding equation

$$\frac{\partial}{\partial t}\hat{\rho}_{\vec{n}}(t) + i\big[\hat{H}, \hat{\rho}_{\vec{n}}(t)\big] = 0; \tag{4}$$

we assume Planck's constant $\hbar = 1$.

The pure state density operators with state vector $\mid \psi\rangle$ read

$$\hat{\rho}_\psi = \mid \psi\rangle\langle\psi \mid, \tag{5}$$

where

$$\mathrm{Tr}\,\hat{\rho}_\psi = \langle\psi \mid \psi\rangle = 1. \tag{6}$$

The energy levels are described by the density operators satisfying the orthogonality conditions

$$\langle\psi_{\vec{n}_1} \mid \psi_{\vec{n}_2}\rangle = \mathrm{Tr}\big(\hat{\rho}_{\vec{n}_1}\,\hat{\rho}_{\vec{n}_2}\big) = \delta_{\vec{n}_1\,\vec{n}_2}. \tag{7}$$

For symbols of arbitrary operators $\hat{A}_1$ and $\hat{A}_2$, we obtain the relation

$$\mathrm{Tr}\big(\hat{A}_1\,\hat{A}_2\big) = \mathrm{Tr}\int f_{A_1}(X_1)\, f_{A_2}(X_2)\,\hat{D}(X_1)\,\hat{D}(X_2)\, dX_1\, dX_2, \tag{8}$$

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which can be rewritten as follows:

$$\mathrm{Tr}\big(\hat{A}_1\,\hat{A}_2\big) = \int f_{A_1}(X_1)\,f_{A_2}(X_2)\,K(X_1,X_2)\,dX_1\,dX_2, \tag{9}$$

where the kernel $K(X_1, X_2)$ reads

$$K(X_1,X_2) = \mathrm{Tr}\big(\hat{D}(X_1)\,\hat{D}(X_2)\big). \tag{10}$$

An important partial case of this relation takes place for density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ of the orthogonal states $\mid\psi_1\rangle$ and $\mid\psi_2\rangle$, namely,

$$\mathrm{Tr}(\hat{\rho}_1\,\hat{\rho}_2) = 0; \tag{11}$$

this relation in terms of symbols of the density operators reads

$$\int K(X_1,X_2) f_{A_1}(X_1)\,f_{A_2}(X_2)\,dX_1\,dX_2 = 0 \tag{12}$$

and corresponds to the orthogonality relation for arbitrary symbols of quantum system orthogonal states, including the probability distributions.

If we have density operators of orthogonal energy states $\hat{\rho}_{E_k}$; $\hat{\rho}_{E_k} \equiv \hat{\rho}_k$ and consider symbols of the operators, we arrive at the equality

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}_k\hat{\rho}_k\cdots\hat{\rho}_k) = \int & f_k(X_1) f_k(X_2)\cdots f_k(X_n) \\ &\times K(X_1,X_2,\ldots,X_n) dX_1\,dX_2\cdots dX_n = 1,\end{aligned} \tag{13}$$

where the kernel reads

$$K(X_1,X_2,\ldots,X_n) = \mathrm{Tr}\big(\hat{D}(X_1)\hat{D}(X_2)\cdots\hat{D}(X_n)\big). \tag{14}$$

If we have the equality $\hat{\rho}_n^2 = \hat{\rho}_n$ for pure states $\mid\psi_n\rangle\langle\psi_n\mid = \hat{\rho}_n$, we obtain the equality of symbols of the operator $\hat{\rho}_n$, which provides the equality of symbols of operators $\hat{\rho}_n$ of the form

$$\int K(X_1,X_2,X_3)\,f_n(X_1)\,f_n(X_2)\,dX_1\,dX_2 = f_n(X_3), \tag{15}$$

where $f_n(X_3)$ is the same symbol of the density operator $\hat{\rho}_n$ of the pure state $\mid\psi_n\rangle$.

3. A *Chance* Operator

In the probability representation of quantum mechanics, dequantizer has the physical meaning of a *chance* operator.

Now we discuss the physical meaning of operator $\hat{U}$, which is the dequantizer (*chance*) operator for the system state with the density operator $\hat{\rho}$ given by the relation $f_\rho = \mathrm{Tr}\big(\hat{\rho}\hat{U}\big)$ determining symbol of the density operator $\hat{\rho}$. For the pure system state with the state vector $\mid\psi\rangle$, symbol of the operator $\hat{\rho}$ reads

$$f_\rho = \langle\psi\mid\hat{U}\mid\psi\rangle, \tag{16}$$

which is the mean value of the operator $\hat{U}$. If the operator $\hat{U}$ has the properties of the density operator, as it takes place in the probability representation of quantum states [42], symbol f_ρ has the meaning of probability distribution function of some physical observable. In turn, this means that the dequantizer operator determines a chance to have a large or a small value of the observable in such a state, and the dequantizer operator can be called as a *chance* operator.

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Let us consider a particular system of the one-dimensional harmonic oscillator and the probability representation of quantum states known as symplectic tomographic probability representation [15]. In this representation, the pure oscillator quantum state $\mid \psi\rangle$ is determined by the formula

$$G_\psi(X|\mu,\nu) = \langle \psi \mid \delta\big(X\hat{\mathbf{1}} - \mu\hat{q} - \nu\hat{p}\big) \mid \psi\rangle, \tag{17}$$

using the tomogram $G_\psi(X|\mu,\nu)$ [43]. This symbol is the probability distribution of the position X measured in a set of fixed reference frames of positions q and momenta p (the phase space) determined by parameters μ and ν; namely, by real parameters X, μ, and ν, for which

$$X\hat{\mathbf{1}} = \mu\hat{q} + \nu\hat{p}. \tag{18}$$

The tomogram is the probability distribution of position X, providing a chance to obtain this position in the reference frame in the phase space determined by the parameters μ and ν. In view of this chance, the dequantizer operator $\hat{U}(X,\mu,\nu)$ can be called the *chance* variable operator.

For mixed states with the density operator $\hat{\rho}$, the formula for the probability $G_\psi(X|\mu,\nu)$ changes to the following one:

$$G_\rho(X|\mu,\nu) = \mathrm{Tr}\hat{\rho}\;\delta\big(X\hat{\mathbf{1}} - \mu\hat{q} - \nu\hat{p}\big). \tag{19}$$

The states of other systems like, for example, a charge in magnetic field can also be described by dequantizers, and it provides the possibility to evaluate the system states by probability distribution functions.

In the case of ground oscillator state with the wave function $\psi_0(x) = \pi^{-1/4}e^{-x^2/2}$, its tomogram is

$$G_0(X|\mu,\nu) = \frac{\pi^{-1/2}}{\sqrt{\mu^2+\nu^2}}\exp\left(-\frac{X^2}{\mu^2+\nu^2}\right). \tag{20}$$

The general formula for any pure state of the oscillator system was obtained in [15]; it reads

$$G_\psi(X|\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(x)\exp\left(\frac{i\mu x^2}{2\nu} - \frac{ixX}{\nu}\right)dx\right|^2. \tag{21}$$

The inverse relation for the density matrix $\rho_\psi(x,x')$ of the pure state in the position representation is

$$(\hat{\rho}_\psi)_{xx'} = \frac{1}{2\pi}\int G(X|\mu,\nu)[\exp\{i\,(X\hat{\mathbf{1}} - \mu\hat{q} - \nu\hat{p})\}]_{xx'}dXd\mu\,d\nu, \tag{22}$$

where $\rho_\psi(x,x') \equiv (\hat{\rho}_\psi)_{x\,x'} = \psi(x)\psi^*(x')$.

Thus, the mean value of *chance* dequantizer operator is the conditional probability distribution function determining the existing way to evaluate the value of the position X measured in the oscillator phase space with given axes of the positions and given axes of momenta.

The density operator of the oscillator mixed state $\hat{\rho}$ is determined by the probability distribution $G(X|\mu,\nu)$ and the quantizer operator

$$\hat{\rho} = \frac{1}{2\pi}\int G(X|\mu,\nu)\exp[i(X-\mu\hat{q}-\nu\hat{p})]dX\,d\mu\,d\nu, \tag{23}$$

and the pure-state density operator $(\hat{\rho}_\psi)_{xx'}$ has the matrix elements in the position representation given by equation (22). One can check that, in view of equation (22), it is not difficult to obtain for the ground state of the harmonic oscillator the following equality:

$$\frac{1}{\sqrt{\pi}}\exp\left(-\frac{x^2}{2}-\frac{x'^2}{2}\right) = \frac{1}{2\pi}\int\frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}$$
$$\times\exp\left(-\frac{X^2}{\mu^2+\nu^2}\right)\exp[i(X-\mu\hat{q}-\nu\hat{p})]_{xx'}dX\,d\mu\,d\nu. \tag{24}$$

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4. Energy Levels and Probabilities Describing the Oscillator Stationary States

It is well known [44] that the energy levels of the oscillator state $\mid n\rangle$, such that $\psi_n(x) = \langle x \mid n\rangle$, are given in terms of the creation operator $\hat{a}^\dagger$ and the ground state $\mid 0\rangle$ as follows:

$$\mid n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} \mid 0\rangle; \quad n = 0, 1, 2. \dots \tag{25}$$

and the ground state satisfies the equality

$$\hat{a} \mid 0\rangle = 0, \tag{26}$$

where the annihilation operator $\hat{a}$ reads

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}) = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right). \tag{27}$$

Thus, we have

$$\psi_n(x) = \langle x \mid n\rangle = \pi^{-1/4}\frac{e^{-x^2/2}}{\sqrt{2^n n!}} H_n(x), \tag{28}$$

where $H_n(x)$ is the Hermite polynomial. Using the relationship between the probability distribution $G_n(X|\mu,\nu)$ and the wave function ψ_n given by (22), we have the tomogram (conditional probability distribution) of the form

$$G_n(X|\mu,\nu) = G_0(X|\mu,\nu)\frac{1}{2^n n!}\left|H_n\left(\frac{X}{\sqrt{\mu^2+\nu^2}}\right)\right|^2 \tag{29}$$

expressed in terms of Hermite polynomials. The probability distributions satisfy the integral equalities

$$\hat{\rho}_n\hat{\rho}_n = \hat{\rho}_n \quad \text{or} \quad \mathrm{Tr}(\hat{\rho}_n\hat{\rho}_n) = \int G_n(X|\mu,\nu)dX = 1.$$

Also, there exists the orthogonality relation $\mathrm{Tr}(\hat{\rho}_n\hat{\rho}_{n'}) = 0$, namely,

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}_n\hat{\rho}_n) = \frac{1}{2\pi}\int & G_n(X|\mu,\nu)G_n(Y|-\mu,-\nu) \\ & \times \exp(iX+iY)dX\,dY\,d\mu\,d\nu = 1.\end{aligned} \tag{30}$$

Finally, we arrive at the integral relation for Hermite polynomials; it reads

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}_n\hat{\rho}_{n'}) = \frac{1}{2\pi}\int G_0(X|\mu,\nu)G_0(Y|-\mu,-\nu)\frac{\exp(iX+iY)}{2^{n+n'}n!n'!} \\ \times\left|H_n\left(\frac{X}{\sqrt{\mu^2+\nu^2}}\right)H_{n'}\left(\frac{Y}{\sqrt{\mu^2+\nu^2}}\right)\right|^2 dX\,dY\,d\mu\,d\nu = \delta_{nn'},\end{aligned} \tag{31}$$

where $G_0(X|\mu,\nu)$ is given by equation (20).

Thus, we obtained the integral relations for Hermite polynomial functions describing the properties of the set of orthogonal conditional probability distributions of quantum oscillator states.

5. Superposition Principle and Probability Representation of Quantum States

Using the introduced construction of the probability representation of quantum states, we consider a combination of probability distributions not yet employed in the probability theory. Namely, for given two-state vectors $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$ belonging to the Hilbert space $\mathcal{H}$, the vector $\mid \psi\rangle$,

$$\mid \psi\rangle = c_1 \mid \psi_1\rangle + c_2 \mid \psi_2\rangle, \tag{32}$$

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being a superposition of these two vectors, also describes the quantum state, and this is the superposition principle of quantum mechanics. Here, numbers c_1 and c_2 are such complex numbers that, for normalized vectors $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$, the vector $\mid \psi\rangle$ is also normalized.

Due to our constructed map (19), any state vector $\mid \psi\rangle$ as well as vectors $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$ have symbols, which are the probability-distribution functions. The idea is to apply the map given by (19). We arrive at the probability distribution describing the probability distributions associated with state vectors $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$; it reads

$$\begin{aligned}\langle \psi \mid \hat{U} \mid \psi\rangle &= |c_1|^2\langle \psi_1 \mid \hat{U} \mid \psi_1\rangle + |c_2|^2\langle \psi_2 \mid \hat{U} \mid \psi_2\rangle \\ &+c_1c_2^*\langle \psi_2 \mid \hat{U} \mid \psi_1\rangle + c_2c_1^*\langle \psi_1 \mid \hat{U} \mid \psi_2\rangle. \end{aligned} \tag{33}$$

In the case of one-dimensional harmonic oscillator, dequantizer operator $\hat{U}$ is given by equation (19), and symbols of the states $\mid \psi\rangle$ with the wave function $\psi(x)$ are given by the following integral relation:

$$\begin{aligned}G_\psi(X|\mu,\nu) &= \frac{1}{2\pi|\nu|} \\ &\times\left|\int [c_1\psi_1(x) + c_2\psi_2(x)] \exp\left(\frac{i\mu x^2}{2\nu} - \frac{ixX}{\nu}\right) dx\right|^2. \end{aligned} \tag{34}$$

The function $G_\psi(X|\mu,\nu)$ is the probability distribution describing the state with the wave function ψ in the probability representation of quantum states. Finally, we have

$$G(X|\mu,\nu) = |c_1|^2G_1(X|\mu,\nu) + |c_2|^2G_2(X|\mu,\nu) + \delta\, G_3(X|\mu,\nu). \tag{35}$$

In such a way, we can provide the rule for the superposition of probability distribution functions determined by quantum-system wave functions. For example, the Hermite polynomials determining the energy state of harmonic oscillator $\mid n\rangle$ satisfy this superposition principle, and this fact provides the possibility to obtain new relations for Hermite polynomials.

6. Dequantizers and Superposition Principle for Symbols of Density Operators

The discussed rule for superpositions of probability distributions can be extended to arbitrary kinds of symbols of density operators including the Wigner functions and Husimi functions. Using the dequantizer operator $\hat{U}$, we can obtain symbol of the operator $\hat{\rho}_\psi =\mid \psi\rangle\langle\psi \mid$,

$$f_{\rho_\psi} = \langle \psi \mid \hat{U} \mid \psi\rangle, \tag{36}$$

where $\mid \psi\rangle = c_1 \mid \psi_1\rangle + c_2 \mid \psi_2\rangle$. In this way, we arrive at new general formula for any kinds of operators; it reads

$$f_{\rho_\psi} = |c_1|^2 f_1 + |c_2|^2 f_2 + \delta f_{12}, \tag{37}$$

where δf_{12}

$$\delta f_{12} = c_1^*c_2\langle \psi_1 \mid \hat{U} \mid \psi_2\rangle + c_1c_2^*\langle \psi_2 \mid \hat{U} \mid \psi_1\rangle. \tag{38}$$

Thus, we formulated the superposition principle in the case of any kinds of operator symbols. Analogous formulas for the superposition of two-mode analytical signals in the tomographic-probability representation was discussed in [45], where the entanglement phenomenon for the superposition of two-mode wave functions was considered. In such states, the entangled probability distributions appeared as was shown in [20,21].

For the Husimi function $H_\rho(\alpha)$, the dequantizer $\hat{U}$ is described by the coherent state vectors $\mid \alpha\rangle$; it reads

$$\hat{U} =\mid \alpha\rangle\langle\alpha \mid .$$

For the Wigner function $W(q, p)$, the dequantizer $\hat{U}$ is described by the position state vectors $\mid x\rangle$; $\hat{U}$ has the form

$$\hat{U} = \frac{1}{2\pi} \int \mid q + u/2\rangle\langle q - u/2 \mid e^{-ipu}\, du.$$

For the states with density operators $\hat{\rho} = p_1\hat{\rho}_1 + p_2\hat{\rho}_2$, where p_1 and p_2 are the positive probabilities, such that $p_1 + p_2 = 1$, all kinds of symbols, like probabilities, Wigner functions, Husimi functions, correspond to the conditions

$$\mathrm{Tr}\big(\hat{\rho}\hat{U}\big) = p_1\mathrm{Tr}\big(\hat{\rho}_1\hat{U}\big) + p_2\mathrm{Tr}\big(\hat{\rho}_2\hat{U}\big).$$

For pure states $\hat{\rho}_1 = \mid \psi_1\rangle\langle\psi_1 \mid$ and $\hat{\rho}_2 = \mid \psi_2\rangle\langle\psi_2 \mid$, the superposition principle of quantum mechanics provides the extra rule for combination of symbols of operators corresponding to the superposition principle for wave functions of pure quantum states; this rule was introduced in our consideration, and such a rule does not exist in classical statistical mechanics.

Thus, we pointed out that the superposition of quantum states formulated in the probability representation of quantum system states provides new formulas in the probability theory; they can be checked, in view of quantum information and quantum physics technologies.

7. Conclusions

Concluding, we pointed out the main results presented here. We considered the superposition principle of quantum states in quantum mechanics, using the probability and quaprobability representations of quantum system states. Our consideration of the probabilities associated with this principle was shown on the example of oscillator's quantum states, with the wave functions being the sums of other wave functions. The states are described by a new kind of probability distributions (called the entangled probability distributions); they were unknown in the probability theory. The new structure of such probabilities was demonstrated to be available for Wigner functions, Husimi functions and other symbols of density operators. Since the superposition principle of quantum mechanics is valid for any systems and any wave functions, relationships between the superposed states and symbols of the density operators can be used to construct different kinds of operator symbols, including the entangled probability distribution [20,21]. This kind of the superposed entangled probability distribution can be extended to construct the superposed Wigner function as well as other superposed density operator symbols of quantum states; this will be done in future publications. This approach can be also applied to the case of a charge moving in the time-dependent magnetic field discussed in [38–41].

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