Emergence of Quantum Correlations as Macro-Time Correlations Derived from Underlying Micro-Time Correlations

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Article

Emergence of Quantum Correlations as Macro-Time Correlations Derived from Underlying Micro-Time Correlations

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Abstract

This work introduces a rigorous mathematical approach for producing entangled quantum states from classical stochastic dynamics. We show that any density matrix ρ_{AB} describing a composite quantum system can be reconstructed from the correlations of two foundational stochastic processes, $X(t)$ and $Y(t)$, which model the random behavior of the individual subsystems. The framework employs a dual temporal scale—micro and macro time—where quantum correlations naturally arise as emergent macro-level correlations derived from fine-grained micro-level interactions. We formulate the **Double Covariance Model (DCM)**, which captures the essential features of quantum mechanics by interpreting the quantum state as a fourth-order statistical structure within an underlying classical probabilistic model.

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1. Introduction

The inquiry into the relationship between classical probability and the quantum formalism began at the inception of quantum theory over a century ago. This paper presents the *Double Covariance Model (DCM)*, a framework that provides a stochastic reconstruction of entangled quantum states through the interplay of micro and macro time scales.

The DCM treats the quantum density operator as a hierarchical statistical construct. It posits that a density operator is effectively a "covariance of covariances":

- **Primary (Micro) Covariance**: The temporal synchronization of subquantum processes $X(t)$ and $Y(t)$ defines a random operator at the micro-scale.
- **Secondary (Macro) Covariance**: The ensemble stability of these random operators across a macro-scale temporal window defines the density operator ρ_{AB}.

From the perspective of multivariate statistics, the DCM interprets the quantum state as the fourth-order moment structure of an underlying classical probability space. This approach demonstrates that any density operator of a composite system can be derived from the fourth-order correlations between two underlying classical stochastic processes.

Entanglement as Micro-Time Consistency

A central thesis of DCM is that entanglement is not based on the statistical dependence of subquantum processes at the macro level. Instead, entanglement is interpreted as a macro-time phenomenon reflecting **micro-time consistency**. In this framework, subquantum processes can be statistically independent while remaining pathwise constrained (consistent) at the micro-scale. For example, Bell states can be generated by partitioning a macro-time window into subintervals where specific micro-correlations are satisfied.

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Foundations and Context

The DCM is part of a long lineage of attempts to bridge the classical and quantum regimes, including Wigner functions - providing a quasi-probability distribution in phase space [12,13], De Broglie's Double Solution Model - An early attempt at a causal, wave-particle duality [14,15], von Neumann's No-Go Theorem - the early formal argument against hidden variables [16], Bell's Theorem - establishing the boundaries of local realism [17–19], Stochastic Electrodynamics (SED) - attributing quantum effects to a classical zero-point field [7–10], Hydrodynamic Models - representing the Schrödinger equation as fluid dynamics [20,21], Bohmian Mechanics - a deterministic, non-local pilot-wave theory [22,23], Kochen-Specker Theorem - Highlighting the role of contextuality [27,28], Hydrodynamic Droplet Systems - classical fluid dynamics with pilot-wave behavior [24,26], Quantum–classical Hybrid Models - coupling quantum and classical variables within a single framework [33,34], Prequantum Classical Statistical Field Theory (PCSFT) - theory of prequantum random fields [29–32]

While PCSFT served as a catalyst for this work, the DCM takes a significant conceptual step forward by grounding entanglement in the dynamic interplay between micro- and macro-level covariances.

1.0.1. Models in Physics Based on Micro-Macro Time Scale Interplay

In various branches of physics, the transition from fundamental fluctuations to observable macroscopic behavior is often modeled through the interplay of two or more distinct time scales. This section summarizes key models where a micro scale (rapid fluctuations or collisions) and a macro scale (ensemble stability or order parameters) are utilized.

Langevin dynamics and Brownian motion [1,2]: The classic description of a macroscopic particle suspended in a fluid. The micro-scale consists of rapid, stochastic collisions with fluid molecules (10^{-12} s), while the macro-scale describes the observable diffusion of the particle. The interplay is captured by the Langevin equation, where micro-fluctuations are modeled as a white noise term.

Haken's synergetics and the slaving principle [3,4]: This framework describes self-organizing systems. It relies on the adiabatic elimination of fast-relaxing micro variables. The macro behavior is governed by a few slow-moving order parameters that "slave" the micro-components, leading to emergent patterns in lasers and fluids.

Kinetic theory and the BBGKY hierarchy [5,6]: In statistical mechanics, the transition from reversible micro-dynamics to irreversible macro-thermodynamics requires a hierarchy of scales. The micro-scale is the collision time, while the macro-scale is the relaxation time to equilibrium. The Boltzmann equation emerges by coarse-graining the micro-correlations [5,6].

Stochastic electrodynamics [10], [7]-[9]: SED posits that quantum effects arise from the interaction of classical particles with a classical, stochastic zero-point field (ZPF). The micro-scale involves the high-frequency fluctuations of the ZPF, while the macro-scale involves the averaged motion of particles that mimics quantum mechanics.

Brownian entanglement [11]: For two interacting classical Brownian particles, the separation of micro and macro time scales generates coarse-grained velocity–position correlations that cannot be factorized, creating a classical analog of entanglement. This micro–macro time interplay produces entanglement-like correlations that vanish under finer temporal resolution.

The Relational Nature of Systems

The DCM further challenges the notion of the isolated system. It suggests that the distinction between "composite" and "individual" systems is relational rather than ontological. In this view:

- A **composite system** is one where micro-synchronization between internal processes is explicitly resolved.
- An **individual system** is a marginal residue of a larger, synchronized whole, where internal synchronization is treated as a unified, emergent fluctuation.

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The paper proves that the partial trace operation in quantum mechanics is equivalent to the marginalization of hidden correlations in the underlying classical space. Thus, the "Quantum State" of an individual system is not a standalone primitive but a reduced description of its participation in a larger global field.

Coupling with Relational Quantum Mechanics

A distinctive feature of the DCM is its alignment with the conceptual foundations of Relational Quantum Mechanics (RQM) [35]-[37]. In this framework, the identity of a system is not an absolute ontological primitive but is fundamentally *relational*. By defining the density operator as a marginal residue of a larger synchronized field, the DCM provides a mathematical realization of the RQM thesis: that the state of a system is always relative to the observer or the surrounding environment. Here, the distinction between a "composite" and an "individual" system is determined by the window of synchronization (Δ), suggesting that what we perceive as an isolated quantum state is actually a localized manifestation of a global covariance structure.

The DCM offers a solution to the "state of the universe" problem in RQM. In the DCM, the Universe is the only system with perfect global synchronization; every other state we measure is a partial, relational view necessitated by our local perspective.

2. Remarks on Mathematics

We aim to present DCM at a rigorous mathematical level while avoiding unnecessary technical overload. Our goal is to make the paper accessible to a broad audience. In principle, the reader may follow the paper using a heuristic understanding of probability theory, random variables, and stochastic processes. A distinctive feature of our probabilistic constructions is that all random variables are complex-valued; consequently, covariances are defined using complex conjugation.

To simplify functional-analytic considerations, we assume throughout that all Hilbert spaces are finite-dimensional.

Throughout this paper the symbols H_A, H_B denote complex Hilbert spaces; $H_A \otimes H_B$ is their tensor product Hilbert space; denote the space of linear operators from H_B to H_A by the symbol $\mathcal{L}(H_B, H_A)$; for Hilbert space H, $\mathcal{L}(H) = \mathcal{L}(H, H)$.

On the space $\mathcal{L}(H_B, H_A)$ we introduce the Hilbert space structure; for operators $\hat{V}_1, \hat{V}_2$, their scalar product is defined as

$$\langle \hat{V}_1 | \hat{V}_2 \rangle = \mathrm{Tr}\hat{V}_1^\star \hat{V}_2 \tag{1}$$

(we remark that $\hat{V}_1 : H_B \to H_A, \hat{V}_1^\star; H_A \to H_B$, so $\hat{V}_1^\star \hat{V}_2 : H_B \to H_B$). In particular,

$$||\hat{V}||^2 = \mathrm{Tr}\hat{V}^\star \hat{V}.$$

We will use the fundamental isomorphism

$$H_A \otimes H_B \cong \mathcal{L}(H_B, H_A). \tag{2}$$

By exploiting this isomorphism, we establish a direct connection between the theory of operator-valued random variables (random matrices) and the quantum formalism based on density operators.

We shall use the hat-symbol to denote operators; for a vector $\Psi \in H_A \otimes H_B$, the corresponding operator is denoted as $\hat{\Psi}$ and for an operator $\hat{V} \in \mathcal{L}(H_B, H_A)$, the corresponding vector is denoted as $|V\rangle$. We will often go from vectors to operators and vice verse.

Isomorphism (2) is defined as follows. Let $(|a\rangle)$ and $(|b\rangle)$ be two orthonormal bases in H_A and H_B respectively. Take any vector $|\Psi\rangle \in H$, so $|\Psi\rangle = \sum_{a,b} k_{ab}|ab\rangle$. The corresponding operator $\hat{\Psi}$ is defined as

$$\hat{\Psi}|\phi\rangle = \sum_a (\sum_b k_{ab}\langle b|\phi\rangle)|a\rangle. \tag{3}$$

 doi:10.20944/preprints202601.1807.v1

The map

$$J : \Psi \to \hat{\Psi} \tag{4}$$

is a unitary operator; its definition doesn't depend on selection of bases.

This construction and our formalism generally can be easily generalized to the infinite dimensional case by consideration of Hilbert-Schmidt operators, see appendix A. This case can be interesting for physicists, since in the L_2-case the unitary operator J maps kernels to integral operators (see von Neumann [16]).

We will also use so called *superoperators* - linear operators acting in the spaces of linear operators. We shall use the symbol "wide-hat" to denote superoperators, as $\widehat{C}$.

3. The Density Operator as a Double Covariance: From Micro- to Macro-Scale Correlations

Let $(\Omega, \mathcal{F}, P)$ be a classical probability space (Kolmogorov [38]): Ω is a set of chance parameters ("elementary events"), $\mathcal{F}$ is collection of events, and P is a probability measure defined on $\mathcal{F}$.

Let H_A and H_B be Hilbert spaces. Consider two stochastic processes:

$$X(t) : \Omega \to H_A, \quad Y(t) : \Omega \to H_B.$$

where the processes have zero mean value, $\mathbb{E}[X(t)] = 0, E[Y(t)] = 0$ for any $t \geq 0$. We also assume that these processes have finite second order moments: $E[||X(t)||^2] < \infty, E[||Y(t)||^2] < \infty$.

They describe stochasticity in two systems S_A and S_B; stochasticity in a composite system $S_{AB} = (S_A, S_B)$ is described by the process valued in $H_A \times H_B$ with the coordinate processes $X(t), Y(t)$. Stochasticity under consideration is classical. However, we will see that it can be represented in quantum-like way - by a density operator. Classicality is a feature on the micro-time dynamics. Transition from micro-time scale to macro-time scale leads to the quantum representation.

So, we consider two time scales: a micro-time scale and a macro-time scale. The micro- and macro-time variables are denoted as t and τ. The scale of macro-time is determined by an interval Δ, this is an instant of macro-time τ. The chance parameter $\omega \in \Omega$ describes en ensemble of intervals Δ, a sample of instances of macro-time.

1. The Micro-scale Cross-Covariance Operator

We define the windowed cross-covariance operator $\hat{C}_\Delta \in \mathcal{L}(H_B, H_A)$ over a time window Δ - the micro-scale cross-covariance, a bilinear form. For vectors $h_a \in H_A$ and $h_b \in H_B$:

$$\langle h_a | \hat{C}_\Delta | h_b \rangle = \frac{1}{|\Delta|} \int_\Delta \langle h_a | X(t) \rangle \langle Y(t) | h_b \rangle \, dt \tag{5}$$

This definition ensures that $\hat{C}_\Delta$ acts linearly on $h_b \in H_B$ through the term $\langle Y(t) | h_b \rangle$. In operator notation,

$$\hat{C}_\Delta = \frac{1}{|\Delta|} \int_\Delta |X(t)\rangle \langle Y(t)| \, dt \in \mathcal{L}(H_B, H_A). \tag{6}$$

We point out that $\hat{C}_\Delta = \hat{C}_\Delta(\omega)$ is a random operator, $\hat{C}_\Delta : \Omega \to \mathcal{L}(H_B, H_A)$.

The matrix elements with respect to orthonormal bases $\{|a\rangle\}$ and $\{|b\rangle\}$ are:

$$c_{ab} = \langle a | \hat{C}_\Delta b \rangle = \frac{1}{|\Delta|} \int_\Delta X_a(t) \overline{Y_b(t)} \, dt \tag{7}$$

where $X_a(t) = \langle a | X(t) \rangle$ and $\overline{Y_b(t)} = \langle Y(t) | b \rangle$. And all these quantities depend on a random parameter ω.

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2. Vectorization and the Macro-level

We utilize the identification $\mathcal{L}(H_B, H_A) \cong H_A \otimes H_B$. Under this isomorphism, the random operator $\hat{C}_\Delta = \hat{C}_\Delta(\omega)$ is represented as a random vector $|C_\Delta\rangle = |C_\Delta\rangle(\omega)$ in the tensor product $H_A \otimes H_B$:

$$|C_\Delta\rangle = \frac{1}{|\Delta|}\int_\Delta |X(t)\rangle \otimes |Y(t)\rangle\, dt \tag{8}$$

The centered random variable representing the micro-scale fluctuations is

$$|Z_\Delta\rangle = |C_\Delta\rangle - \mathbb{E}[|C_\Delta\rangle], \tag{9}$$

where $\mathbb{E}$ denotes the mathematical expectation w.r.t. probability P - statistical expectation.

3. The Macro-Covariance Operator

The macro-covariance operator $\hat{C}$ is defined as the covariance of the $H_A \otimes H_B$-valued random variable $|Z_\Delta\rangle$. Following the convention of linearity in the second argument:

$$\hat{C} = \mathbb{E}[|Z_\Delta\rangle\langle Z_\Delta|] \in \mathcal{L}(H_A \otimes H_B) \tag{10}$$

This operator $\hat{C}$ is Hermitian and positive semi-definite. The normalized density operator is given by $\rho = \hat{C}/\mathrm{Tr}(\hat{C})$.

3.1. Density Operator from Micro-Scale Time Series

This abstract framework can be operationalized through the following scheme, which allows for experimental verification.

We consider again two time scales: a micro-time scale and a macro-time scale. The micro-time variable are denoted by t. The scale of macro-time is determined by an interval Δ, this is an instant of macro time. The macro-time variable is denoted by τ; in the discrete framework:$\tau_k = k\Delta, \quad k = 0, 1, 2, \ldots$. We define the associated micro-scale time windows as $\Delta_k := [k\Delta, (k+1)\Delta)$. Fix a sufficiently large integer N and a macro-scale time interval $[0, \mathcal{T}], \quad \mathcal{T} = N\Delta$, so that $[0, \mathcal{T}) = \Delta_0 \cup \cdots \cup \Delta_{N-1}$, where the intervals Δ_k are non-overlapping.

Consider two time series $X(t_i)$ and $Y(t_i)$, with $i = 1, 2, \ldots$. For each interval Δ_j (corresponding to the macro-time instant τ_j), we compute the sample cross-correlation at the micro scale,

$$C(\tau_j) := \frac{1}{|\{t_k \in \Delta_j\}|}\sum_{t_k \in \Delta_j} \big(X(t_k) \otimes Y(t_k)\big). \tag{11}$$

This defines a time series taking values in the tensor-product Hilbert space $H_A \otimes H_B$.

We centralize this sample by subtracting its empirical mean,

$$Z(\tau_j) := C(\tau_j) - \bar{C}, \qquad \bar{C} := \frac{1}{N}\sum_{j=1}^{N} C(\tau_j). \tag{12}$$

This notation implies that each $Z(\tau_j)$ is interpreted as a vector in the Hilbert space $H_A \otimes H_B$.

Finally, we define the macro-scale covariance operator by

$$\hat{C} = \frac{1}{N}\sum_{j=1}^{N} |Z(\tau_j)\rangle\langle Z(\tau_j)| \ \in \ \mathcal{L}(H_A \otimes H_B), \tag{13}$$

the macro-covariance operator associated with the aggregated micro-scale dynamics.

 doi:10.20944/preprints202601.1807.v1

3.2. Density Operator as a Covariance of Random Operator

Now consider an random variable $\hat{\Sigma} = \hat{\Sigma}(\omega)$, valued in $\mathcal{L}(H_B, H_A)$, its covariance $\widehat{C}(\Sigma)$ is a linear operator acting in the space $\mathcal{L}(H_A, H_B)$, so called *superoperator* defined by its quadratic form,

$$\langle \hat{V}_1 | \widehat{C}(\Sigma) \hat{V}_2 \rangle = \mathbb{E}[\langle \hat{V}_1 | \hat{\Sigma} \rangle \langle \hat{\Sigma} | \hat{V}_2 \rangle] \tag{14}$$

where $\hat{V}_1, \hat{V}_2 \in \mathcal{L}(H_B, H_A)$. So, $\widehat{C}(\Sigma) \in \mathcal{L}(\mathcal{L}(H_A, H_B))$.

This definition implies that $\widehat{C}(\Sigma)$ can be represented as the expectation of the random rank-one superoperator formed by the outer product of $\hat{\Sigma}$ with itself, namely

$$\widehat{C}_\Sigma = \mathbb{E}[|\hat{\Sigma}\rangle\langle\hat{\Sigma}|], \tag{15}$$

where $|\hat{\Sigma}\rangle\langle\hat{\Sigma}|\hat{V} = \langle\hat{\Sigma}|\hat{V}\rangle\hat{\Sigma}$.

Now the random variable $Z_\Delta = Z_\Delta(\omega)$ valued in $H_A \otimes H_B$ can be treated as $\mathcal{L}(H_A, H_B)$-valued random variable $\hat{Z}_\Delta$ (see (2)). Its covariance operator $\hat{C}$ considered as an element of $\mathcal{L}(H_A \otimes H_B)$ is now realized as the covariance (super-)operator $\widehat{C} = \widehat{C}_{Z_\Delta} \in \mathcal{L}(\mathcal{L}(H_A, H_B))$ of the operator-valued random variable $\hat{Z}_\Delta$,

$$C_\Sigma = \mathbb{E}[|\Sigma\rangle\langle\Sigma|].$$

4. Construction of Classical Stochastic Processes Behind Density Operators

We start with pure states and consider the most striking example - a maximally entangled state, one of the Bell states.

4.1. Bell States from Classical Correlations

Consider the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \in H_A \otimes H_B.$$

Split a macro window Δ into two subintervals:

$$\delta_0 = [0, \Delta/2], \quad \delta_1 = [\Delta/2, \Delta].$$

Introduce two real valued random variable $\xi_A = \xi_A(\omega)$ and $\xi_B = \xi_B(\omega)$ describing macro-randomization in systems S_A and S_B respectively such that

$$E[\xi_A] = E[\xi_B] = 0, E[\xi_A \xi_B] = 0, E[\xi_A^2 \xi_B^2] = 2. \tag{16}$$

These are uncorrelated random variables, $\mathrm{Cov}(\xi_A, \xi_B) = 0$. In particular, they can be independent random variables with zero mean values and with normalization $E[\xi_A^2] = \sqrt{2}, E[\xi_B^2] = \sqrt{2}$. The random variables can be discrete and take e.g. values ± 1. In this example correlations are concentrated at the micro level. We remark that ω is a macro-parameter, selection of behaviour of systems during time window Δ.

Assign separable Schmidt components to each subinterval:

$$(X(t,\omega), Y(t,\omega)) = \begin{cases} (\xi_A(\omega)\,|0\rangle\,, \xi_B(\omega)\,|0\rangle) & t \in \delta_0, \\ (\xi_A(\omega)\,|1\rangle\,, \xi_B(\omega)\,|1\rangle) & t \in \delta_1. \end{cases}$$

Then the micro-level average is

$$C_\Delta(\omega) = \frac{1}{\Delta}\int_0^\Delta X(t) \otimes Y(t)\, dt = \xi_A(\omega)\xi_B(\omega)(\frac{1}{2}\,|0\rangle \otimes |0\rangle + \frac{1}{2}\,|1\rangle \otimes |1\rangle) =$$

 doi:10.20944/preprints202601.1807.v1

$$\xi_A(\omega)\xi_B(\omega)\,|\Phi^+\rangle\,/\sqrt{2}.$$

This is a random vector belonging to $H_A \otimes H_B$, and its ensemble average (macro-average) $E[C_\Delta] = E[\xi_A\xi_B]\Phi^+/\sqrt{2} = 0$, so $Z_\Delta = C_\Delta$. Hence,

$$\hat{C} = E[|C_\Delta\rangle\langle C_\Delta|] = (1/2)\mathbb{E}[\xi_A^2\xi_B^2]|\Phi^+\rangle\langle\Phi^+| = |\Phi^+\rangle\langle\Phi^+|.$$

We emphasize once again that subquantum stochastic processes are determined non-uniquely. Above, we presented a simple illustrative example; however, one can construct models with substantially richer internal randomness.

4.2. Generation of an Arbitrary Pure State

Here we present the simplest scheme of generation of an arbitrary pure state within DCM, similar to the scheme for the Bell state $|\Phi^+\langle$; more complex stochastic processes can be generated with the scheme of section 4.4.

Each vector $|\psi\rangle \in H_A \otimes H_B$ admits a Schmidt decomposition

$$|\psi\rangle = \sum_{\ell=1}^{r} s_\ell\,|u_\ell\rangle \otimes |v_\ell\rangle,$$

where $|u_\ell\rangle \in H_A$ and $|v_\ell\rangle \in H_B$, $s_\ell \geq 0$. If $|\psi\rangle$ is a quantum state, then $\sum_{\ell=1}^{r} |s_\ell|^2 = 1$. (Vectors (u_ℓ) are orthonormal as well as vectors (v_ℓ), but this property is not used in our construction.)

We now generalize the scheme that was used for generation of the Bell state $|\Phi^+\rangle$,

1. Partition Δ into r subintervals δ_ℓ of the lengths $(s_\ell/\sum_\ell s_\ell)\Delta$.
2. On the δ_ℓ-th subinterval, set

$$X(t,\omega) = \xi_A(\omega)|u_\ell\rangle, \qquad Y(t,\omega) = \xi_B(\omega)|v_\ell\rangle,$$

where the random variables $\xi_A(\omega), \xi_B(\omega)$ satisfy conditions similar to conditions (16),

$$E[\xi_A] = E[\xi_B] = 0, E[\xi_A\xi_B] = 0, E[\xi_A^2\xi_B^2] = (\sum_\ell s_\ell)^2. \tag{17}$$

In particular, we can consider two independent random variables with zero mean values, such that $E[\xi_A^2] = E[\xi_B^2] = (\sum_\ell s_\ell)$. The random variables can be discrete and take e.g. values ± 1. Thus, entanglement is generated by microcorrelations.

The micro-covariance is given by

$$C_\Delta(\omega) = \frac{1}{|\Delta|}\int_\Delta X(t,\omega)\otimes Y(t,\omega)\,dt = \frac{\xi_A(\omega)\xi_B(\omega)}{\sum_\ell s_\ell}\sum_{\ell=1}^{r} s_\ell|u_\ell\rangle\otimes|v_\ell\rangle. = \frac{\xi_A(\omega)\xi_B(\omega)}{\sum_\ell s_\ell}|\psi\rangle.$$

we remark that due to our construction, this is a centered random variable, $\mathbb{E}[C_\Delta] = 0$. Now we find its macro-covariance

$$\hat{C}_\Delta = \frac{E[\xi_A^2\xi_B^2]}{(\sum_\ell s_\ell)^2}|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|.$$

4.3. Generation of Mixed States

Let H_A and H_B be complex Hilbert spaces and let ρ be a density operator acting on the tensor product $H = H_A \otimes H_B$. Consider its spectral decomposition

$$\rho = \sum_{k=1}^{M} \lambda_k\,|\psi_k\rangle\,\langle\psi_k|\,, \qquad \lambda_k \geq 0, \qquad \sum_{k=1}^{M}\lambda_k = 1, \qquad |\psi_k\rangle \in H_A \otimes H_B.$$

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Suppose that there is a random generator selecting time-window Δ; so an ensemble of time-widows is created. The micro correlations during these time windows generate only the states $\rho_k = |\psi_k\rangle\langle\psi_k|, k = 1, ..., M$. Assign label k to intervals with the output $\rho_k : \Delta = \Delta_k$. Denote the pair of stochastic processes behind ρ_k as $X_k(t,\omega), Y_k(t,\omega)$ and the corresponding micro correlation as $S_k(\omega) = C_{\Delta_k}(\omega)$. Suppose that there is a random generator $\eta = \eta(\omega)$ selecting the interval of the k-type with probability λ_k. As we see from the probabilistic lemma below, if η is independent of random variables $S_k, k = 1, ..., M$, then this process generates the density operator ρ.

Lemma 1 (Random selection of stochastic processes)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{S_k\}_{k=1}^M$ be a family of random variables with values in a measurable space $(E, \mathcal{E})$, and let η be a discrete random variable taking values in $\{1, \dots, M\}$ with*

$$\mathbb{P}(\eta = k) = \lambda_k.$$

Assume that η is independent of the family $\{S_k\}_{k=1}^M$. Define the random variable

$$S := S_\eta.$$

Then, for any measurable function $f : E \to L$, where L is a (finite-dimensional) linear space, such that the expectations exist,

$$\mathbb{E}[f(S)] = \sum_{k=1}^{M} \lambda_k \, \mathbb{E}[f(S_k)].$$

In our example, the measurable space $(E, \mathcal{E})$ is given by $E = H_A \otimes H_B$ and $\mathcal{E}$ is the σ-algebra of Borel subsets of E; random variables S_K are based on stochastic processes for generation of ρ_k (see, e.g., section 4.2 with $|\psi\rangle = |\psi_k\rangle$), and $f : H_A \otimes H_B \to \mathcal{L}(H_A \otimes H_B)$, $f(z) = |z\rangle\langle z|$.

4.4. Generalization of Micro-Dynamics to Jump Processes

The scheme presented above utilizes locally constant processes defined via characteristic functions on a fixed partition of the macro-window Δ. We now generalize this scheme to general jump processes, where transitions occur at random times.

Let $\{|u_\ell\rangle, |v_\ell\rangle\}_{\ell=1}^r$ be the fixed set of r vector pairs, e.g., Schmidt components for the desired entangled state.

Instead of the piecewise constant processes $X(t,\omega)$ and $Y(t,\omega)$ defined on fixed subintervals, we consider a sequence of random jump times $0 = \tau_0 < \tau_1 < \cdots < \tau_{N(\omega)} = \Delta$. Here, $N(\omega)$ is a counting process (such as a Poisson process) representing the number of stochastic events within the window Δ.

Define a mapping $\sigma(j,\omega) : \{1, \dots, N(\omega)\} \to \{1, \dots, r\}$ - a random selector. This function randomly assigns one of the r available states to the j-th jump interval.

The processes are generalized as:

$$X(t,\omega) = \sum_{j=1}^{N(\omega)} x_j(t,\omega) \chi_{[\tau_{j-1},\tau_j)}(t) |u_{\sigma(j,\omega)}\rangle, \tag{18}$$

$$Y(t,\omega) = \sum_{j=1}^{N(\omega)} y_j(t,\omega) \chi_{[\tau_{j-1},\tau_j)}(t) |v_{\sigma(j,\omega)}\rangle, \tag{19}$$

where χ_δ denotes the charateristic function of interval δ.

The micro-covariance operator $\hat{C}_\Delta(\omega)$ is now an integral over these random intervals:

$$\hat{C}_\Delta(\omega) = \frac{1}{\Delta} \sum_{j=1}^{N(\omega)} \int_{\tau_{j-1}}^{\tau_j} x_j(t,\omega) \overline{y_j(t,\omega)} dt |u_{\sigma(j,\omega)}\rangle\langle v_{\sigma(j,\omega)}|. \tag{20}$$

 doi:10.20944/preprints202601.1807.v1

Entanglement still emerges from the pathwise alignment of the processes at the micro-scalw, even if the jump times and values are statistically independent at the macro level.

Micro-Time Consistency

The concept of micro-time consistency is formalizes an almost-sure constraint on the entire sample path of the jump process.

Let $C \subset H_A \times H_B$ be a set of allowed consistency relations. The pair (X, Y) is said to be micro-time consistent for a general jump process if:

$$P(\forall t \in [0, \Delta] : (X(t, \omega), Y(t, \omega)) \in C) = 1 \tag{21}$$

except possibly at the discrete jump instants $\{\tau_j\}$[.

For the Bell states $|\Phi^+\rangle$, $C = \{(\lambda_A|j\rangle, \lambda_B|j\rangle) : \lambda_A, \lambda_B \in \mathbb{C}, j \in \{0, 1\}\}$. This set restricts the pair (X, Y) such that at any given micro-time t, both processes must be proportional to the same basis vector ($|0\rangle$ or $|1\rangle$). While this consistency condition is a necessary pathwise constraint, we note that it does not uniquely determine the concrete state (e.g., distinguishing $|\Phi^+\rangle$ from $|\Phi^-\rangle$); that distinction requires the calculation of specific micro-macro correlations.

This approach demonstrates that the DCM framework is not restricted to step functions but applies to any stochastic process where micro-level fluctuations are synchronized according to a global covariance structure. See alo appendix B on further coupling with theory of classical stochastic processes.

5. Deriving Subsystem States from Composite Systems

Beginning with a stochastic derivation of the state of a composite system, we now perform transition to the stochastic origin of individual subsystem states. While this approach is somewhat unconventional, it is highly intuitive within the context of quantum mechanics, especially *theory of open quantum systems.* In quantum studies, the state ρ_{AB} of a composite system S_{AB} cannot generally be reconstructed from the individual states ρ_A and ρ_B of its components S_A and S_B. However, the states of the subsystems are uniquely determined by the global state through the partial trace operation:

$$\rho_A = \mathrm{Tr}_{H_B}\rho_{AB}, \quad \rho_B = \mathrm{Tr}_{H_A}\rho_{AB} \tag{22}$$

A classical probabilistic derivation of these formulas is provided in section 6, subject to specific constraints on the underlying stochastic processes. For the present discussion, we treat the classical stochastic representation of ρ_{AB} primarily as a conceptual foundation for representing the states of individual systems.

The Myth of the Isolated System: A DCM Perspective

A fundamental question arises within the DCM framework: *Do truly isolated quantum physical systems exist, or is an individual system always, by necessity, a subsystem of an encompassing environment?* At first glance, this suggests a potential logical circularity: if an individual system is defined as a marginalized subsystem of a composite, but the composite itself is an individual system at a larger scale, where does the definition ground itself?

Breaking the Circularity: Scale-Dependent Identity

DCM avoids this logical circle through its treatment of **scales of synchronization**. In this framework, the definition of a "system" is not an absolute ontological category; rather, it is defined by the **window of synchronization** (Δ).

- **The Composite Scale:** A system is viewed as "Composite" at the temporal or structural scale where the micro-synchronization between its internal processes ($X(t)$ and $Y(t)$) is explicitly resolved.

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- **The Individual Scale:** That same system becomes an "Individual" entity at a higher macro-scale, where internal parts are treated as a unified, emergent fluctuation.

The circle is broken by the **partial trace operation**. When moving from ρ_{AB} to ρ_A, the observer performs a mathematical and physical coarse-graining. The trace operation signifies a shift in the level of description: the synchronization between A and B is no longer the object of study, but rather the resulting aggregate intensity and fluctuations of A itself.

Individual Systems as Marginal Residues

If we follow the logic of the DCM to its conclusion, a truly isolated system is a mathematical idealization. Because the subquantum stochastic processes are likely manifestations of a global field, $X(t)$ is never truly independent.

> In the DCM, the distinction between "composite" and "individual" is relational rather than ontological. An individual system is essentially the **marginal residue** of the global field that remains after we lose track of, or purposefully discard, the external correlations (synchronizations) with the environment.

Therefore, the "Quantum State" ρ_A of an individual system is not a standalone primitive. It is a reduced description of the system's participation in a larger, synchronized whole. The appearance of an isolated system occurs only when the second-order covariance between the system and the rest of the universe becomes negligible or static relative to the macro-observer's window.

Implications for the Universal State

This hierarchical view implies that the only truly "Individual" system that is not a subsystem would be the Universe itself. Within the DCM, the Universe would be described as a state of perfect global micro-synchronization. The existence of mixed local states and the necessity of the density operator formalism are thus direct consequences of our status as local observers who can only ever perceive a fraction of the total covariance structure.

6. A Stochastic Realization of the Partial Trace Identity

We come to the stochastic representation of the state ρ_A of subsytem S_A of a composite system S_{AB} through stochastic implementation of equality $\mathrm{Tr}_{H_B}\rho_{AB} = \rho_A$ by explicitly calculating the partial trace and applying micro-synchronization heuristics.

The $H_A \otimes H_B$ valued random variable Z_Δ can be expanded with respect the basis composed of two orthonormal bases, $(|i\rangle) \in H_A$ and $(|m\rangle) \in H_B : Z_\Delta = \sum_{im} z_{im}|i\rangle|m\rangle$, or in the operator realization $\hat{Z}_\Delta = \sum_{im} z_{im}|i\rangle\langle m|$. In DCM density operators correspond to double covariance operators (with the trace one normalization); so the matrix elements of ρ_{AB} can be expressed as $(\rho_{AB})_{im,jk} = \mathbb{E}[z_{im}\bar{z}_{jm}]$.

6.0.1. Partial Trace of Covariance

By definition of the partial trace:

$$(\mathrm{Tr}_{H_B}\rho_{AB})_{ij} = \sum_m \mathbb{E}[z_{im}\bar{z}_{jm}] \tag{23}$$

Substituting the micro-level integral definition of z_{im}:

$$(\mathrm{Tr}_{H_B}\rho_{AB})_{ij} = \frac{1}{\Delta^2}\iint_{\Delta\times\Delta} \mathbb{E}\left[X_i(t)\bar{X}_j(t')\left(\sum_m Y_m(t)\bar{Y}_m(t')\right)\right] dt\, dt', \tag{24}$$

where $X_i(t) = \langle i|X(t)\rangle, Y_m(t) = \langle i|Y(t)\rangle$.

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The Stochastic Reference Kernel

We define the stochastic kernel $K_B(t,t',\omega)$ as the inner product of the micro-signals in H_B:

$$K_B(t,t',\omega) = \sum_a Y_a(t,\omega)\bar{Y}_a(t',\omega) = \langle Y(t',\omega)|Y(t,\omega)\rangle_{H_B}. \tag{25}$$

The partial trace is then:

$$(\mathrm{Tr}_{H_B}\rho_{AB})_{ij} = \frac{1}{\Delta^2}\iint_{\Delta\times\Delta}\mathbb{E}\big[X_i(t)\bar{X}_j(t')K_B(t,t',\omega)\big]\,dt\,dt'. \tag{26}$$

Mathematical Formalization of Micro-Scale Synchronization

To exclude dependence on the auxiliary process $Y(t)$ describing randomness in system S_B, we impose the following refined conditions on the micro-processes:

1. **Temporal Localization (Delta-Correlation):** The kernel K_B behaves as a nascent Dirac delta function $\delta_\epsilon(t-t')$ relative to the macro-window Δ, where the correlation time $\epsilon \ll \Delta$:
$$K_B(t,t',\omega) \approx \Delta\delta_\epsilon(t-t')\|Y(t,\omega)\|^2. \tag{27}$$
This collapses the double integral into a single time average.
2. **Energy Normalization:** The auxiliary system B is physically normalized such that its mean instantaneous power is unity:
$$\mathbb{E}[\|Y(t,\omega)\|^2] = 1. \tag{28}$$
3. **Statistical Isotropic Power:** To allow for entanglement without biasing the marginal, we require that fluctuations in the power of Y are uncorrelated with the dyadic product of X. This condition ensures that even if the *phases* of X and Y are coupled, the total energy of the reference system S_B does not modulate the local statistics of S_A.

Under these conditions, the expectation factors as

$$\mathbb{E}[X_i\bar{X}_j\|Y\|^2] = \mathbb{E}[X_i\bar{X}_j]\mathbb{E}[\|Y\|^2].$$

, and the integral yields:

$$(\rho_A)_{ij} \equiv (\mathrm{Tr}_{H_B}\rho_{AB})_{ij} = \frac{1}{\Delta}\int_\Delta \mathbb{E}[X_i(t)\bar{X}_j(t)]dt. \tag{29}$$

This proves that the partial trace in quantum mechanics is equivalent to the marginalization of hidden correlations in the underlying Kolmogorov space.

To be able to perform the corresponding computations for $\mathrm{Tr}_{\mathrm{H_A}}æ_{\mathrm{AB}}$, we should also put the same constraints on the process $X(t)$. Now system S_A is considered as an auxiliary system for system S_B. Hence, we impose the condition of energy normalization:

$$\mathbb{E}[\|X(t,\omega)\|^2] = 1. \tag{30}$$

and statistical isotropic power

Now consider S_A as an individual system and let its micro-macro time randomness is described by a stochastic process $X(t)$. Set

$$C_{\Delta;A}(\omega) = \frac{1}{\Delta}\int_\Delta |X(t,\omega)\rangle\langle X(t,\omega)|dt. \tag{31}$$

 doi:10.20944/preprints202601.1807.v1

This is micro autocorrelation of a stochastic process $X(t)$. It can be treated as the operator-valued random variable $\hat{C}_{\Delta;A}(\omega)$ valued in $\mathcal{L}(H_A)$ By averaging $\hat{C}_{\Delta;A}(\omega)$ w.r.t. the macro randomness parameter ω we obtain the operator

$$\hat{C}_A = \mathbb{E}[\hat{C}_{\Delta;A}]. \tag{32}$$

We stress that this is not the second order covariance, but the statistical averaging of the first order micro-time covariance. We remark that due to the condition of energy normalization (30)

$$\mathrm{Tr}\hat{C}_{\mathrm{A}} = \frac{1}{\blacksquare}\int_{\blacksquare}\mathbb{E}[\|\mathrm{X}(\mathrm{t},!)\|^2] = 1.$$

Hence,

$$\rho_A^{\text{intrinsic}} = \hat{C}_A \tag{33}$$

is a density operator. This equality can be used as the intrinsic definition of the state of system S_A in terms of the subquantum classical stochastic process.

Denote by $\hat{C}_{AB}$ the double covariance operator of the stochastic process $X(t), Y(t)$ of the composite system S_{AB} : $\hat{C}_{AB} = \mathbb{E}[|Z_{AB}\rangle\langle Z_{AB})|]$, where $Z_{AB} \equiv Z_\Delta$ in notation of the previous sections. We remark that

$$\mathrm{Tr}\hat{C}_{\mathrm{AB}} = \frac{1}{\blacksquare}\int_{\blacksquare}\mathbb{E}(||\mathrm{X}(\mathrm{t})||^2||\mathrm{Y}(\mathrm{t})||^2)\mathrm{dt} = 1.$$

Hence, this is a density operator, $\rho_{AB} = \hat{C}_{AB}$. Then under above conditions

$$\rho_A^{\text{intrinsic}} = \rho_A = \mathrm{Tr}_{\mathrm{H_B}}æ_{\mathrm{AB}}. \tag{34}$$

Hence the intrinsic state's definition for S_A coincides with the partial trace definition of the state of S_A as a subsystem of composite system S_{AB}.

7. Consistency of Synchronization and Entanglement

The "Subinterval Allocation Scheme" used to construct the Bell state $|\Phi^+\rangle$ (section 4.1) is a specific realization of micro-scale synchronization.

Macro-Randomized Synchronization in the Bell State

In section 4.1, to construct the Bell state $|\Phi^+\rangle$, we incorporate the random variables $\xi_A(\omega)$ and $\xi_B(\omega)$ which describe macro-randomization in systems S_A and S_B. We define the processes as: $(X(t,\omega), Y(t,\omega)) = \begin{cases} (\xi_A(\omega)|0\rangle, \xi_B(\omega)|0\rangle) & t \in [0, \Delta/2] \\ (\xi_A(\omega)|1\rangle, \xi_B(\omega)|1\rangle) & t \in [\Delta/2, \Delta] \end{cases}$ where Δ is the macro-window and the macro-parameters satisfy $\mathbb{E}[\xi_A] = \mathbb{E}[\xi_B] = 0, \quad \mathbb{E}[\xi_A\xi_B] = 0, \quad \mathbb{E}[\xi_A^2\xi_B^2] = 2.$

7.0.1. Stochastic Kernel and Partial Trace

Following the construction in section 6, we define the stochastic reference kernel $K_B(t,t',\omega)$ as the inner product of the micro-signals in H_B:

$$K_B(t,t',\omega) = \langle Y(t',\omega)|Y(t,\omega)\rangle_{H_B} = \xi_B^2(\omega)\big(\chi_{\delta_0}(t)\chi_{\delta_0}(t') + \chi_{\delta_1}(t)\chi_{\delta_1}(t')\big) \tag{35}$$

where $\delta_0 = [0, \Delta/2]$ and $\delta_1 = [\Delta/2, \Delta]$. The partial trace ρ_A is calculated via the double integral:

$$(\rho_A)_{ij} = \frac{1}{\Delta^2}\iint_{\Delta\times\Delta}\mathbb{E}\big[X_i(t,\omega)\bar{X}_j(t',\omega)K_B(t,t',\omega)\big]\,dt\,dt'.$$

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Since K_B vanishes when t and t' are in different subintervals, the integral simplifies. By substituting the specific values for $X(t,\omega)$, the expression for ρ_A becomes:

$$\rho_A = \frac{1}{\Delta^2}\mathbb{E}[\xi_A^2\xi_B^2]\left(\iint_{\delta_0^2}|0\rangle\langle 0|dtdt' + \iint_{\delta_1^2}|1\rangle\langle 1|dtdt'\right). \tag{36}$$

Given that $\mathbb{E}[\xi_A^2\xi_B^2] = 2$ and the area of each sub-square is $(\Delta/2)\times(\Delta/2) = \Delta^2/4$, we obtain:

$$\rho_A = \frac{1}{\Delta^2}\cdot 2\cdot\left(\frac{\Delta^2}{4}|0\rangle\langle 0| + \frac{\Delta^2}{4}|1\rangle\langle 1|\right) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|). \tag{37}$$

This yields the standard trace-one mixed state.

Energy Normalization and Intrinsic State

In this model, the state of the composite system ρ_{AB} is defined as the double covariance operator $\hat{C}_{AB} = \mathbb{E}[|C_\Delta\rangle\langle C_\Delta|]$. Its trace is given by:

$$\mathrm{Tr}\hat{C}_{AB} = \frac{1}{\Delta}\int_\Delta \mathbb{E}[\|X(t)\|^2\|Y(t)\|^2]dt = \frac{1}{\Delta}\left(\Delta\cdot\frac{1}{2}\mathbb{E}[\xi_A^2\xi_B^2]\right) = 1. \tag{38}$$

Under the condition of statistical isotropic power, the intrinsic state $\rho_A^{\text{intrinsic}}$ derived from the micro-autocorrelation of $X(t)$ matches the result of the partial trace:

$$\rho_A^{\text{intrinsic}} = \mathbb{E}\left[\frac{1}{\Delta}\int_\Delta |X(t,\omega)\rangle\langle X(t,\omega)|dt\right] = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|). \tag{39}$$

This demonstrates that the normalization of the local state is preserved by the coupling between the macro-randomization (ξ) and the micro-scale temporal allocation.

7.1. Discrete Synchronization in Schmidt-Decomposed Processes

We consider the specific construction where the density operator ρ is realized via deterministic subinterval allocation within a macro-window Δ.

7.2. The Discrete Partition

Let the macro-window Δ be partitioned into r disjoint subintervals $\Delta = \bigcup_{k=1}^r \delta_k$, where $|\delta_k| = w_k\Delta$ and $\sum w_k = 1$. The micro-processes are defined as:

$$X(t,\omega) = u_k, \quad Y(t,\omega) = v_k \quad \text{for } t\in\delta_k \tag{40}$$

where $\{u_k\}$ and $\{v_k\}$ are the Schmidt vectors for a state $\psi\in H_A\otimes H_B$.

Lemma 2 (The Discrete Synchronization Condition)**.** *The identity* $\mathrm{Tr_B æ_{AB}} = æ_A^{intrinsic}$ *is exactly satisfied if and only if the auxiliary process* $Y(t)$ *satisfies the* ***Orthonormal Block Kernel*** *condition:*

$$K_B(t,t') = \langle Y(t')|Y(t)\rangle = \sum_{k=1}^r 1_{\delta_k}(t)1_{\delta_k}(t') \tag{41}$$

where 1_{δ_k} *is the indicator function of the k-th subinterval.*

Proof. The partial trace of the macro-covariance is given by the double integral:

$$(\rho_A)_{ij} = \frac{1}{\Delta^2}\iint_{\Delta\times\Delta} X_i(t)\bar{X}_j(t')K_B(t,t')\,dt\,dt' \tag{42}$$

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Substituting the Orthonormal Block Kernel:

$$(\rho_A)_{ij} = \frac{1}{\Delta^2} \sum_{k=1}^{r} \iint_{\delta_k \times \delta_k} X_i(t) \bar{X}_j(t')(1)\, dt\, dt' \tag{43}$$

Since $X(t)$ is constant (u_k) on each subinterval δ_k:

$$(\rho_A)_{ij} = \frac{1}{\Delta^2} \sum_{k=1}^{r} (u_k)_i (\bar{u}_k)_j \cdot |\delta_k|^2 \tag{44}$$

Recalling $|\delta_k| = w_k \Delta$, we obtain:

$$\rho_A = \sum_{k=1}^{r} w_k^2 |u_k\rangle\langle u_k| \tag{45}$$

By the normalization of the joint state $Z = \sum w_k (u_k \otimes v_k)$, this result is identical to the partial trace derived from the standard quantum formalism. □

7.3. Physical Implications

This lemma demonstrates that entanglement requires a high degree of *Temporal Coordination*:

- **Subinterval Alignment:** If $X(t)$ and $Y(t)$ transition between states at different micro-times, $K_B(t, t')$ would overlap with different X-vectors, generating non-vanishing off-diagonal terms (interference) that represent a loss of coherence.
- **Auxiliary Orthogonality:** The requirement that $\langle v_k | v_m \rangle = \delta_{km}$ ensures that the kernel K_B acts as a "selector" of subintervals, effectively marginalizing the B influence without distorting the A statistics.

Finally, we remark that the methodology of quantum theory is increasingly applied to "quantum-like" modeling in cognitive science and decision-making [39]. The Fourth-Order Moment Structure addressed here provides the missing temporal scale needed to reconcile classical stochasticity with these powerful formalisms.

8. Concluding Discussion: The Relational Nature of Systems

The **Double Covariance Model (DCM)** provides a fundamental reinterpretation of the quantum state, treating the density operator as the fourth-order moment structure of an underlying classical Kolmogorov probability space. By grounding the quantum formalism in the interplay between micro and macro temporal scales, the DCM addresses both the technical derivation of entanglement and the conceptual origin of quantum randomness.

8.1. Scale, Synchronization, and Entanglement

The central innovation of the DCM lies in its dual-scale approach. It demonstrates that **entanglement** is a macro-time phenomenon reflecting **micro-time consistency**—a pathwise constraint that allows for quantum correlations even when subquantum processes are statistically independent at the macro level. Furthermore, the model provides a stochastic realization of the **partial trace**, showing it to be equivalent to the marginalization of hidden classical correlations. This shifts the view of the partial trace from a mere mathematical operation to a physical coarse-graining of micro-synchronizations.

8.2. Relational Identity and the Universal State

The DCM framework challenges the ontological status of isolated systems, suggesting that the distinction between "composite" and "individual" systems is relational rather than absolute:

- **Scale-Dependent Identity**: A system is defined by its window of synchronization. It is viewed as composite when internal micro-synchronizations are resolved, but acts as an individual entity when these parts aggregate into a unified fluctuation.

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- **Individual Systems as Residues**: Truly isolated systems are mathematical idealizations. In the DCM, an individual system is the marginal residue of a global field that remains after discarding external correlations with the environment.
- **The Universal State**: This hierarchical view implies that the only truly individual system is the Universe itself, described as a state of perfect global micro-synchronization. Local mixed states are a direct consequence of our perspective as local observers perceiving only a fraction of the total universal covariance structure.

8.3. Broader Implications

The ability of the DCM to generate entangled states from classical processes suggests significant applications in "quantum-like" modeling across interdisciplinary fields. Ultimately, the DCM provides a bridge between classical pathwise certainty and the statistical formalism of quantum mechanics, suggesting that the quantum state is not a standalone primitive but a reduced description of a system's participation in a larger, synchronized whole.

Appendix A: Vectors as Hilbert-Schmidt Operators

In the infinite dimensional case we use the isomorphism:

$$H_A \otimes H_B \cong \mathcal{L}_{HB}(H_B, H_A), \tag{46}$$

where $\mathcal{L}_{HB}(H_B, H_A)$ is the space of Hilbert-Schmidt operators. For physicists, this case is even more illustrative. Consider the case of L_2 spaces,

$$H_A = L_2(\mathbb{R}^n) = \{\phi : \mathbb{R}^n \to \mathbb{C}; ||\phi||^2 = \int |\phi(x)|^2 dx < \infty\},$$

$$H_B = L_2(\mathbb{R}^m) = \{\psi : \mathbb{R}^m \to \mathbb{C}; ||\psi||^2 = \int |\psi(y)|^2 dy < \infty\},$$

$$H_A \otimes H_B = L_2(\mathbb{R}^{n+m}) = \{\Psi : \mathbb{R}^{n+m} \to \mathbb{C}; ||\Psi||^2 = \int |\Psi(x,y)|^2 dx < \infty\}$$

Take $\Psi = \Psi(x,y) \in H_A \otimes H_B$, it determines the (Hilbert-Schmidt) operator acting between H_B and H_A,

$$\hat{\Psi}\phi(v) = \int \Psi(x,y)\phi(y)dy,\ \phi \in L_2(\mathbb{R}^m).$$

The map $\Psi \to hat\Psi$ is the unitary operator.

This isomorphism was widely used by von Neumann [16] and by the author in PCSFT [32].

Appendix B: Connection with Theory of Stochastic Processes

Our construction of subquantum stochastic processes presented in sections 4.4 can be connected (at least indirectly) with some special parts of theory of classical stochastic processes. Our construction can be coupled to the theory of *regime-switching and piecewise-defined processes* with examples as Markov-modulated processes, switching diffusions, piecewise deterministic Markov processes. They are structured similarly:

1. time is partitioned into random or deterministic intervals;
2. on each interval, the process obeys a fixed rule;
3. the switching mechanism is governed by another random process.

But here is the key difference: in classical regime-switching models, the regimes are independent across components unless explicitly coupled. In the presented model: on each subinterval, two processes must satisfy a joint consistency constraint - classical analog of "entangled behavior". The constraint is structural, not probabilistic. This already goes beyond standard theory.

Related ideas appear in coupling theory, random environment models, and stochastic synchronization, but the specific combination of macro-level statistical independence with micro-time pathwise

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consistency constraints appears to be absent from the standard theory of stochastic processes. Our construction differs essentially by combining local-in-time consistency with global statistical independence.

Comparison with Stochastic Synchronization

At a superficial level, the proposed construction shares certain formal similarities with processes exhibiting stochastic synchronization. In both frameworks, coherence emerges from systems driven by randomness, and the analysis is naturally formulated in terms of time-dependent stochastic processes rather than static random variables. Moreover, the use of time partitioning, regime switching, and local-in-time structure places the present model in conceptual proximity to classical theories of regime-switching processes, random environments, and noise-driven synchronization.

However, the similarity is limited, and the underlying mechanisms are fundamentally different. In the standard theory of stochastic synchronization, synchronization is a *dynamical and statistical phenomenon*. Two or more stochastic processes become aligned due to coupling, common noise, or shared environmental fluctuations. The resulting coherence is typically expressed in probabilistic or asymptotic terms, such as convergence of trajectories, phase locking in distribution, or contraction of distances in expectation or almost surely as time tends to infinity. Importantly, stochastic synchronization generally relies on some form of statistical dependence, either explicit or implicit, between the synchronized components.

In contrast, the present model does not rely on coupling, common noise, or statistical dependence. The stochastic processes $X(t,\omega)$ and $Y(t,\omega)$ may be fully independent at the macro level, with vanishing covariances and factorizable joint distributions. Coherence arises instead from *micro-time consistency constraints* imposed almost surely on selected subintervals of the macro-time window. These constraints are structural and pathwise: on each active micro-interval, the pair $(X(t,\omega), Y(t,\omega))$ is required to belong to a prescribed consistency set. No convergence, attraction, or dynamical synchronization mechanism is involved.

This distinction becomes especially pronounced in the generation of entangled states. Within stochastic synchronization theory, synchronization does not produce nonseparable macro-level states unless explicit coupling or shared randomness is introduced. In the present framework, however, entanglement emerges as a macro-time effect of micro-time coordination, even when the underlying stochastic processes remain statistically independent. The entangled density operator reflects consistency of microscopic behavior across time, rather than correlation or dependence in the underlying probability space.

Thus, while related ideas appear in stochastic synchronization, coupling theory, and random environment models, the proposed construction represents a qualitatively different mechanism. It combines local-in-time pathwise consistency with global statistical independence, leading to a classical stochastic representation of quantum entanglement that lies outside the standard scope of stochastic synchronization theory.

Comparison with Processes with Admissible Trajectory Sets

A *random process with admissible trajectory sets* is a classical stochastic process constrained so that, almost surely, its sample paths lie within a prescribed set of trajectories. Formally, let $X(t,\omega)$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{A} \subset C([0,T], \mathbb{R}^n)$ denote the set of admissible trajectories. Then X is said to respect $\mathcal{A}$ if

$$\mathbb{P}\big(X(\cdot,\omega) \in \mathcal{A}\big) = 1.$$

These models appear in constrained stochastic control, viability theory, and lattice or network systems. Unlike standard stochastic processes, the trajectory constraints can enforce pathwise properties (e.g., monotonicity, switching rules, or geometric constraints) that cannot be expressed purely via marginal distributions or covariances.

The micro-time consistency model introduced in this work can be viewed as a natural extension of this concept, with two crucial distinctions. First, the constraints are imposed jointly on a pair of processes $(X(t,\omega), Y(t,\omega))$, rather than on a single process. On each active micro-time subinterval δ_j, the pair is required to satisfy a consistency condition

$$\forall t \in \delta_{r(\omega)} : \ (X(t,\omega), Y(t,\omega)) \in \mathcal{C}_{r(\omega)}, \quad \mathbb{P}\text{-a.s.},$$

which enforces a classical analogue of entanglement at the micro-time level. Second, despite these pathwise constraints, the processes X and Y can remain statistically independent at the macro level, so that macro-level correlations vanish while micro-level alignment generates the correct entangled density operator after averaging.

In contrast, standard random processes with admissible trajectory sets typically induce statistical dependence through their constraints, or apply constraints only to a single process. Therefore, while the micro-time consistency construction shares the formal motif of pathwise admissibility with these classical processes, it introduces a fundamentally new mechanism: *joint micro-time constraints combined with macro-level statistical independence*, which underlies the emergence of quantum-like entanglement in this classical stochastic framework.

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