Article

A Mathematical Approach to the Theory of Finite Automata

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Article

A Mathematical Approach to the Theory of Finite Automata

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Abstract

There is a lack of rigorous mathematical treatment in the theory of finite automata. This paper provides a rigorous mathematical approach to automata theory which doesn't currently exist in the literature of theoretical computer science. Basic definitions are developed in mathematical terms and used as the foundation for constructing mathematical proofs for theorems. It provides a model for instructors to write better lecture notes and authors to write better textbooks for educational purpose. It also corrects some critical errors and erroneous arguments that can be found in many textbooks which are widely used in the education of theoretical computer science.

Keywords: theoretical computer science; computability theory; finite automata; Pumping Lemma; discrete mathematics; Myhill-Nero theorem

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1. Deterministic Finite Automaton (*DFA*)

Definition 1.
A deterministic finite automaton denoted by DFA is a 5-tuple,
$M = (Q, \Sigma, \delta, q_0, F)$, *where*
(i) Q *is a finite set of states;*
(ii) Σ *is a finite alphabet;*
(iii) $\delta : Q \times \Sigma \longrightarrow Q$ *is the transition function;*
(iv) $q_0 \in Q$ *is the start state; and*
(v) $F \subset Q$ *is the set of accept states.*
Let $w = w_1w_2w_3, \cdots, w_n$ *be a string over* Σ *where each* $w_i \in \Sigma$ *and* $n \geq 1$.
M accepts w if and only if $\exists r_0, r_1, r_2, \cdots, r_n \in Q$ *s.t. the following conditions are satisfied:*
(a) $r_0 = q_0$;
(b) $\delta(r_i, w_{i+1}) = r_{i+1}$ *for* $i = 0, 1, 2, \cdots, n-1$; *and*
(c) $r_n \in F$
For $n = 0, w = \epsilon$. *Only conditions (a) and (c) are applicable and they become* $r_0 = q_0$ *and* $r_0 \in F$. *We therefore define M to accept* ϵ *if the start state is also an accept state.*
On the other hand, since there is no ϵ*-movement in a DFA, the only way the DFA can accept an empty string is to accept it at the start state.*
Accordingly, M accepts ϵ *if and only if the start state is also an accept state.*
If we write $r_i \xrightarrow{w_{i+1},\delta} r_{i+1}$ *instead of* $\delta(r_i, w_{i+1}) = r_{i+1}$ *for* $i = 0, 1, 2, \cdots, n-1$, *then conditions (a), (b) and (c) can be written as follows:*
$q_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_{n-1} \xrightarrow{w_n,\delta} r_n, r_n \in F$
We say M recognizes language A if $A = \{w \in \Sigma^* \mid M \text{ accepts } w\}$ *and it is written as* $L(M) = A$

Definition 2.
A language is called regular if it is recognized by a DFA.

Definition 3.
For any language L,
$L^0 = \{\epsilon\}, L^1 = L, L^2 = LL, \cdots, L^{m+1} = L^m L$ *for* $m > 0$.
$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots = \bigcup_{k=0}^{\infty} L^k$
$= \{w \mid w = w_1 w_2 w_3 \cdots w_n; w_i \in L$ *for* $1 \leq i \leq n; n \geq 1\} \cup \{\epsilon\}$

Definition 4.
Inductive Transition Function
Let $M = (Q, \Sigma, \delta, q_0, F)$ *be a DFA.*
$\hat{\delta} : Q \times \Sigma^* \longrightarrow Q$ *s.t.*
(i) $\hat{\delta}(q, \epsilon) = q \qquad \forall q \in Q$
(ii) $\hat{\delta}(q, wa) = \delta(\hat{\delta}(q, w), a) \qquad \forall a \in \Sigma, w \in \Sigma^*, q \in Q$

Definition 5.
$\forall p, q \in Q, w \in \Sigma^*, p \xrightarrow{w,\hat{\delta}} q \overset{def}{\Longleftrightarrow} q = \hat{\delta}(p, w)$

Proposition 1.
$\hat{\delta}(q, a) = \delta(q, a)\ \forall q \in Q, a \in \Sigma$
$< Proof >$
$\hat{\delta}(q, a) = \hat{\delta}(q, \epsilon a)$
$= \delta(\hat{\delta}(q, \epsilon), a) \qquad (Definition 1.4(ii))$
$= \delta(q, a) \qquad (Definition 1.4(i))$

Theorem 1. ***(DFA Acceptance)***
For any DFA, $M = (Q, \Sigma, \delta, q_0, F)$
$\hat{\delta}(q_0, w) \in F \Longleftrightarrow M$ *accepts* $w \quad \forall w \in \Sigma^*$
$< Proof >$
Claim: If $w = w_1 w_2 \cdots w_n$ *where* $n \geq 0$ *and*
$q_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_i \xrightarrow{w_{i+1},\delta} r_{i+1} \cdots r_{n-1} \xrightarrow{w_n,\delta} r_n$, *then* $\hat{\delta}(r_0, w) = r_n$
This Claim can be proved by induction on n.
For $n = 0, w = \epsilon$ *and the computation becomes* $q_0 = r_0$.
$\hat{\delta}(q_0, w) = \hat{\delta}(q_0, \epsilon)$
$= q_0$ *(By Definition 1.4(i))*
$= r_0$
Therefore, the statement is true for $n = 0$.
Assume the statement is true for $n = k$*, where* $k \geq 0$.
That is, $\hat{\delta}(r_0, w_1 w_2 \cdots w_k) = r_k$
$\hat{\delta}(r_0, w_1 w_2 \cdots w_k w_{k+1}) = \delta(\hat{\delta}(r_0, w_1 w_2 \cdots w_k), w_{k+1})$ *(Definition 1.4(ii))*
$= \delta(r_k, w_{k+1})$ *(Induction Hypothesis)*
$= r_{k+1}$ *(Definition of* r_{i+1}*)*
Therefore, the statement is true for $n = k + 1$.
If M accepts $w = w_1 w_2 \cdots w_n$*, where* $w_i \in \Sigma$ *for* $1 \leq i \leq n$ *and* $n \geq 1$ *or* $(w = \epsilon$ *and* $n = 0)$
$\exists r_0, r_1, r_2, \cdots r_n \in Q$ *st*
$q_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_i \xrightarrow{w_{i+1},\delta} r_{i+1} \cdots r_{n-1} \xrightarrow{w_n,\delta} r_n, r_n \in F$
By Claim, $\hat{\delta}(r_0, w_1 w_2 \cdots w_n) = r_n$
Therefore, $\hat{\delta}(q_0, w) = r_n \qquad (r_0 = q_0; w = w_1 w_2 \cdots w_n)$
Since $r_n \in F$,
$\hat{\delta}(q_0, w) \in F$
Therefore, M accepts $w \Longrightarrow \hat{\delta}(q_0, w) \in F$
Conversely, if $\hat{\delta}(q_0, w) \in F$

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$\hat{\delta}(q_0, w_1w_2\cdots w_n) \in F$
Take $r_0 = q_0$
$r_{i+1} = \delta(r_i, w_{i+1}) \qquad \forall i = 0,1,2,\cdots n-1$
$q_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_i \xrightarrow{w_{i+1},\delta} r_{i+1} \cdots r_{n-1} \xrightarrow{w_n,\delta} r_n$
By Claim, $\hat{\delta}(r_0, w_1w_2\cdots w_n) = r_n$
Since $\hat{\delta}(q_0, w_1w_2\cdots w_n) \in F, r_n \in F.$
Therefore, M accepts w.
Therefore, $\hat{\delta}(q_0, w) \in F \Longrightarrow M$ *accepts w.*
Therefore, $\hat{\delta}(q_0, w) \in F \Longleftrightarrow M$ *accepts w.*
This completes the proof.

Theorem 2.
For any DFAs, M and M' *where*
$M = (Q, \Sigma, \delta, q_0, F)$
$M' = (Q, \Sigma, \delta', q_0, F')$
$\forall q \in Q, a \in \Sigma, w \in \Sigma^*$
$\delta'(q,a) = \delta(q,a) \Longrightarrow \widehat{\delta'}(q,w) = \hat{\delta}(q,w)$
$< Proof >$
The proof is by induction on $|w| \geq 0$.
For $|w| = 0, w = \epsilon$.
By Definition 1.4(i),
$\hat{\delta}(q,\epsilon) = q$ *and* $\widehat{\delta'}(q,\epsilon) = q$
Therefore, $\hat{\delta}(q,\epsilon) = \widehat{\delta'}(q,\epsilon)$
Assume the statement is true for $|w| = k \geq 0$.
$\hat{\delta}(q, wa) = \delta(\hat{\delta}(q,w), a)$
$= \delta'(\hat{\delta}(q,w), a) \qquad (\delta'(q,a) = \delta(q,a))$
$= \delta'(\widehat{\delta'}(q,w), a) \qquad$ *(Induction Hypothesis)*
$= \widehat{\delta'}(q, wa) \qquad$ *(Definition 1.4(ii))*
The statement is also true for $|w| = k+1$

2. Nondeterministic Finite Automaton (*NFA*)

Definition 6.
A nondeterministic finite automaton (NFA) is a 5-tuple,
$N = (Q, \Sigma, \delta, q_0, F)$, *where*
(i) Q is a finite set of states;
(ii) Σ *is a finite alphabet;*
(iii) $\delta : Q \times \Sigma_\epsilon \longrightarrow P(Q)$ *is the transition function, where*
$\Sigma_\epsilon = \Sigma \cup \{\epsilon\}, P(Q) =$ *the power set of* $Q = \{S \mid S \subset Q\}$.
(iv) $q_0 \in Q$ *is the start state; and*
(v) $F \subset Q$ *is the set of accept states.*
Let $w = w_1w_2w_3\cdots w_m$ *where* $w_i \in \Sigma_\epsilon$ *for* $1 \leq i \leq m$ *and* $m \geq 1$.
N accepts w if and only if $\exists r_0, r_1, r_2, \cdots, r_m \in Q$ *s.t. the following conditions are satisfied:*
(a) $r_0 \in \{q_0\}$
(b) $r_{i+1} \in \delta(r_i, w_{i+1})$ *for* $\qquad i = 0,1,2,\cdots,m-1$
(c) $r_m \in F$
For $m = 0, w = \epsilon$.
Only conditions (a) and (c) are applicable and they become $r_0 = q_0$ *and* $r_0 \in F$.
We therefore define N to accept ϵ *if the start state is also an accept state.*
If we write $r_i \xrightarrow{w_{i+1},\delta} r_{i+1}$ *instead of* $r_{i+1} \in \delta(r_i, w_{i+1})$ *for* $i = 0,1,2,\cdots,m-1$*, then conditions (a), (b) and (c) can be written as follows:*

$q_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_i \xrightarrow{w_{i+1},\delta} r_{i+1} \cdots r_{m-1} \xrightarrow{w_m,\delta} r_m, \quad r_m \in F.$
Note that when $m = 0$, this computation becomes $q_0 = r_0$ and $r_0 \in F$.

Definition 7. *(Inductive Transition Function)*
Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA.
$\hat{\delta} : P(Q) \times \Sigma_\epsilon^* \longrightarrow P(Q)$ *such that*

(i) $\hat{\delta}(A, \epsilon) = A \qquad \forall A \in P(Q)$

(ii) $\hat{\delta}(A, wa) = \bigcup_{q \in \hat{\delta}(A,w)} \delta(q, a) \qquad \forall a \in \Sigma_\epsilon, w \in \Sigma_\epsilon^*, A \in P(Q).$

Definition 8.
$\forall p, q \in Q, w \in \Sigma_\epsilon^*, \quad p \xrightarrow{w,\hat{\delta}} q \stackrel{def}{\Longleftrightarrow} q \in \hat{\delta}(\{p\}, w).$

Proposition 2.
If $N = (Q, \Sigma, \delta, q_0, F)$ is an NFA, then
$\forall a \in \Sigma_\epsilon, p \in Q, \hat{\delta}(\{p\}, a) = \delta(p, a).$
$< Proof >$
$\hat{\delta}(\{p\}, a) = \hat{\delta}(\{p\}, \epsilon a)$
$= \bigcup_{q \in \hat{\delta}(\{p\},\epsilon)} \delta(q, a)$ *(Definition 1.10 (ii))*
$= \bigcup_{q \in \{p\}} \delta(q, a)$ *(Definition 1.10 (i))*
$= \delta(p, a)$

Proposition 3.
If $N = (Q, \Sigma, \delta, s_0, F)$ is an NFA,
$\forall w \in \Sigma_\epsilon^*$ *where* $w = w_1 w_2 w_3 \cdots w_n$; $w_i \in \Sigma_\epsilon$ *for* $1 \le i \le n$ *and* $n \ge 1$ *or* $w = \epsilon$ *for* $n = 0$.
$(\exists r_0, r_1, r_2, \cdots, r_n \in Q \text{ s.t. } s_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_{n-1} \xrightarrow{w_n,\delta} r_n) \Longleftrightarrow r_n \in \hat{\delta}(\{s_0\}, w)$
$< Proof >$
This proposition can be proved by induction on n.
Let $P(n)$ denote the statement:
$(\exists r_0, r_1, r_2, \cdots, r_n \in Q \text{ s.t. } s_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_{n-1} \xrightarrow{w_n,\delta} r_n)$*; and*
$Q(n)$ denote the statement: $r_n \in \hat{\delta}(\{s_0\}, w)$.
For $n = 0, w = \epsilon$.
$P(0) \Longleftrightarrow (\exists r_0 \in Q \text{ s.t. } s_0 = r_0)$
$\Longleftrightarrow r_0 \in \{s_0\}$
$\Longleftrightarrow r_0 \in \hat{\delta}(\{s_0\}, \epsilon)$ *(Definition 1.10(i))*
$\Longleftrightarrow r_0 \in \hat{\delta}(\{s_0\}, w)$ $(w = \epsilon)$
$\Longleftrightarrow Q(0)$
Assume $P(k) \Longleftrightarrow Q(k)$ for any $k \ge 0$.
$P(k+1) \Longleftrightarrow (\exists r_0, r_1, r_2, \cdots r_k, r_{k+1} \in Q \text{ s.t. } s_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_{k-1} \xrightarrow{w_k,\delta} r_k \xrightarrow{w_{k+1},\delta} r_{k+1})$
$P(k+1) \Longrightarrow P(k)$ *(From computation path of $P(k+1)$)*
$\Longrightarrow Q(k)$ *(Induction Hypothesis)*
$\Longrightarrow r_k \in \hat{\delta}(\{s_0\}, w)$ *where* $w_1 w_2 w_3 \cdots w_k = w$ *(Definition of $Q(k)$)*
Since
$\hat{\delta}(\{s_0\}, w w_{k+1}) = \bigcup_{q \in \hat{\delta}(\{s_0\},w)} \delta(q, w_{k+1})$ *(Definition 1.10(ii)*
and $r_k \in \hat{\delta}(\{s_0\}, w)$,
$\delta(r_k, w_{k+1}) \subset \hat{\delta}(\{s_0\}, w w_{k+1})$
$r_k \xrightarrow{w_{k+1},\delta} r_{k+1}$ *(From computation path of $P(k+1)$)*
Therefore, $r_{k+1} \in \delta(r_k, w_{k+1}) \subset \hat{\delta}(\{s_0\}, w w_{k+1})$

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Therefore, $r_{k+1} \in \hat{\delta}(\{s_0\}, ww_{k+1})$
Therefore, $P(k+1) \implies Q(k+1)$.
Conversely,
$Q(k+1) \implies r_{k+1} \in \hat{\delta}(\{s_0\}, ww_{k+1})$
$\implies r_{k+1} \in \bigcup_{q \in \hat{\delta}(\{s_0\}, w)} \delta(q, w_{k+1})$
$r_{k+1} \in \delta(r_k, w_{k+1})$ *for some* $r_k \in \hat{\delta}(\{s_0\}, w)$
$r_k \in \hat{\delta}(\{s_0\}, w) \implies Q(k)$
$\implies P(k)$ *(Induction Hypothesis)*
$\implies (\exists r_0, r_1, r_2, \cdots r_k \in Q \text{ s.t. } s_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_{k-1} \xrightarrow{w_k,\delta} r_k)$
$r_{k+1} \in \delta(r_k, w_{k+1}) \implies r_k \xrightarrow{w_{k+1},\delta} r_{k+1}$
Combining the two computation paths,
$s_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_{k-1} \xrightarrow{w_k,\delta} r_k \xrightarrow{w_{k+1},\delta} r_{k+1}$
Therefore, $Q(k+1) \implies P(k+1)$ *and the proof is complete.*

Proposition 4.
$\forall x, y \in \Sigma_\epsilon^* \ \& \ A \in P(Q), \hat{\delta}(A, xy) = \hat{\delta}(\hat{\delta}(A, x), y)$
$< Proof >$
The proof is by induction on $n = |y|$.
Let $T(n)$ *denote the statement corresponding to* $n = 0, 1, 2, \cdots$
For $|y| = 0$, $y = \epsilon$.
$\hat{\delta}(A, x\epsilon) = \hat{\delta}(A, x)$
$= \hat{\delta}(\hat{\delta}(A, x), \epsilon)$ *(Definition 1.10(i))*
$T(0)$ *is true.*
Assume $T(k)$ *is true for* $|y| = k \geq 0$.
That is $\hat{\delta}(A, xy) = \hat{\delta}(\hat{\delta}(A, x), y)$ *for* $|y| = k \geq 0$
For any $a \in \Sigma_\epsilon$, $y \in \Sigma_\epsilon^*$, $|y| = k$
LHS of $T(k+1) = \hat{\delta}(A, xya)$
$= \bigcup_{q \in \hat{\delta}(A, xy)} \delta(q, a)$ *(By Definition 1.10(ii))*
$= \bigcup_{q \in \hat{\delta}(\hat{\delta}(A, x), y)} \delta(q, a)$ *(By Induction Hypothesis)*
$= \hat{\delta}(\hat{\delta}(A, x), ya)$ *(By Definition 1.10(ii))*
$=$ *RHS of* $T(k+1)$
Therefore, $T(k) \implies T(k+1)$.

Proposition 5.
$\forall A_i \subset Q, x \in \Sigma_\epsilon^*, i = 1, 2, \cdots n, n \in N, \hat{\delta}\left(\bigcup_{i=1}^{n} A_i, x\right) = \bigcup_{i=1}^{n} \hat{\delta}(A_i, x)$
$< Proof >$
The proof is by induction on $|x|$.
For $|x| = 0$, $x = \epsilon$.
$\hat{\delta}\left(\bigcup_{i=1}^{n} A_i, \epsilon\right) = \bigcup_{i=1}^{n} A_i$ *(Definition 1.10(i))*
$= \bigcup_{i=1}^{n} \hat{\delta}(A_i, \epsilon)$ *(Definition 1.10(i))*
Claim: $\forall n \in N$, *sets* A_i *and* S_x
$\bigcup_{x \in \cup_{i=1}^{n} A_i} S_x = \bigcup_{i=1}^{n} \left(\bigcup_{x \in A_i} S_x\right)$
<Proof of Claim>
LHS $= \bigcup_{x \in A_1 \cup A_2 \cup \cdots A_n} S_x$

$= \left(\bigcup_{x\in A_1} S_x\right) \cup \left(\bigcup_{x\in A_2} S_x\right) \cup \cdots \cup \left(\bigcup_{x\in A_n} S_x\right)$

$= \bigcup_{i=1}^{n} \left(\bigcup_{x\in A_i} S_x\right)$

$= RHS$

Assume the statement is true for $|x| = k$ *for* $k \geq 0$.

$\forall a \in \Sigma_\epsilon, |xa| = k+1.$

$\hat{\delta}\left(\bigcup_{i=1}^{n} A_i, xa\right) = \bigcup_{p\in\hat{\delta}((\cup_{i=1}^{n} A_i),x)} \delta(p,a)$ *(Definition 1.10(ii))*

$= \bigcup_{p\in\cup_{i=1}^{n}\hat{\delta}(A_i,x)} \delta(p,a)$ *(Induction Hypothesis)*

$= \bigcup_{i=1}^{n} \left(\bigcup_{p\in\hat{\delta}(A_i,x)} \delta(p,a)\right)$ *(Claim)*

$= \bigcup_{i=1}^{n} \hat{\delta}(A_i, xa)$ *(Definition 1.10(ii))*

Therefore, the statement is also true for $|x| = k+1$.

Proposition 6.

$\hat{\delta}(A,x) = \bigcup_{q\in A} \hat{\delta}(\{q\},x)$ *for all* $A \subset Q$.

$< Proof >$

$LHS = \hat{\delta}\left(\bigcup_{q\in A} \{q\}, x\right)$

$= \bigcup_{q\in A} \hat{\delta}(\{q\},x)$ *(Proposition 1.15)*

$= RHS$

Proposition 7.

$\forall A, B,$ *where* $A \subset B \subset Q, \hat{\delta}(A,x) \subset \hat{\delta}(B,x)$

$< Proof >$

$B = A \cup (B \setminus A)$ *(Set Theory)*

$\hat{\delta}(B,x) = \hat{\delta}(A \cup (B\setminus A), x)$

$= \hat{\delta}(A,x) \cup \hat{\delta}(B\setminus A, x)$ *(Proposition 1.15)*

Therefore, $\hat{\delta}(A,x) \subset \hat{\delta}(B,x)$

Proposition 8.

For any two NFAs N_1 *and* N_2*, where*

$N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$

$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ *and* $Q_1 \subset Q_2$

$\forall q \in Q_1, a \in \Sigma_\epsilon, \delta_1(q,a) \subset \delta_2(q,a) \Rightarrow \hat{\delta}_1(\{q\},w) \subset \hat{\delta}_2(\{q\},w) \forall w \in \Sigma_\epsilon^*$

$< Proof >$

The proof is by induction on $|w|$.

For $|w| = 0$, $w = \epsilon$.

$\hat{\delta}_1(\{q\},\epsilon) = \{q\}$ *and* $\hat{\delta}_2(\{q\},\epsilon) = \{q\}$ *(By Definition 1.10(i))*

Therefore, $\hat{\delta}_1(\{q\},\epsilon) \subset \hat{\delta}_2(\{q\},\epsilon)$.

The statement is true for $|w| = 0$.

Assume the statement is true for $|w| = k \geq 0$.

That is, $\hat{\delta}_1(\{q\},w) \subset \hat{\delta}_2(\{q\},w)$ *for* $|w| = k \geq 0$.

For $k+1$,

$\hat{\delta}_1(\{q\},wa) = \bigcup_{p\in\hat{\delta}_1(\{q\},w)} \delta_1(p,a)$

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$$\subset \bigcup_{p \in \hat{\delta}_2(\{q\},w)} \delta_2(p,a) \quad \text{(By Induction Hypothesis and } \delta_1(q,a) \subset \delta_2(q,a))$$
$$= \hat{\delta}_2(\{q\},wa) \quad \text{(By Definition 1.10(ii))}$$

Theorem 3. ***(NFA acceptance)***
$N = (Q, \Sigma, \delta, s_0, F)$ is an NFA.
$\forall w \in \Sigma_\epsilon^*$ where $w = w_1 w_2 w_3 \cdots w_n$; and
$(w_i \in \Sigma_\epsilon$ for $1 \le i \le n$ and $n \ge 1)$ or $(w = \epsilon$ and $n = 0)$.
N accepts w if and only if $\hat{\delta}(\{s_0\}, w) \cap F \neq \emptyset$
In other words, N accepts w if and only if $(\exists r \in F$ s.t. $s_0 \xrightarrow{w,\hat{\delta}} r)$
$< Proof >$
If N accepts w
$\exists r_0, r_1, r_2, \cdots r_n \in Q$ s.t. $s_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_{n-1} \xrightarrow{w_n,\delta} r_n$ and $r_n \in F$.
$r_n \in \hat{\delta}(\{s_0\}, w)$ (By Proposition 1.13)
Since r_n is also in F,
$\hat{\delta}(\{s_0\}, w) \cap F \neq \emptyset$
Conversely, if $\hat{\delta}(\{s_0\}, w) \cap F \neq \emptyset$,
$\exists r_n \in \hat{\delta}(\{s_0\}, w)$ and $r_n \in F$.
$\exists r_0, r_1, r_2, \cdots r_n \in Q$ s.t. $s_0 = r_0 \xrightarrow{w_1,\delta} r_1 \xrightarrow{w_2,\delta} r_2 \cdots r_{n-1} \xrightarrow{w_n,\delta} r_n$; $r_n \in F$ (Proposition 1.13)
Therefore, N accepts w.

3. Epsilon-Closure

The ϵ-Closure of a set of states is a collection of states that can be reached from a member of the given set of states via zero or a finite number of ϵ transitions.
Formally, we define ϵ-Closure as follows.

Definition 9.
Let $N = (Q, \Sigma, \delta, s_0, F)$ be an NFA.
For any $R \subset Q$, the ϵ-Closure of R is
$E(R) = \{q \in Q \mid p \xrightarrow{\epsilon^i,\hat{\delta}} q$ for some $p \in R\}$ where
i is an integer ≥ 0 and $p \xrightarrow{\epsilon^0,\hat{\delta}} q$ means $p = q$.

Proposition 9.
$\forall A_i \subset Q, i = 1, 2, \cdots n, n \in N, \quad E\left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} E(A_i)$
$< Proof >$
Claim. $E(A_1 \cup A_2) = E(A_1) \cup E(A_2)$
<Proof of Claim>
$q \in E(A_1 \cup A_2) \Leftrightarrow \exists p \in A_1 \cup A_2$ s.t. $p \xrightarrow{\epsilon^i,\hat{\delta}} q$ where $i \ge 0$
$\Leftrightarrow ((\exists p \in A_1) \vee (\exists p \in A_2)) \wedge (p \xrightarrow{\epsilon^i,\hat{\delta}} q$ where $i \ge 0)$
$\Leftrightarrow (q \in E(A_1)) \vee (q \in E(A_2))$
$\Leftrightarrow q \in E(A_1) \cup E(A_2)$
Therefore, $E(A_1 \cup A_2) = E(A_1) \cup E(A_2)$
With this Claim and an induction argument, we can conclude Proposition 1.21.

4. The Equivalence of *DFA* and *NFA*

Lemma 1.
Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA, $M = (Q', \Sigma, \delta', q_0', F')$ be a DFA.
$Q' = P(Q), q_0' = E(\{q_0\}), F' = \{R \in Q' \mid R \cap F \neq \emptyset\}$
$\delta' : Q' \times \Sigma \longrightarrow Q'$ such that

$\delta^{'}(R,a) = \bigcup_{r \in R} E(\delta(r,a)) \quad \forall a \in \Sigma, R \in Q^{'}$
Let $w = w_1 w_2 w_3 \cdots w_n$ *such that if* $n = 0, w = \epsilon$ *and if* $n \geq 1$, *then* $w_i \neq \epsilon \quad \forall 1 \leq i \leq n$.
Let $i_0, i_1, i_2, \cdots i_n$ *be integers* $\geq 0, q_0, q_1, q_2, \cdots q_n \in Q, p_1, p_2, p_3, \cdots p_n \in Q$ *and* $q \in Q$.
The following holds:
$q_0 \xrightarrow{\epsilon^{i_0},\hat{\delta}} q_1 \xrightarrow{w_1,\delta} p_1 \xrightarrow{\epsilon^{i_1},\hat{\delta}} q_2 \xrightarrow{w_2,\delta} p_2 \xrightarrow{\epsilon^{i_2},\hat{\delta}} q_3 \cdots q_{n-1} \xrightarrow{w_{n-1},\delta} p_{n-1} \xrightarrow{\epsilon^{i_{n-1}},\hat{\delta}} q_n \xrightarrow{w_n,\delta} p_n \xrightarrow{\epsilon^{i_n},\hat{\delta}} q.$
$\Longleftrightarrow q \in \hat{\delta}^{'}(q_0^{'}, w)$
$< Proof >$
Proof is by induction on $|w| = n$.
Let $P(n)$ *denote the statement of*
$q_0 \xrightarrow{\epsilon^{i_0},\hat{\delta}} q_1 \xrightarrow{w_1,\delta} p_1 \xrightarrow{\epsilon^{i_1},\hat{\delta}} q_2 \xrightarrow{w_2,\delta} p_2 \xrightarrow{\epsilon^{i_2},\hat{\delta}} q_3 \cdots q_{n-1} \xrightarrow{w_{n-1},\delta} p_{n-1} \xrightarrow{\epsilon^{i_{n-1}},\hat{\delta}} q_n \xrightarrow{w_n,\delta} p_n \xrightarrow{\epsilon^{i_n},\hat{\delta}} q.$
and $Q(n)$ *denote the statement of* $q \in \hat{\delta}^{'}(q_0^{'}, w)$ *corresponding to* $n \geq 0$.
For $|w| = n = 0, w = \epsilon$.
$P(0) \Longleftrightarrow q_0 \xrightarrow{\epsilon^{i_0},\hat{\delta}} q$
$\Longleftrightarrow q \in E(\{q_0\})$
$\Longleftrightarrow q \in q_0^{'} \qquad (q_0^{'} = E(\{q_0\}))$
$\Longleftrightarrow q \in \hat{\delta}^{'}(q_0^{'}, \epsilon) \qquad$ *(Definition 1.4(i))*
$\Longleftrightarrow q \in \hat{\delta}^{'}(q_0^{'}, w) \qquad (w = \epsilon)$
$\Longleftrightarrow Q(0)$
Assume $P(k) \Leftrightarrow Q(k)$ *for* $k \geq 0$.
$P(k+1)$
$\Leftrightarrow q_0 \xrightarrow{\epsilon^{i_0},\hat{\delta}} q_1 \xrightarrow{w_1,\delta} p_1 \xrightarrow{\epsilon^{i_1},\hat{\delta}} q_2 \xrightarrow{w_2,\delta} p_2 \xrightarrow{\epsilon^{i_2},\hat{\delta}} q_3 \cdots q_k \xrightarrow{w_k,\delta} p_k \xrightarrow{\epsilon^{i_k},\hat{\delta}} q_{k+1} \xrightarrow{w_{k+1},\delta} p_{k+1} \xrightarrow{\epsilon^{i_{k+1}},\hat{\delta}} q.$
$\Leftrightarrow P(k)\ \&\ q_{k+1} \xrightarrow{w_{k+1},\delta} p_{k+1} \xrightarrow{\epsilon^{i_{k+1}},\hat{\delta}} q$
$\Leftrightarrow Q(k)\ \&\ q_{k+1} \xrightarrow{w_{k+1},\delta} p_{k+1} \xrightarrow{\epsilon^{i_{k+1}},\hat{\delta}} q \qquad$ *(Induction Hypothesis)*
$\Leftrightarrow q_{k+1} \in \hat{\delta}^{'}(q_0^{'}, w)$ *where* $|w| = k\ \&\ q_{k+1} \xrightarrow{w_{k+1},\delta} p_{k+1} \xrightarrow{\epsilon^{i_{k+1}},\hat{\delta}} q$
$\Leftrightarrow q_{k+1} \in \hat{\delta}^{'}(q_0^{'}, w)$ *where* $|w| = k\ \&\ q \in E(\delta(q_{k+1}, w_{k+1}))$
$\Leftrightarrow q \in \bigcup_{r \in \hat{\delta}^{'}(q_0^{'}, w)} E(\delta(r, w_{k+1}))$ *where* $|w| = k$
$\Leftrightarrow q \in \delta^{'}(\hat{\delta}^{'}(q_0^{'}, w), w_{k+1})$ *where* $|w| = k$
(*Consider* $R = \hat{\delta}^{'}(q_0^{'}, w), w_{k+1} = a\ \&\ \delta^{'}(R,a) \stackrel{def}{=} \bigcup_{r \in R} E(\delta(r,a))$)
$\Leftrightarrow q \in \hat{\delta}^{'}(q_0^{'}, ww_{k+1})$ *where* $|w| = k \qquad$ *(Definition 1.4(ii))*
$\Leftrightarrow Q(k+1)$
This completes the proof of Lemma 1.22.

Theorem 4.
Every NFA can be converted to an equivalent DFA.
$< Proof >$
Let $N = (Q, \Sigma, \delta, q_0, F)$ *be an NFA.*
Construct a DFA as follows.
$M = (Q^{'}, \Sigma, \delta^{'}, q_0^{'}, F^{'})$ *where*
$Q^{'} = P(Q), q_0{'} = E(\{q_0\}), F^{'} = \{R \in Q^{'} \mid R \cap F \neq \emptyset\}$
$\delta^{'} : Q^{'} \times \Sigma \longrightarrow Q^{'}$ *such that*
$\delta^{'}(R,a) = \bigcup_{r \in R} E(\delta(r,a)) \quad \forall a \in \Sigma, R \in Q^{'}$
We claim that N *and* M *are equivalent by showing that*
$\forall w \in \Sigma_{\epsilon}^{*}, N$ *accepts* $w \Leftrightarrow M$ *accepts* w
The proof is divided into two cases, one with $w = \epsilon$ *and one with* $w \neq \epsilon$.
(i) $w = \epsilon$
If N *accepts* w,

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$\exists j \geq 0$ s.t. $q_0 \xrightarrow{\epsilon^j, \hat{\delta}} p$ *and* $p \in F$.
Therefore, $p \in E(\{q_0\})$ & $p \in F$.
Therefore, $p \in q_0^{'}$ & $p \in F$.
Therefore, $q_0^{'} \cap F \neq \emptyset$.
Therefore, $q_0^{'} \in F^{'}$.
Therefore, the start state of M is also an accept state of M.
By definition, M accepts $\epsilon (= w)$.
Conversely, if M accepts $w = \epsilon$,
$q_0^{'} \in F^{'}$ *(A DFA accepts* ϵ *iff its start state is also an accept state.)*
$q_0^{'} \cap F \neq \emptyset$ *(By definition of* $F^{'}$*)*
$\exists p \in q_0^{'}$ *and* $p \in F$.
Since $q_0{'} = E(\{q_0\}), q_0 \xrightarrow{\epsilon^j, \hat{\delta}} p$ *for some* $j \geq 0$.
Since $p \in F$, *N accepts* ϵ^j, *which is same as* ϵ.
(ii) $w \neq \epsilon$
$\exists w_i \neq \epsilon, \forall 1 \leq i \leq n, n \geq 1$ *and*
$w = \epsilon^{i_0} w_1 \epsilon^{i_1} w_2 \epsilon^{i_2} w_3 \epsilon^{i_3} \cdots w_n \epsilon^{i_n}$ *for some integers* $i_0, i_1, i_2, \cdots i_n \geq 0$
If N accepts w,
$\exists q_0, q_1, q_2, \cdots q_n \in Q, p_1, p_2, p_3, \cdots p_n \in Q$ *and* $q \in Q$. *s.t.*
$q_0 \xrightarrow{\epsilon^{i_0}, \hat{\delta}} q_1 \xrightarrow{w_1, \delta} p_1 \xrightarrow{\epsilon^{i_1}, \hat{\delta}} q_2 \xrightarrow{w_2, \delta} p_2 \xrightarrow{\epsilon^{i_2}, \hat{\delta}} q_3 \cdots q_{n-1} \xrightarrow{w_{n-1}, \delta} p_{n-1} \xrightarrow{\epsilon^{i_{n-1}}, \hat{\delta}} q_n \xrightarrow{w_n, \delta} p_n \xrightarrow{\epsilon^{i_n}, \hat{\delta}} q$ & $q \in F$.
By Lemma 1.22, $q \in \hat{\delta}^{'}(q_0^{'}, w)$ *where* $w = w_1 w_2 w_3 \cdots w_n$.
Therefore, $\hat{\delta}^{'}(q_0^{'}, w) \cap F \neq \emptyset$.
Therefore, $\hat{\delta}^{'}(q_0^{'}, w) \in F'$.
Therefore, M accepts w *(DFA acceptance)*
Conversely, if M accepts $w = w_1 w_2 w_3 \cdots w_n$,
$\hat{\delta}^{'}(q_0^{'}, w) \in F'$ *(DFA acceptance)*
$\hat{\delta}^{'}(q_0^{'}, w) \cap F \neq \emptyset$ *(Definition of* F'*)*
$\exists q \in \hat{\delta}^{'}(q_0^{'}, w)$ *and* $q \in F$.
By Lemma 1.22,
$q_0 \xrightarrow{\epsilon^{i_0}, \hat{\delta}} q_1 \xrightarrow{w_1, \delta} p_1 \xrightarrow{\epsilon^{i_1}, \hat{\delta}} q_2 \xrightarrow{w_2, \delta} p_2 \xrightarrow{\epsilon^{i_2}, \hat{\delta}} q_3 \cdots q_{n-1} \xrightarrow{w_{n-1}, \delta} p_{n-1} \xrightarrow{\epsilon^{i_{n-1}}, \hat{\delta}} q_n \xrightarrow{w_n, \delta} p_n \xrightarrow{\epsilon^{i_n}, \hat{\delta}} q$ & $q \in F$.
Therefore, N accepts $w = \epsilon^{i_0} w_1 \epsilon^{i_1} w_2 \epsilon^{i_2} w_3 \epsilon^{i_3} \cdots w_n \epsilon^{i_n}$
This completes the proof of Theorem 1.23.

Corollary 1.
A language is regular iff some NFA recognizes it.

5. Regular Operators

Regular Languages are closed under the operation of Regular Operators.

Theorem 5.
L is regular $\Rightarrow \Sigma^* \setminus L$ *is regular.*
$< Proof >$
Let $M = (Q, \Sigma, \delta, q_0, F)$ *be the DFA that recognizes L.*
That is, $L(M) = L$.
Define $M^{'} = (Q, \Sigma, \delta^{'}, q_0, Q \setminus F)$ *where*
$\delta^{'} : Q \times \Sigma \longrightarrow Q$ *s.t.* $\forall q \in Q, a \in \Sigma, \delta^{'}(q, a) = \delta(q, a)$
$\forall w \in \Sigma^* \setminus L$,
$w \notin L \Longrightarrow \hat{\delta}(q_0, w) \notin F$
$\Longrightarrow \hat{\delta}(q_0, w) \in Q \setminus F$
$\Longrightarrow \hat{\delta}^{'}(q_0, w) \in Q \setminus F$ *(Theorem 1.8)*

$\Longrightarrow M^{'}$ *accepts* w
Conversely, if $M^{'}$ *accepts* w, $\hat{\delta}^{'}(q_0, w) \in Q \setminus F$
$\hat{\delta}(q_0, w) \in Q \setminus F$ *(Theorem 1.8)*
Therefore, $\hat{\delta}(q_0, w) \notin F$
Therefore, $w \notin L$ *(because* $w \in L \Longrightarrow M$ *accepts* $w \Longrightarrow \hat{\delta}(q_0, w) \in F$*)*
$w \in \Sigma^* \setminus L$
Therefore, $w \in \Sigma^* \setminus L \Longleftrightarrow M^{'}$ *accepts* w.
$L(M^{'}) = \Sigma^* \setminus L$
$\Sigma^* \setminus L$ *is regular.*

Theorem 6.
L_1 *and* L_2 *are regular* $\Longrightarrow L_1 \cap L_2$ *is regular.*
$< Proof >$
$\exists$*DFAs* M_1 *and* M_2 *s.t.* $L(M_1) = L_1$ *and* $L(M_2) = L_2$
Let $M_1 = (Q_1, \Sigma, \delta_1, s_0, F_1)$
$M_2 = (Q_2, \Sigma, \delta_2, s_0^{'}, F_2)$
Define M_3 *as follows.*
$M_3 = (Q_3, \Sigma, \delta_3, s_0^{''}, F_3)$
where $s_0^{''} = (s_0, s_0^{'}), Q_3 = Q_1 \times Q_2, F_3 = F_1 \times F_2$
$\delta_3 : Q_3 \times \Sigma \longrightarrow Q_3$ *s.t.*
$\delta_3((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a)) \quad \forall q_1 \in Q_1, q_2 \in Q_2, a \in A.$
Claim. $\forall n \in N \cup \{0\}, w \in \Sigma^*$, *where* $|w| = n$, *if*
(i) $s_0 = r_0 \xrightarrow{w_1,\delta_1} r_1 \xrightarrow{w_2,\delta_1} r_2 \cdots r_{n-1} \xrightarrow{w_n,\delta_1} r_n$
(ii) $s_0^{'} = r_0^{'} \xrightarrow{w_1,\delta_2} r_1^{'} \xrightarrow{w_2,\delta_2} r_2^{'} \cdots r_{n-1}^{'} \xrightarrow{w_n,\delta_2} r_n^{'}$
(iii) $s_0^{''} = r_0^{''} \xrightarrow{w_1,\delta_3} r_1^{''} \xrightarrow{w_2,\delta_3} r_2^{''} \cdots r_{n-1}^{''} \xrightarrow{w_n,\delta_3} r_n^{''}$
then $r_n^{''} = (r_n, r_n^{'})$.
Proof of Claim is by induction on n.
For $n = 0$, *(i), (ii) and (iii) become* $s_0 = r_0, s_0^{'} = r_0^{'}$, *and* $s_0^{''} = r_0^{''}$.
$s_0^{''} = (s_0, s_0^{'})$ *(By definition of* M_3*.)*
Therefore, $r_0^{''} = (r_0, r_0^{'})$
Assume the statement is true for $n = k \geq 0$.
(i), (ii) & (iii) for $n = k + 1 \Rightarrow$
$s_0 = r_0 \xrightarrow{w_1,\delta_1} r_1 \xrightarrow{w_2,\delta_1} r_2 \cdots r_{k-1} \xrightarrow{w_k,\delta_1} r_k \xrightarrow{w_{k+1},\delta_1} r_{k+1}$
$s_0^{'} = r_0^{'} \xrightarrow{w_1,\delta_2} r_1^{'} \xrightarrow{w_2,\delta_2} r_2^{'} \cdots r_{k-1}^{'} \xrightarrow{w_k,\delta_2} r_k^{'} \xrightarrow{w_{k+1},\delta_2} r_{k+1}^{'}$
$s_0^{''} = r_0^{''} \xrightarrow{w_1,\delta_3} r_1^{''} \xrightarrow{w_2,\delta_3} r_2^{''} \cdots r_{k-1}^{''} \xrightarrow{w_k,\delta_3} r_k^{''} \xrightarrow{w_{k+1},\delta_3} r_{k+1}^{''}$
$\Rightarrow$ *(i), (ii) & (iii) for* $n = k$ & $r_k \xrightarrow{w_{k+1},\delta_1} r_{k+1}$ & $r_k^{'} \xrightarrow{w_{k+1},\delta_2} r_{k+1}^{'}$ & $r_k^{''} \xrightarrow{w_{k+1},\delta_3} r_{k+1}^{''}$
$\Rightarrow r_k^{''} = (r_k, r_k^{'})$ & $r_{k+1} = \delta_1(r_k, w_{k+1})$ & $r_{k+1}^{'} = \delta_2(r_k^{'}, w_{k+1})$ & $r_{k+1}^{''} = \delta_3(r_k^{''}, w_{k+1})$
(Induction Hypothesis)
$\Rightarrow r_{k+1} = \delta_1(r_k, w_{k+1})$ & $r_{k+1}^{'} = \delta_2(r_k^{'}, w_{k+1})$ & $r_{k+1}^{''} = \delta_3((r_k, r_k^{'}), w_{k+1})$
$\Rightarrow r_{k+1} = \delta_1(r_k, w_{k+1})$ & $r_{k+1}^{'} = \delta_2(r_k^{'}, w_{k+1})$ & $r_{k+1}^{''} = (\delta_1(r_k, w_{k+1}), \delta_2(r_k^{'}, w_{k+1}))$
(Definition of δ_3*)*
$\Rightarrow r_{k+1}^{''} = (r_{k+1}, r_{k+1}^{'})$
We now need to show $L_1 \cap L_2 = L(M_3)$.
$\forall w \in L_1 \cap L_2, w \in L_1$ *and* $w \in L_2$.
$w \in L_1 \Rightarrow \exists s_0 = r_0, r_1, r_2, \cdots r_n$ *s.t.* $s_0 = r_0 \xrightarrow{w_1,\delta_1} r_1 \xrightarrow{w_2,\delta_1} r_2 \cdots\cdots r_{n-1} \xrightarrow{w_n,\delta_1} r_n$ & $r_n \in F_1$
$w \in L_2 \Rightarrow \exists s_0^{'} = r_0^{'}, r_1^{'}, r_2^{'}, \cdots r_n^{'}$ *s.t.* $s_0^{'} = r_0^{'} \xrightarrow{w_1,\delta_2} r_1^{'} \xrightarrow{w_2,\delta_2} r_2^{'} \cdots\cdots r_{n-1}^{'} \xrightarrow{w_n,\delta_2} r_n^{'}$ & $r_n^{'} \in F_2$
Let $r_0^{''} = s_0^{''} = (s_0, s_0^{'})$
$r_1^{''} = \delta_3(r_0^{''}, w_1), \cdots r_{i+1}^{''} = \delta_3(r_i^{''}, w_{i+1}), \cdots r_n^{''} = \delta_3(r_{n-1}^{''}, w_n)$.

Therefore, $s_0'' = r_0'' \xrightarrow{w_1,\delta_3} r_1'' \xrightarrow{w_2,\delta_3} r_2'' \cdots r_{n-1}'' \xrightarrow{w_n,\delta_3} r_n''$
By Claim, $r_n'' = (r_n, r_n')$
Since $r_n \in F_1$ *and* $r_n' \in F_2, r_n'' \in F_1 \times F_2 = F_3$.
Therefore, M_3 *accepts* w.
$w \in L(M_3)$
$L_1 \cap L_2 \subset L(M_3)$
Conversely, if $w \in L(M_3)$,
$\exists r_0'', r_1'', r_2'', \cdots r_n'' \in Q_3$ *s.t.* $s_0'' = r_0'' \xrightarrow{w_1,\delta_3} r_1'' \xrightarrow{w_2,\delta_3} r_2'' \cdots\cdots r_{n-1}'' \xrightarrow{w_n,\delta_3} r_n''$ & $r_n'' \in F_3$
Take
$r_0 = s_0;$
$r_{i+1} = \delta_1(r_i, w_{i+1}) \forall i = 0, 1, 2, \cdots n-1;$
$r_0' = s_0';$
$r_{i+1}' = \delta_2(r_i', w_{i+1}) \forall i = 0, 1, 2, \cdots n-1.$
Therefore,
$s_0 = r_0 \xrightarrow{w_1,\delta_1} r_1 \xrightarrow{w_2,\delta_1} r_2 \cdots\cdots r_{n-1} \xrightarrow{w_n,\delta_1} r_n$
$s_0' = r_0' \xrightarrow{w_1,\delta_2} r_1' \xrightarrow{w_2,\delta_2} r_2' \cdots\cdots r_{n-1}' \xrightarrow{w_n,\delta_2} r_n'$
By Claim, $r_n'' = (r_n, r_n')$
Since $r_n'' \in F_3 = F_1 \times F_2, r_n \in F_1$ *and* $r_n' \in F_2$.
M_1 *accepts* w *and* M_2 *accepts* w.
$w \in L(M_1)$ *and* $w \in L(M_2)$
$w \in L_1$ *and* $w \in L_2$
$w \in L_1 \cap L_2$
$L(M_3) \subset L_1 \cap L_2$
Combining both directions, $L(M_3) = L_1 \cap L_2$
$L_1 \cap L_2$ *is regular.*

Theorem 7.
L_1 *and* L_2 *are regular* $\Longrightarrow L_1 \cup L_2$ *is regular.*
$< Proof >$
From set theory, $\Sigma^* \setminus (L_1 \cup L_2) = (\Sigma^* \setminus L_1) \cap (\Sigma^* \setminus L_2)$
L_1 *is regular* $\Longrightarrow \Sigma^* \setminus L_1$ *is regular.* *(Theorem 1.25)*
L_2 *is regular* $\Longrightarrow \Sigma^* \setminus L_2$ *is regular.* *(Theorem 1.25)*
$\Sigma^* \setminus L_1$ *and* $\Sigma^* \setminus L_2$ *are regular* $\Longrightarrow (\Sigma^* \setminus L_1) \cap (\Sigma^* \setminus L_2)$ *is regular.* *(Theorem 1.26)*
Therefore, $\Sigma^* \setminus (L_1 \cup L_2)$ *is regular.*
Therefore, $L_1 \cup L_2$ *is regular.* *(Theorem 1.25)*

Theorem 8.
Every NFA can be converted to another NFA with the following properties.
(i) There is only one accept state which has transition arrows coming in and no transition arrows going out.
(ii) The accept state is different from the start state.
(iii) The start state has no arrows coming in from other states but only transition arrows going out.
$< Proof >$
Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ *be the NFA to be converted.*
Define NFA, $N = (Q, \Sigma, \delta, q_0, \{q_a\})$ *where* $Q = Q_1 \cup \{q_0, q_a\}, q_0 \neq q_a$ *and*

$$\delta(q,x) = \begin{cases} \{q_1\} & \text{if} \quad (q,x,) = (q_0,\epsilon) \\ \emptyset & \text{if} \quad q = q_0 \text{ and } x \neq \epsilon \\ \emptyset & \text{if} \quad q = q_a \\ \delta_1(q,x) & \text{if} \quad q \in Q_1 \setminus F_1 \\ \delta_1(q,x) & \text{if} \quad q \in F_1 \text{ and } x \neq \epsilon \\ \delta_1(q,x) \cup \{q_a\} & \text{if} \quad q \in F_1 \text{ and } x = \epsilon \end{cases}$$

It is clear that N satisfies conditions (i), (ii) and (iii).
Furthermore, $\delta_1(q,x) \subset \delta(q,x)\ \forall x \in \Sigma_\epsilon, q \in Q_1$ *and hence*
$\hat{\delta}_1(\{q\},w) \subset \hat{\delta}(\{q\},w)\ \forall w \in \Sigma_\epsilon^*$ *by Proposition 1.18.*
It remains to show that $\forall w \in \Sigma_\epsilon^*, N_1$ *accepts* $w \Leftrightarrow N$ *accepts* w.
For forward direction $'' \Rightarrow ''$,
Let N_1 *accepts* w.
$q_1 \xrightarrow{w,\hat{\delta}_1} r, r \in F_1$.
Since $\hat{\delta}_1(q_1,w) \subset \hat{\delta}(q_1,w), q_1 \xrightarrow{w,\hat{\delta}} r, r \in F_1$.
Since $\delta(q_0,\epsilon) = \{q_1\}, q_0 \xrightarrow{\epsilon,\delta} q_1$.
Furthermore, since $\delta(q,\epsilon) = \delta_1(q,\epsilon) \cup \{q_a\}\ \forall q \in F_1$, $\delta(r,\epsilon) = \delta_1(r,\epsilon) \cup \{q_a\}$.
Therefore, $q_a \in \delta(r,\epsilon)$.
That is, $r \xrightarrow{\epsilon,\delta} q_a$
Therefore, $q_0 \xrightarrow{\epsilon,\delta} q_1 \xrightarrow{w,\hat{\delta}} r \xrightarrow{\epsilon,\delta} q_a$.
Therefore N accepts $\epsilon w \epsilon$ *which is the same as* w.
Therefore, N_1 *accepts* $w \Rightarrow N$ *accepts* w.
Conversely, if N accepts $w = x_1x_2\cdots x_n$ *where* $x_i \in \Sigma_\epsilon$ *for* $n \geq 1$ & $1 \leq i \leq n$.
(Note that $w = \epsilon$ *if* $x_i = \epsilon\ \forall i$.*)*
$\exists r_0, r_1, r_2, \cdots r_n \in Q$ *s.t.*
$q_0 = r_0 \xrightarrow{x_1,\delta} r_1 \xrightarrow{x_2,\delta} r_2 \cdots r_{n-1} \xrightarrow{x_n,\delta} r_n$ & $r_n \in \{q_a\}$
Since the only way to transition to q_a *using* δ *is from a state in* F_1 *via the* ϵ *arrow, we must have* $r_{n-1} \in F_1$ & $x_n = \epsilon$.
Since the only way to transition out of $q_0 (= r_0)$ *using* δ *is via an* ϵ *arrow, we must have* $x_1 = \epsilon$.
Since $\delta(q_0,\epsilon) = \{q_1\}$, *we must have* $r_1 = q_1$.
We now can rewrite the above computation as
$q_0 = r_0 \xrightarrow{\epsilon,\delta} q_1 \xrightarrow{x_2,\delta} r_2 \cdots r_{n-2} \xrightarrow{x_{n-1},\delta} r_{n-1} \xrightarrow{\epsilon,\delta} q_a$ & $r_{n-1} \in F_1$.
For all $1 \leq j \leq n-2, r_j \notin \{q_0, q_a\}$ *because* r_j *has both incoming and outgoing arrows.*
Therefore, $r_j \in Q_1$.
Claim. $r_j \xrightarrow{x_{j+1},\delta_1} r_{j+1} \quad \forall 1 \leq j \leq n-2$.
Since $r_j \in Q_1, \delta(r_j, x_{j+1}) = \delta_1(r_j, x_{j+1})$ *or* $\delta_1(r_j, x_{j+1}) \cup \{q_a\}$ *by definition of* δ.
$r_j \xrightarrow{x_{j+1},\delta} r_{j+1}$
$\Rightarrow r_{j+1} \in \delta(r_j, x_{j+1})$
$\Rightarrow r_{j+1} \in \delta_1(r_j, x_{j+1})$ *or* $r_{j+1} \in \delta_1(r_j, x_{j+1}) \cup \{q_a\}$
$\Rightarrow r_{j+1} \in \delta_1(r_j, x_{j+1})$ *or* $r_{j+1} \in \delta_1(r_j, x_{j+1}) \quad$ *(because* $r_{j+1} \neq q_a$*)*
$\Rightarrow r_{j+1} \in \delta_1(r_j, x_{j+1})$
$\Rightarrow r_j \xrightarrow{x_{j+1},\delta_1} r_{j+1}$
The computation now becomes
$q_0 = r_0 \xrightarrow{\epsilon,\delta} q_1 \xrightarrow{x_2,\delta_1} r_2 \cdots r_{n-2} \xrightarrow{x_{n-1},\delta_1} r_{n-1} \xrightarrow{\epsilon,\delta} q_a$ & $r_{n-1} \in F_1$.
Therefore, $q_1 \xrightarrow{x_2,\delta_1} r_2 \cdots r_{n-2} \xrightarrow{x_{n-1},\delta_1} r_{n-1}$ & $r_{n-1} \in F_1$.
Therefore, N_1 *accepts* $x_2x_3 \cdots\cdots x_{n-1}$.
Therefore, N_1 *accepts* $w = x_1x_2x_3 \cdots\cdots x_{n-1}x_n$ *because* $x_1 = x_n = \epsilon$.

Therefore, N accepts $w \Rightarrow N_1$ *accepts* w.
This completes the proof of Theorem 1.28.

Theorem 9.
For any regular languages L_1 *and* L_2*, the language* L_1L_2 *is regular.*
$<$ *Proof* $>$
Since L_1 *and* L_2 *are regular, there exist NFAs* N_1, N_2 *that recognize* L_1 *and* L_2.
By Theorem 1.28, we can start with N_1 *and* N_2 *defined as follows.*
$N_1 = (Q_1, \Sigma, \delta_1, q_{1s}, \{q_{1a}\})$ *where*
$q_{1s} \neq q_{1a}, q_{1s} \notin \delta_1(q,x)\ \forall q \in Q_1, x \in \Sigma_\epsilon$ *and* $\delta_1(q_{1a}, x) = \emptyset\ \forall x \in \Sigma_\epsilon$.
$N_2 = (Q_2, \Sigma, \delta_2, q_{2s}, \{q_{2a}\})$ *where*
$q_{2s} \neq q_{2a}, q_{2s} \notin \delta_2(q,x)\ \forall q \in Q_2, x \in \Sigma_\epsilon$ *and* $\delta_2(q_{2a}, x) = \emptyset\ \forall x \in \Sigma_\epsilon$.
We can further assume that $Q_1 \cap Q_2 = \emptyset$ *because we can always replace* Q_1 *with a set of objects which are completely different from those in* Q_2 *without affecting the function of* N_1.
Now construct $N = (Q, \Sigma, \delta, q_{1s}, \{q_{2a}\})$ *where* $Q = Q_1 \cup Q_2$.

$$\delta(q,x) = \begin{cases} \delta_1(q,x) & \text{if} \quad q \in Q_1 \setminus \{q_{1a}\} \\ \delta_1(q_{1a},x) & \text{if} \quad q = q_{1a}\ \&\ x \neq \epsilon \\ \delta_1(q_{1a},x) \cup \{q_{2s}\} & \text{if} \quad q = q_{1a}\ \&\ x = \epsilon \\ \delta_2(q,x) & \text{if} \quad q \in Q_2 \end{cases}$$

We now need to show $L(N) = L_1L_2$.
If $w \in L_1L_2, w = w_1w_2$ *where* $w_1, w_2 \in \Sigma_\epsilon^*$ *and* $w_1 \in L_1, w_2 \in L_2$.
Since N_1 *recognizes* L_1 *and* N_2 *recognizes* L_2, N_1 *accepts* w_1 *and* N_2 *accepts* w_2.
$\exists r_1 \in \{q_{1a}\}$ *and* $r_2 \in \{q_{2a}\}$ *such that*
$q_{1s} \xrightarrow{w_1, \hat{\delta}_1} r_1$ *and* $q_{2s} \xrightarrow{w_2, \hat{\delta}_2} r_2$ *(By Theorem 1.19 of NFA Acceptance)*
$q_{1s} \xrightarrow{w_1, \hat{\delta}} q_{1a}$ *and* $q_{2s} \xrightarrow{w_2, \hat{\delta}} q_{2a}$ *(Proposition 1.18 and* $r_1 = q_{1a}; r_2 = q_{2a}$*).*
By definition of δ, $q_{2s} \in \delta(q_{1a}, \epsilon)$.
Therefore,
$q_{1s} \xrightarrow{w_1, \hat{\delta}} q_{1a} \xrightarrow{\epsilon, \delta} q_{2s} \xrightarrow{w_2, \hat{\delta}} q_{2a}$.
Therefore, N accepts $w_1 \epsilon w_2$*, which is the same as* w_1w_2.
$L_1L_2 \subset L(N)$
Conversely, if N accepts $w = x_1x_2 \cdots x_n$*, where* $x_1, x_2, \cdots x_n \in \Sigma_\epsilon$ *for* $n \geq 1$,
$\exists r_0, r_1, r_2, \cdots r_n \in Q$ *such that*
$q_{1s} = r_0 \xrightarrow{x_1, \delta} r_1 \xrightarrow{x_2, \delta} r_2 \cdots r_{n-1} \xrightarrow{x_n, \delta} r_n\ \&\ r_n = q_{2a}$.
(Note that $w = \epsilon$ *if* $x_i = \epsilon \quad \forall i$*).*
Since the only way to transition from a state of N_1 *to a state of* N_2 *is via* q_{1a} *to* q_{2s} *using the* ϵ *arrow,* $\exists$ *an* $r_i = q_{1a}$ *and* $r_{i+1} = q_{2s}$ *such that* $x_{i+1} = \epsilon$ *and the computation becomes*
$q_{1s} = r_0 \xrightarrow{x_1, \delta} r_1 \xrightarrow{x_2, \delta} r_2 \cdots r_{i-1} \xrightarrow{x_i, \delta} q_{1a} \xrightarrow{\epsilon, \delta} q_{2s} \xrightarrow{x_{i+2}, \delta} r_{i+2} \cdots r_{n-1} \xrightarrow{x_n, \delta} r_n; r_n = q_{2a}$.
Claim 1. $r_0, r_1, r_2, \cdots r_{i-1} \in Q_1$.
<*Proof of Claim 1*>
$q_{1s} = r_0 \Rightarrow r_0 \in Q_1$.
Assume for contradiction that $r_{i-1} \notin Q_1$.
Then $r_{i-1} \in Q_2$.
$r_{i-1} \xrightarrow{x_i, \delta} q_{1a}$
$\Rightarrow r_{i-1} \xrightarrow{x_i, \delta_2} q_{1a}$ $\quad (\delta(q,x) = \delta_2(q,x)$ *if* $q \in Q_2)$
$\Rightarrow q_{1a} \in Q_2$
$\Rightarrow$ *Contradiction*
Therefore, $r_{i-1} \in Q_1$.
With similar and inductive argument, we can conclude $r_{i-2}, \cdots r_2, r_1$ *are all in* Q_1.
Claim 2. $r_j \neq q_{1a}\ \forall\ 0 \leq j \leq i-1$.

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Assume for contradiction $r_j = q_{1a}$ *for some* $0 \le j \le i-1$.
Therefore, $r_j \xrightarrow{x_{j+1},\delta} r_{j+1} \Leftrightarrow q_{1a} \xrightarrow{x_{j+1},\delta} r_{j+1} \Leftrightarrow r_{j+1} \in \delta(q_{1a}, x_{j+1})$.
By definition of $\delta, \delta(q_{1a}, x_{j+1})$
$= \delta_1(q_{1a}, x_{j+1})$ *or* $\delta_1(q_{1a}, x_{j+1}) \cup \{q_{2s}\}$
$= \emptyset$ or $\emptyset \cup \{q_{2s}\}$
$= \emptyset$ or $\{q_{2s}\}$
Therefore, $r_{j+1} \in \emptyset$ *or* $r_{j+1} \in \{q_{2s}\}$.
Either of these leads to a contradiction.
Therefore, $r_j \ne q_{1a} \ \forall\ 0 \le j \le i-1$.
Combining Claim 1 and Claim 2, $r_j \in Q_1 \setminus \{q_{1a}\} \ \forall\ 0 \le j \le i-1$.
By definition of $\delta, \delta(r_j, x) = \delta_1(r_j, x) \ \forall\ 0 \le j \le i-1$.
Therefore, computation $q_{1s} = r_0 \xrightarrow{x_1,\delta} r_1 \xrightarrow{x_2,\delta} r_2 \cdots r_{i-1} \xrightarrow{x_i,\delta} q_{1a}$ *can be replaced by computation*
$q_{1s} = r_0 \xrightarrow{x_1,\delta_1} r_1 \xrightarrow{x_2,\delta_1} r_2 \cdots r_{i-1} \xrightarrow{x_i,\delta_1} q_{1a}$.
Therefore, N_1 *accepts* $w_1 = x_1x_2\cdots x_i$.
$w_1 \in L(N_1) = L_1$.
Claim 3. $r_j \in Q_2 \ \forall\ i+2 \le j \le n-1$.
<Proof of Claim 3>
$q_{2s} \xrightarrow{x_{i+2},\delta} r_{i+2}$
$\Rightarrow r_{i+2} \in \delta(q_{2s}, x_{i+2})$
$\Rightarrow r_{i+2} \in \delta_2(q_{2s}, x_{i+2}) \qquad (\delta(q,x) = \delta_2(q,x)$ *if* $q \in Q_2)$
$\Rightarrow q_{2s} \xrightarrow{x_{i+2},\delta_2} r_{i+2}$
$\Rightarrow r_{i+2} \in Q_2$.
With similar and inductive argument, we can show that $r_{i+3}, \cdots r_{n-1}$ *are all in* Q_2.
Therefore, computation $q_{2s} \xrightarrow{x_{i+2},\delta} r_{i+2} \cdots r_{n-1} \xrightarrow{x_n,\delta} r_n; r_n = q_{2a}$ *can be replaced by computation*
$q_{2s} \xrightarrow{x_{i+2},\delta_2} r_{i+2} \cdots r_{n-1} \xrightarrow{x_n,\delta_2} r_n; r_n = q_{2a}$
Therefore, N_2 *accepts* $w_2 = x_{i+2}x_{i+3}\cdots x_n$.
$w_2 \in L(N_2) = L_2$.
$w_1w_2 \in L_1L_2$.
$w = x_1x_2\cdots x_ix_{i+1}x_{i+2}\cdots x_n$
$= w_1x_{i+1}w_2$
$= w_1w_2 \qquad (x_{i+1} = \epsilon)$
Therefore, $w \in L_1L_2$.
Therefore, $L(N) \subset L_1L_2$.
Combining both directions, $L(N) = L_1L_2$.

Theorem 10.
For any regular language L, L^* *is regular.*
< Proof >
Let N_1 *be the NFA that recognizes* L.
By Theorem 1.28, we can start with an N_1 *defined as follows.*
$N_1 = (Q_1, \Sigma, T_1, q_1, \{q_a\})$ *where*
$q_1 \ne q_a, q_1 \notin T_1(q,x)\ \forall q \in Q_1, x \in \Sigma_\epsilon$ *and* $T_1(q_a, x) = \emptyset\ \forall x \in \Sigma_\epsilon$.
Let $N = (Q, \Sigma, T, q_0, \{q_a, q_0\})$ *such that* $Q = Q_1 \cup \{q_0\}$.

$$T(q,x) = \begin{cases} T_1(q,x) & \text{if} \quad q \in Q_1 \setminus \{q_a\} \\ \{q_1\} \cup T_1(q_a, \epsilon) & \text{if} \quad q = q_a \ \&\ x = \epsilon \\ T_1(q_a, x) & \text{if} \quad q = q_a \ \&\ x \ne \epsilon \\ \{q_1\} & \text{if} \quad q = q_0 \ \&\ x = \epsilon \\ \emptyset & \text{if} \quad q = q_0 \ \&\ x \ne \epsilon \end{cases}$$

We need to show $w \in L^* \Leftrightarrow N$ *accepts* w.

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If $w \in L^*$,
$w \in L^M$ *for some* $M \geq 0$.
If $M = 0, w \in L^0 = \{\epsilon\}$.
Therefore, $w = \epsilon$.
ϵ *is accepted by N because N has a start state that is also an accept state.*
For $M \geq 1$,
let $w = w_1 w_2 \cdots w_M$ *with each* $w_i \in L$ *for* $1 \leq i \leq M$.
Therefore, N_1 *accepts* w_i *for each i.*
For each $i, q_1 \xrightarrow{w_i, \hat{T}_1} q_a$ *(By Theorem 1.19 of NFA Acceptance)*
For each $i, q_1 \xrightarrow{w_i, \hat{T}} q_a$ *(Proposition 1.18)*
Since $T(q_a, \epsilon) = \{q_1\} \cup T_1(q_a, \epsilon)$,
$q_1 \in \{q_1\} \cup T_1(q_a, \epsilon) \Rightarrow q_1 \in T(q_a, \epsilon) \Rightarrow q_a \xrightarrow{\epsilon, T} q_1$.
Therefore,
$q_0 \xrightarrow{\epsilon, T} q_1 \xrightarrow{w_1, \hat{T}} q_a \xrightarrow{\epsilon, T} q_1 \xrightarrow{w_2, \hat{T}} q_a \xrightarrow{\epsilon, T} q_1 \cdots q_a \xrightarrow{\epsilon, T} q_1 \xrightarrow{w_M, \hat{T}} q_a$
Therefore, N accepts $\epsilon w_1 \epsilon w_2 \cdots \epsilon w_M = w_1 w_2 \cdots w_M = w$.
Therefore, $w \in L^* \Rightarrow$ *N accepts* w.
Conversely, if N accepts $w = x_1 x_2 x_3 \cdots x_n$ *where* $x_i \in \Sigma_\epsilon$ *for* $1 \leq i \leq n$ & $n \geq 1$.
(Note that $w = \epsilon$ *if* $x_i = \epsilon$ $\forall i$.*)*
$\exists r_0, r_1, r_2, \cdots r_n \in Q$ *such that*
$q_0 = r_0 \xrightarrow{x_1, T} r_1 \xrightarrow{x_2, T} r_2 \cdots r_{n-1} \xrightarrow{x_n, T} r_n$ & $r_n \in \{q_0, q_a\}$.
Since $T(q_0, x) = \emptyset$ *if* $x \neq \epsilon, x_1 = \epsilon$.
Furthermore, $T(q_0, \epsilon) = \{q_1\}$.
Therefore, $r_1 = q_1$.
$r_n \in \{q_0, q_a\} \Rightarrow r_n = q_a$ *because* q_0 *has no incoming arrows.*
The computation now becomes
$q_0 \xrightarrow{\epsilon, T} q_1 \xrightarrow{x_2, T} r_2 \cdots r_{n-1} \xrightarrow{x_n, T} q_a$.
Therefore, $q_1 \xrightarrow{x_2, T} r_2 \cdots r_{n-1} \xrightarrow{x_n, T} q_a$.
Claim 1:
For the computation, $q_0 = r_0 \xrightarrow{x_1, T} r_1 \xrightarrow{x_2, T} r_2 \cdots r_i \xrightarrow{x_{i+1}, T} r_{i+1} \cdots r_{n-1} \xrightarrow{x_n, T} r_n$ & $r_n = q_a$,
if $\exists r_i = q_a$ *for* $1 < i < n-1$, *then* $r_{i+1} = q_1$ & $x_{i+1} = \epsilon$.
<Proof of Claim 1>
$r_i \xrightarrow{x_{i+1}, T} r_{i+1} \Rightarrow q_a \xrightarrow{x_{i+1}, T} r_{i+1} \Rightarrow r_{i+1} \in T(q_a, x_{i+1})$.
$T(q_a, x_{i+1})$
$= T_1(q_a, x_{i+1})$ *or* $T_1(q_a, x_{i+1}) \cup \{q_1\}$ *(by definition of T)*
$= \emptyset$ *or* $\emptyset \cup \{q_1\}$ *(by definition of* N_1*)*
$= \emptyset$ *or* $\{q_1\}$
Therefore, $r_{i+1} \in \emptyset$ *or* $r_{i+1} \in \{q_1\}$.
Therefore, $r_{i+1} \in \{q_1\}$ *and hence* $r_{i+1} = q_1$.
Therefore, $r_i \xrightarrow{x_{i+1}, T} r_{i+1}$
$\Rightarrow q_a \xrightarrow{x_{i+1}, T} q_1$
$\Rightarrow q_1 \in T_1(q_a, x_{i+1})$ *if* $x_{i+1} \neq \epsilon$
$\Rightarrow q_1 \in \emptyset$ *if* $x_{i+1} \neq \epsilon$ *(by definition of* N_1*)*
$\Rightarrow$ *Contradiction if* $x_{i+1} \neq \epsilon$.
Therefore, $x_{i+1} = \epsilon$.
Claim 2:
For any computation $q_1 \xrightarrow{x_1, T} s_1 \xrightarrow{x_2, T} s_2 \cdots s_i \xrightarrow{x_{i+1}, T} s_{i+1} \cdots s_{n-1} \xrightarrow{x_n, T} s_n \xrightarrow{x_{n+1}, T} q_a$,
if $\exists$ *no* q_a *in between* q_1 *and* q_a, *that is* $s_i \neq q_a$ *for* $1 \leq i \leq n$, *then*
$q_1 \xrightarrow{w, \hat{T}_1} q_a$ *for some* $w \in \Sigma_\epsilon^*$.

<Proof of Claim 2>
$q_0 \notin \{s_1, s_2, \cdots s_n\}$ *because* q_0 *has no incoming arrows.*
Therefore, $s_1, s_2, \cdots s_n \in Q_1$.
Therefore, $s_1, s_2, \cdots s_n \in Q_1 \setminus \{q_a\}$.
By definition of $T, T(q, x) = T_1(q, x)$ *if* $q \in Q_1 \setminus \{q_a\}$.
Therefore, $T(s_i, x) = T_1(s_i, x)$ *for* $1 \le i \le n$ & $x \in \Sigma_\epsilon$.
The given computation can be replaced by
$q_1 \xrightarrow{x_1,T_1} s_1 \xrightarrow{x_2,T_1} s_2 \cdots s_i \xrightarrow{x_{i+1},T_1} s_{i+1} \cdots s_{n-1} \xrightarrow{x_n,T_1} s_n \xrightarrow{x_{n+1},T_1} q_a$,
$q_1 \xrightarrow{w,\hat{T}_1} q_a$ *where* $w = x_1x_2x_3 \cdots x_{n+1}$.
Back to computation $q_1 \xrightarrow{x_2,T} r_2 \xrightarrow{x_3,T} r_3 \cdots r_{n-1} \xrightarrow{x_n,T} q_a$.
Let m *be the number of* q_a*'s in between* q_1 & q_a.
If $m = 0$*, by Claim 2,* $q_1 \xrightarrow{w',\hat{T}_1} q_a$ *where* $w' = x_2x_3 \cdots x_n$.
N_1 *accepts* w'.
$w = x_1w' = \epsilon w' = w'$.
N_1 *accepts* w.
$w \in L \subset L^*$.
For $m \ge 1, \exists r_{j_1} = r_{j_2} = \cdots r_{j_m} = q_a$.
By Claim 1, $r_{j_1+1} = r_{j_2+1} = \cdots r_{j_m+1} = q_1$.
$q_1 \xrightarrow{w_1,\hat{T}_1} r_{j_1} = q_a$ *(Claim 2)*
$q_a = r_{j_1} \xrightarrow{\epsilon,T} r_{j_1+1} = q_1$ *(Claim 1)*
$q_1 = r_{j_1+1} \xrightarrow{w_2,\hat{T}_1} r_{j_2} = q_a$ *(Claim 2)*
$q_a = r_{j_2} \xrightarrow{\epsilon,T} r_{j_2+1} = q_1$ *(Claim 1)*
$\vdots$
$\vdots$
$q_1 = r_{j_{m-1}+1} \xrightarrow{w_m,\hat{T}_1} r_{j_m} = q_a$ *(Claim 2)*
$q_a = r_{j_m} \xrightarrow{\epsilon,T} q_1 \xrightarrow{w_{m+1},\hat{T}_1} q_a$ *(Claim 1 & Claim 2)*
Therefore, N_1 *accepts* $w_1, w_2, \cdots w_m, w_{m+1}$.
$w_1, w_2, \cdots w_m, w_{m+1} \in L$.
$w_1w_2 \cdots w_mw_{m+1} \in L^{m+1}$
However, $x_2x_3 \cdots x_n = w_1\epsilon w_2\epsilon \cdots \epsilon w_m\epsilon w_{m+1} = w_1w_2 \cdots w_mw_{m+1}$.
$w = x_1x_2x_3 \cdots x_n = \epsilon x_2x_3 \cdots x_n = x_2x_3 \cdots x_n = w_1w_2 \cdots w_mw_{m+1}$.
Therefore, $w \in L^{m+1} \subset L^*$.
Therefore, N *accepts* $w \Rightarrow w \in L^*$.
Combining both directions, $w \in L^* \Leftrightarrow N$ *accepts* w.
This completes the proof of Theorem 1.30.

Definition 10.
For any string $w = x_1x_2 \cdots x_n$*, where* $x_i \in \Sigma_\epsilon$ *for each* i*, the reverse of* w*, written* w^R *is the string* $x_nx_{n-1} \cdots x_1$.
For any language A, $A^R \overset{def}{=} \{w^R \mid w \in A\}$.

Theorem 11.
For any language A*,* A *is regular iff* A^R *is regular.*
< Proof >
Since A *is regular, there is an NFA,* N_A *that recognizes it.*
Let $N_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$.
Construct $N_{A^R} = (Q_A \cup \{q_s\}, \Sigma, \delta_{A^R}, q_s, \{q_A\})$ *where* $q_s \notin Q_A$ *such that*

$$\delta_{A^R}(q,x) = \begin{cases} F_A & \text{if} \quad (q,x) = (q_s,\epsilon) \\ \emptyset & \text{if} \quad q = q_s \;\&\; x \neq \epsilon \\ \{p \in Q_A \mid q \in \delta_A(p,x)\} & \text{if} \quad q \in Q_A \end{cases}$$

From the third row of this definition, it immediately follows that
$p \in \delta_{A^R}(q,x) \Leftrightarrow q \in \delta_A(p,x)$ *or*
$q \xrightarrow{x,\delta_{A^R}} p \Leftrightarrow p \xrightarrow{x,\delta_A} q \cdots\cdots (*)$.
Claim: $\exists$ *a computation path for* w *from* p *to* q *via transition function* δ_A *iff* $\exists$ *a computation path for* w^R *from* q *to* p *via transition function* δ_{A^R}.
That is, $p \xrightarrow{w,\hat{\delta_A}} q \Leftrightarrow q \xrightarrow{w^R,\hat{\delta_{A^R}}} p$.
This Claim can be proved by induction on $|w|$.
For $|w| = 1$, $w = w^R = x$ *where* $x \in \Sigma_\epsilon$.
From $(*)$, $q \xrightarrow{x,\delta_{A^R}} p \Leftrightarrow p \xrightarrow{x,\delta_A} q$
Therefore, the statement is true for $|w| = 1$.
Assume the statement is true for $|w| = k$ *where* $k \geq 1$.
That is , $p \xrightarrow{w,\hat{\delta_A}} q \Leftrightarrow q \xrightarrow{w^R,\hat{\delta_{A^R}}} p$ *for* $|w| = k$.
$p \xrightarrow{wx,\hat{\delta_A}} q$
$\Leftrightarrow p \xrightarrow{w,\hat{\delta_A}} q' \xrightarrow{x,\delta_A} q$ *(Proposition 1.12)*
$\Leftrightarrow p \xrightarrow{w,\hat{\delta_A}} q'$ and $q' \xrightarrow{x,\delta_A} q$
$\Leftrightarrow q' \xrightarrow{w^R,\hat{\delta_{A^R}}} p$ *and* $q \xrightarrow{x,\delta_{A^R}} q'$ *(Induction Hypothesis and (*))*
$\Leftrightarrow q \xrightarrow{x,\delta_{A^R}} q' \xrightarrow{w^R,\hat{\delta_{A^R}}} p$
$\Leftrightarrow q \xrightarrow{xw^R,\hat{\delta_{A^R}}} p$ *(Proposition 1.12)*
$\Leftrightarrow q \xrightarrow{(wx)^R,\hat{\delta_{A^R}}} p$ $(xw^R = (wx)^R)$
The statement is true for $|w| = k+1$ *and the proof of Claim is complete.*
To prove that A^R *is regular, we need to prove that*
$w^R \in A^R$ *iff* N_{A^R} *accepts* w^R.
If $w^R \in A^R$, $w \in A$.
Since N_A *accepts* w, $q_A \xrightarrow{w,\hat{\delta_A}} q$, $q \in F_A$ *(Theorem 1.19 – NFA acceptance)*
By Claim, $q \xrightarrow{w^R,\hat{\delta_{A^R}}} q_A$
Since $\delta_{A^R}(q_s,\epsilon) = F_A$, *and* $q \in F_A$,
$q_s \xrightarrow{\epsilon,\delta_{A^R}} q$.
Therefore, $q_s \xrightarrow{\epsilon,\delta_{A^R}} q \xrightarrow{w^R,\hat{\delta_{A^R}}} q_A$.
N_{A^R} *accepts* ϵw^R *(Theorem 1.19 – NFA acceptance)*
N_{A^R} *accepts* w^R.
Conversely, if N_{A^R} *accepts* w^R,
N_{A^R} *accepts* ϵw^R.
$q_s \xrightarrow{\epsilon w^R,\hat{\delta_{A^R}}} q_A$ *(Theorem 1.19 – NFA acceptance)*
$q_s \xrightarrow{\epsilon,\delta_{A^R}} q \xrightarrow{w^R,\hat{\delta_{A^R}}} q_A$ *(Proposition 1.12)*
Since $\delta_{A^R}(q_s,\epsilon) = F_A$, $q \in F_A$.
$q_A \xrightarrow{w,\hat{\delta_A}} q$, *and* $q \in F_A$ *(Claim)*
Therefore, N_A *accepts* w *(Theorem 1.19 – NFA acceptance)*
$w \in A$ *(N_A recognizes A)*
$w^R \in A^R$.
$w^R \in A^R$ *iff* N_{A^R} *accepts* w^R.
We have proved that A *is regular* $\Rightarrow$ A^R *is regular.*

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On the other hand, sine $(A^R)^R = A$,
A^R *is regular* $\Rightarrow (A^R)^R$ *is regular* $\Rightarrow A$ *is regular.*
Therefore, A *is regular iff* A^R *is regular.*

6. Regular Expression

Definition 11. *(Regular Expression)*
Let Σ *be a finite alphabet.*
$\Re_\Sigma$ *is a set with the following properties:*
(a) $R \in \Re_\Sigma$ *iff* R *is one of the following:*
(i) a *for some* $a \in \Sigma$
(ii) $\hat{\epsilon}$
(iii) $\hat{\emptyset}$
(iv) $R_1 \hat{\cup} R_2$ *for some* $R_1, R_2 \in \Re_\Sigma$
(v) $R_1 \hat{\bullet} R_2$ *for some* $R_1, R_2 \in \Re_\Sigma$
(vi) $R_1^{\hat{*}}$ *for some* $R_1 \in \Re_\Sigma$
where $\hat{\cup}, \hat{\bullet}$ *and* $\hat{*}$ *are operations in* $\Re_\Sigma$ *with*
$\hat{\cup} : \Re_\Sigma \times \Re_\Sigma \longrightarrow \Re_\Sigma$
$\hat{\bullet} : \Re_\Sigma \times \Re_\Sigma \longrightarrow \Re_\Sigma$
$\hat{*} : \Re_\Sigma \longrightarrow \Re_\Sigma$
(b) $\exists$ *an injective (one-to-one) mapping* $L : \Re_\Sigma \longrightarrow P(\Sigma^*)$ *s.t.*
(i) $L(a) = \{a\}\ \forall a \in \Sigma$
(ii) $L(\hat{\epsilon}) = \{\epsilon\}$
(iii) $L(\hat{\emptyset}) = \emptyset$
(iv) $L(R_1 \hat{\cup} R_2) = L(R_1) \cup L(R_2)\ \forall R_1, R_2 \in \Re_\Sigma$
(v) $L(R_1 \hat{\bullet} R_2) = L(R_1) \bullet L(R_2)\ \forall R_1, R_2 \in \Re_\Sigma$
(vi) $L(R_1^{\hat{*}}) = (L(R_1))^*\ \forall R_1 \in \Re_\Sigma$
$\Re_\Sigma$ *is called the set of all regular expressions over the alphabet* Σ.
Any member of $\Re_\Sigma$ *is called a regular expression over* Σ.
For any regular expression R, $L(R)$ *is called the language described by* R.
While $\hat{\cup}, \hat{\bullet}$ *and* $\hat{*}$ *are operations in* $\Re_\Sigma$, $\cup, \bullet$ *and* $*$ *are set operations in* $P(\Sigma^*)$.
When there is no danger of confusion, $\hat{\cup}, \hat{\bullet}$ *and* $\hat{*}$ *are usually written same as* $\cup, \bullet$ *and* $*$.
While $\hat{\epsilon}$ *and* $\hat{\emptyset}$ *are regular expressions,* ϵ *is the empty string and* $\emptyset$ *is the empty language. When there is no danger of confusion, they are all written as* ϵ *and* $\emptyset$.

Proposition 10.
Let Σ *be a finite alphabet and* $\Re_\Sigma$ *be the set of all regular expressions over* Σ.
The following statements are true.
(a) $\forall R_1, R_2 \in \Re_\Sigma, R_1 \cup R_2 = R_2 \cup R_1$
(b) $\exists$ *regular expressions* $\hat{\Sigma}$ *and* $\hat{\Sigma^*}$ *such that* $L(\hat{\Sigma}) = \Sigma$ *and* $L(\hat{\Sigma^*}) = \Sigma^*$. *When there is no danger of confusion,* $\hat{\Sigma}$ *and* $\hat{\Sigma^*}$ *are usually written same as* Σ *and* Σ^*.
$< Proof >$
(a) $L(R_1 \cup R_2) = L(R_1) \cup L(R_2$
$L(R_2 \cup R_1) = L(R_2) \cup L(R_1)$
$L(R_1) \cup L(R_2) = L(R_2) \cup L(R_1)$ *from set theory.*
Therefore, $L(R_1 \cup R_2) = L(R_2 \cup R_1)$
Therefore, $R_1 \cup R_2 = R_2 \cup R_1$ *(L is one-one)*
(b) Define $\hat{\Sigma} = \bigcup_{a \in \Sigma} a$
$\hat{\Sigma}$ *is a regular expression by Definition 1.33(a)(i) and 1.33(a)(iv).*
By Definition 1.33(b)(iv)

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$L(\hat{\Sigma}) = L\left(\bigcup_{a\in\Sigma} a\right) = \left(\bigcup_{a\in\Sigma} L(a)\right) = \left(\bigcup_{a\in\Sigma} \{a\}\right) = \Sigma$

Define $\hat{\Sigma^*} = (\hat{\Sigma})^{\hat{*}}$.

$\hat{\Sigma^*}$ *is a regular expression by Definition 1.33(a)(vi).*

By Definition 1.33(b)(vi),

$L(\hat{\Sigma^*}) = L((\hat{\Sigma})^{\hat{*}}) = (L(\hat{\Sigma}))^* = \Sigma^*$ $\qquad (L(\hat{\Sigma}) = \Sigma)$

Example 1.

Find the language described by $\Sigma^*1\Sigma^*$ *where* $\Sigma = \{0,1\}$.

$L(\Sigma^*1\Sigma^*) = L(\Sigma^*)L(1)L(\Sigma^*) = \Sigma^*\{1\}\Sigma^* = \{w \mid w \text{ has at least one } 1\}$.

Example 2.

Find the language described by $(\Sigma\Sigma\Sigma)^*$ *where* $\Sigma = \{0,1\}$.

$L((\Sigma\Sigma\Sigma)^*) = (L(\Sigma\Sigma\Sigma))^* = (L(\Sigma)L(\Sigma)L(\Sigma))^* = (\Sigma\Sigma\Sigma)^*$

$= \{xyz \mid x,y,z \in \Sigma\}^* = \{w \mid |w| \text{ is a multiple of three}\}$.

Lemma 2.

If a language is described by a regular expression, then it is regular. That is, if $A = L(R)$ *for some* $R \in \Re_\Sigma$, *then* $A = L(N)$ *for some finite automaton N.*

$< Proof >$

From the formal definition of regular expressions, R is one of the following:

(i) a *for some* $a \in \Sigma$

(ii) $\hat{\epsilon}$

(iii) $\hat{\emptyset}$

(iv) $R_1 \hat{\cup} R_2$ *for some* $R_1, R_2 \in \Re_\Sigma$

(v) $R_1 \hat{\bullet} R_2$ *for some* $R_1, R_2 \in \Re_\Sigma$

(vi) $R_1^{\hat{*}}$ *for some* $R_1 \in \Re_\Sigma$

In case (i), $L(a) = \{a\}$ *and* $\{a\}$ *can be recognized by the NFA defined as follows:*

$N = (\{q_1, q_2\}, \Sigma_\epsilon, \delta, q_1, \{q_2\})$ *such that* $\delta(q_1, a) = \{q_2\}$, $\delta(q, b) = \emptyset \ \forall q \neq q_1, b \neq a$.

In case (ii), $L(\epsilon) = \{\epsilon\}$ *and* $\{\epsilon\}$ *can be recognized by the following NFA:*

$N = (\{q_1\}, \Sigma_\epsilon, \delta, q_1, \{q_1\})$, *where* $\delta(q_1, b) = \emptyset \ \forall b \neq \epsilon$ *and* $\delta(q_1, \epsilon) = \{q_1\}$.

In case (iii), $L(\emptyset) = \emptyset$, *which is recognized by the following NFA:*

$N = (\{q\}, \Sigma_\epsilon, \delta, q, \emptyset)$ *where* $\delta(q, b) = \emptyset \ \forall b \in \Sigma_\epsilon$.

In cases (iv), (v) and (vi), R is repeated operations of $\hat{\cup}$, $\hat{\bullet}$ *and* $\hat{*}$ *on* a, ϵ *and* $\emptyset$. *Since we have shown above* $L(a)$, $L(\epsilon)$ *and* $L(\emptyset)$ *are regular and we have proved before that regular languages are closed under* $\cup$, $\bullet$ *and* $*$, $L(R)$ *is regular.*

Definition 12.

A generalized nondeterministic finite automaton (denoted by GNFA) has all the properties as described in Theorem 1.28 and is a 5-tuple, $(Q, \Sigma, \delta, q_{start}, \{q_{accept}\})$ *where*

(i) Q is a finite set of states;

(ii) Σ *is a finite alphabet;*

(iii) $\delta : (Q \setminus \{q_{accept}\}) \times (Q \setminus \{q_{start}\}) \longrightarrow \Re_\Sigma$ *is the transition function;*

(iv) q_{start} *is the start state; and*

(v) q_{accept} *is the accept state.*

A GNFA accepts a string $w \in \Sigma^*$, *if* $w = w_1 w_2 \cdots w_n$, *where each* w_i *is in* Σ^* *and a sequence of states* $q_0, q_1, q_2, \cdots q_n$ *exist such that*

(1) $q_0 = q_{start}$;

(2) $q_n = q_{accept}$; *and*

(3) For each i, $w_i \in L(R_i)$ *where*

$R_i = \delta(q_{i-1}, q_i)$ *and* $L(R_i)$ *is the language described by expression* R_i.

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If we write $q_i \xrightarrow{R,\delta} q_j$ *instead of* $\delta(q_i, q_j) = R$*, the definition of acceptance can be written as*
$q_{start} = q_0 \xrightarrow{R_1,\delta} q_1 \xrightarrow{R_2,\delta} q_2 \cdots \xrightarrow{R_n,\delta} q_n = q_{accept}$ *with* $w_i \in L(R_i)$ *for* $i = 1, 2, \cdots, n$.

Lemma 3.
Every NFA can be converted into an equivalent GNFA.
$< Proof >$
Because of Theorem 1.28, we can start with an NFA defined as follows.
$N = (Q, \Sigma, \delta, q_{start}, \{q_{accept}\})$ *where*
$q_{start} \neq q_{accept}$; $\delta(q_{accept}, a) = \emptyset\ \forall a \in \Sigma_\epsilon$; *and* $q_{start} \notin \delta(q, a)\ \forall a \in \Sigma_\epsilon, q \in Q$.
Define GNFA, N_G *as follows:*
$N_G = (Q, \Sigma, \delta_G, q_{start}, \{q_{accept}\})$ *where*
$\delta_G : (Q \setminus \{q_{accept}\}) \times (Q \setminus \{q_{start}\}) \longrightarrow \Re_\Sigma$ *such that:*
$\forall (q_i, q_j) \in (Q \setminus \{q_{accept}\}) \times (Q \setminus \{q_{start}\})$
$\delta_G(q_i, q_j) = R_{i,j}$ *where*
$R_{i,j} = \bigcup_{w \in S_{i,j}} w$ *; and*
$S_{i,j} = \{w \in \Sigma^* \mid q_i \xrightarrow{w,\hat{\delta}} q_j\}$.
Note that if $i = j$, $\delta_G(q_i, q_i) = R_{i,i}$, $S_{i,i} = \{w \in \Sigma^* \mid q_i \xrightarrow{w,\hat{\delta}} q_i\}$; *and*
$R_{i,i} = \bigcup_{w \in S_{i,i}} w^*$
$\forall (q_i, q_j), S_{i,j}$ *is unique and therefore* $R_{i,j}$ *is unique.*
Since w *is the concatenation of symbols from* Σ*, and every symbol in* Σ *is a regular expression,* w *is a regular expression.*
Therefore, $R_{i,j} = \bigcup_{w \in S_{i,j}} w$ *is a regular expression.*
Therefore, $\delta_G(q_i, q_j) = R_{i,j}$ *is well defined.*
Claim 1. For any string w *in* Σ^*, $L(w) = \{w\}$.
$< Proof\ of\ Claim\ 1 >$
$$\begin{aligned} L(w) &= L(a_1 a_2 \cdots a_n) \text{ where } a_i \in \Sigma \\ &= L(a_1)L(a_2)\cdots L(a_n) \\ &= \{a_1\}\{a_2\}\cdots\{a_n\} \\ &= \{a_1 a_2 \cdots a_n\} \\ &= \{w\} \end{aligned}$$
Claim 2. $\forall w \in \Sigma^*$, *N accepts* $w \Leftrightarrow N_G$ *accepts* w.
$< Proof\ of\ Claim\ 2 >$
For forward direction " $\Rightarrow$ *"*
Let N accepts w *where* $w = w_1 w_2 \cdots w_n$, $n \geq 1$, *and each* w_i *is in* Σ^* *for* $1 \leq i \leq n$.
By theorem of acceptance, $\exists q_0, q_1, q_2, \cdots q_n \in Q$ *such that*
$q_{start} = q_0 \xrightarrow{w_1,\hat{\delta}} q_1 \xrightarrow{w_2,\hat{\delta}} q_2 \cdots q_{n-1} \xrightarrow{w_n,\hat{\delta}} q_n = q_{accept}$.
Since $q_{i-1} \xrightarrow{w_i,\hat{\delta}} q_i, w_i \in S_{i-1,i}$.
By definition of δ_G,
$\delta_G(q_{i-1}, q_i) = R_{i-1,i} = \bigcup_{w \in S_{i-1,i}} w$
$$\begin{aligned} &L(\delta_G(q_{i-1}, q_i)) \\ &= L(R_{i-1,i}) \\ &= L\left(\bigcup_{w \in S_{i-1,i}} w\right) \\ &= \bigcup_{w \in S_{i-1,i}} L(w) \\ &= \bigcup_{w \in S_{i-1,i}} \{w\} \qquad \text{(By Claim 1)} \end{aligned}$$

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$= S_{i-1,i}$.
Since $w_i \in S_{i-1,i}$, $w_i \in L(R_{i-1,i})$.
Since $q_{i-1} \xrightarrow{R_{i-1,i},\delta_G} q_i\ \forall i = 1, 2, \cdots n$,
$q_{start} = q_0 \xrightarrow{R_{0,1},\delta_G} q_1 \xrightarrow{R_{1,2},\delta_G} q_2 \cdots q_{i-1} \xrightarrow{R_{i-1,i},\delta_G} q_i \cdots q_{n-1} \xrightarrow{R_{n-1,n},\delta_G} q_n = q_{accept}$.
N_G *accepts* w.
Conversely, if N_G *accepts* w *for* $w = w_1 w_2 \cdots w_n$, $n \geq 1$, *and each* w_i *is in* Σ^*,
$\exists q_0, q_1, q_2, \cdots q_n \in Q$ *such that*
$q_{start} = q_0 \xrightarrow{R_{0,1},\delta_G} q_1 \xrightarrow{R_{1,2},\delta_G} q_2 \cdots q_{i-1} \xrightarrow{R_{i-1,i},\delta_G} q_i \cdots q_{n-1} \xrightarrow{R_{n-1,n},\delta_G} q_n = q_{accept}$
with $w_i \in L(R_{i-1,i})\ \forall i \in \{1, 2, 3, \cdots n\}$,
$R_{i-1,i} = \bigcup_{w \in S_{i-1,i}} w$
and
$S_{i-1,i} = \{w \in \Sigma^* \mid q_{i-1} \xrightarrow{w,\hat{\delta}} q_i\}$
$L(R_{i-1,i})$
$= L\left(\bigcup_{w \in S_{i-1,i}} w\right)$
$= \bigcup_{w \in S_{i-1,i}} L(w)$
$= \bigcup_{w \in S_{i-1,i}} \{w\}$ *(By Claim 1)*
$= S_{i-1,i}$.
$\forall i \in \{1, 2, 3, \cdots n\}$,
$w_i \in L(R_{i-1,i})$
$\Rightarrow w_i \in S_{i-1,i}$
$\Rightarrow q_{i-1} \xrightarrow{w_i,\hat{\delta}} q_i$ *(Definition of* $S_{i,j}$*)*
Therefore, $q_{start} = q_0 \xrightarrow{w_1,\hat{\delta}} q_1 \xrightarrow{w_2,\hat{\delta}} q_2 \cdots q_{n-1} \xrightarrow{w_n,\hat{\delta}} q_n = q_{accept}$.
Therefore, N *accepts* $w = w_1 w_2 \cdots w_n$.
N *and* N_G *are equivalent and the Lemma is proved.*

Lemma 4.
Every GNFA of n *states* ($n \geq 2$) *can be reduced to an equivalent GNFA of* 2 *states.*
$< Proof >$
This lemma can be proved by induction on n.
It is trivial that the statement is true for $n = 2$.
Assume that the statement is true for $n = k \geq 2$.
Let $G = (Q, \Sigma, \delta, q_{start}, \{q_{accept}\})$ *be a GNFA with* $k + 1$ *states.*
$\exists q_{rip} \in Q \setminus \{q_{start}, q_{accept}\}$ *because* $k + 1 \geq 3$.
Construct $G' = (Q', \Sigma, \delta', q_{start}, \{q_{accept}\})$ *such that*
$Q' = Q \setminus \{q_{rip}\}$
$\forall (q_i, q_j) \in (Q \setminus \{q_{accept}\}) \times (Q \setminus \{q_{start}\})$,
$\delta'(q_i, q_j) = \delta(q_i, q_{rip})(\delta(q_{rip}, q_{rip}))^* \delta(q_{rip}, q_j) \cup \delta(q_i, q_j)$.
Therefore, Q' *is a GNFA with* k *states.*
Let G *accept* $w = w_1 w_2 \cdots w_n$ *where each* $w_i \in \Sigma^*$.
$\exists q_0, q_1, q_2, \cdots q_n \in Q$ *such that*
$q_{start} = q_0 \xrightarrow{R_1,\delta} q_1 \xrightarrow{R_2,\delta} q_2 \cdots q_{i-1} \xrightarrow{R_i,\delta} q_i \cdots q_{n-1} \xrightarrow{R_n,\delta} q_n = q_{accept}$; *and*
$w_i \in L(R_i) = L(\delta(q_{i-1}, q_i))$.
If none of $q_0, q_1, q_2, \cdots q_n$ *is* q_{rip}, *then they are all in* Q'.
Also,
$w_i \in L(\delta(q_{i-1}, q_i))$
$\Rightarrow w_i \in L\big(\delta(q_{i-1}, q_{rip})(\delta(q_{rip}, q_{rip}))^* \delta(q_{rip}, q_i)\big) \cup L(\delta(q_{i-1}, q_i))$

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$\Rightarrow w_i \in L\big(\delta(q_{i-1}, q_{rip})(\delta(q_{rip}, q_{rip}))^* \delta(q_{rip}, q_i) \cup \delta(q_{i-1}, q_i)\big)$
$\Rightarrow w_i \in L(\delta'(q_{i-1}, q_i))$
$\Rightarrow w_i \in L(R_i')$ *where* $R_i' = \delta'(q_{i-1}, q_i)$
$q_{start} = q_0 \xrightarrow{R_1',\delta'} q_1 \xrightarrow{R_2',\delta'} q_2 \cdots q_{i-1} \xrightarrow{R_i',\delta'} q_i \cdots q_{n-1} \xrightarrow{R_n',\delta'} q_n = q_{accept}$
with $w_i \in L(R_i')$.
Therefore, G' *accepts* $w = w_1 w_2 \cdots w_n$.
If $\exists$ *some* q*'s in the sequence* $q_0, q_1, q_2, \cdots q_n$ *which are* q_{rip},
let q_i *be the first such* q_{rip} *and* q_j *be the first state in the sequence after* q_i *such that* $q_j \neq q_{rip}$.
$q_{i-1} \xrightarrow{R_i} q_i = q_{rip} \xrightarrow{R_{i+1}} q_{rip} \cdots q_{rip} \xrightarrow{R_{j-1}} q_{rip} \xrightarrow{R_j} q_j$.
$R_{i+1} = \delta(q_i, q_{i+1}) = \delta(q_{rip}, q_{rip}) \Rightarrow w_{i+1} \in L(\delta(q_{rip}, q_{rip}))$
$\vdots$
$R_{j-1} = \delta(q_{j-2}, q_{j-1}) = \delta(q_{rip}, q_{rip}) \Rightarrow w_{j-1} \in L(\delta(q_{rip}, q_{rip}))$
$w_{i+1} \cdots w_{j-1} \in L^{j-i-1}(\delta(q_{rip}, q_{rip}))$
$w_{i+1} \cdots w_{j-1} \in L^*(\delta(q_{rip}, q_{rip}))$
Let $w_j' = w_i w_{i+1} \cdots w_{j-1} w_j$
$w_i \in L(\delta(q_{i-1}, q_i))$ *and* $q_i = q_{rip} \Rightarrow w_i \in L(\delta(q_{i-1}, q_{rip}))$
$w_j \in L(\delta(q_{j-1}, q_j))$ *and* $q_{j-1} = q_{rip} \Rightarrow w_j \in L(\delta(q_{rip}, q_j))$
$w_j' \in L(\delta(q_{i-1}, q_{rip}))L^*(\delta(q_{rip}, q_{rip}))L(\delta(q_{rip}, q_j))$
$w_j' \in L(\delta(q_{i-1}, q_{rip}))L^*(\delta(q_{rip}, q_{rip}))L(\delta(q_{rip}, q_j)) \cup L(\delta(q_{i-1}, q_j))$
Therefore, $w_j' \in L\big(\delta(q_{i-1}, q_{rip})(\delta(q_{rip}, q_{rip}))^* \delta(q_{rip}, q_j) \cup \delta(q_{i-1}, q_j)\big)$
$w_j' \in L(\delta'(q_{i-1}, q_j))$
$w_j' \in L(R_j')$ *where* $R_j' = \delta'(q_{i-1}, q_j)$
If there are no more q_{rip}*'s in the sequence,*
$q_{start} = q_0 \xrightarrow{R_1',\delta'} q_1 \xrightarrow{R_2',\delta'} q_2 \cdots q_{i-1} \xrightarrow{R_j',\delta'} q_j \xrightarrow{R_{j+1}',\delta'} q_{j+1} \cdots q_{n-1} \xrightarrow{R_n',\delta'} q_n = q_{accept}$
is the path of acceptance in G' *for* $(w_1 w_2 \cdots w_{i-1})(w_j')(w_{j+1} \cdots w_n)$,
which is the same as $(w_1 w_2 \cdots w_{i-1})(w_i w_{i+1} \cdots w_{j-1} w_j)(w_{j+1} \cdots w_n)$ *because*
$w_j' = w_i w_{i+1} \cdots w_{j-1} w_j$.
Therefore, G' *accepts* $w = w_1 w_2 \cdots w_n$.
If there are some more q_{rip}*'s in the sequence, repeat the above process until all* q_{rip}*'s are removed and the resulting computation path is the path of acceptance of* w *in* G'.
Conversely, if G' *accepts* $w = w_1 w_2 \cdots w_n$ *where* $w_i \in \Sigma^*$,
$\exists q_0, q_1, q_2, \cdots q_n \in Q'$ *such that*
$q_{start} = q_0 \xrightarrow{R_1',\delta'} q_1 \xrightarrow{R_2',\delta'} q_2 \cdots q_{i-1} \xrightarrow{R_i',\delta'} q_i \cdots q_{n-1} \xrightarrow{R_n',\delta'} q_n = q_{accept}$
with $w_i \in L(R_i')$ *where* $R_i' = \delta'(q_{i-1}, q_i)$.
Therefore, $w_i \in L(\delta(q_{i-1}, q_{rip})(\delta(q_{rip}, q_{rip}))^* \delta(q_{rip}, q_i) \cup \delta(q_{i-1}, q_i))$
Therefore, $w_i \in L(\delta(q_{i-1}, q_{rip})(\delta(q_{rip}, q_{rip}))^* \delta(q_{rip}, q_i))$ *or* $w_i \in L(\delta(q_{i-1}, q_i))$.
If $w_i \in L(\delta(q_{i-1}, q_i))$,
$q_{start} = q_0 \xrightarrow{R_1,\delta} q_1 \xrightarrow{R_2,\delta} q_2 \cdots q_{i-1} \xrightarrow{R_i,\delta} q_i \cdots q_{n-1} \xrightarrow{R_n,\delta} q_n = q_{accept}$ *where* $w_i \in L(R_i)$
is the acceptance path for $w = w_1 w_2 \cdots w_n$ *in* G.
If $w_i \in L(\delta(q_{i-1}, q_{rip})(\delta(q_{rip}, q_{rip}))^* \delta(q_{rip}, q_i))$,
let $w_i = w_{i1} w_{i2} w_{i3}$ *where*
$w_{i1} \in L(\delta(q_{i-1}, q_{rip})) = L(R_{i-1,rip})$,
$w_{i2} \in L^*(\delta(q_{rip}, q_{rip})) = L^*(R_{rip,rip})$, *and*
$w_{i3} \in L(\delta(q_{rip}, q_i)) = L(R_{rip,i})$.
$\exists m \geq 0$ *such that* $w_{i2} \in L^m(\delta(q_{rip}, q_{rip}))$.
$w_{i2} = w_{i2}(1) w_{i2}(2) \cdots w_{i2}(m)$ *where each* $w_{i2}(j) \in L(\delta(q_{rip}, q_{rip})) = L(R_{rip,rip})$.
$q_{i-1} \xrightarrow{R_{i-1,rip},\delta} q_{rip} \xrightarrow{R_{rip,rip},\delta} q_{rip} \cdots q_{rip} \xrightarrow{R_{rip,i},\delta} q_i$

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is a computation path in G for $w_{i1}w_{i2}w_{i3} = w_i$.
This is true for all $1 \leq i \leq n$.
Therefore, there is a computation path in G from q_0 *to* q_n *for* $w_1w_2 \cdots w_n = w$.
Therefore, G accepts $w = w_1w_2 \cdots w_n$.
So G and G' *are equivalent.*
Since G' *has k states, by induction hypothesis,* G' *can be reduced to an equivalent GNFA of* 2 *states.*
Hence, G can be reduced to an equivalent GNFA of 2 *states.*
This completes the proof.

Lemma 5.
If an NFA, $N = (Q, \Sigma, \delta, q_0, F)$ *is equivalent to a 2-state GNFA,* $N_G = (Q, \Sigma, \delta_G, q_{start}, \{q_{accept}\})$, *then* $L(N) = L(R)$ *where* $R = \delta_G(q_{start}, q_{accept})$.
$< Proof >$
$w \in L(N)$
$\Leftrightarrow$ *N accepts w*
$\Leftrightarrow$ N_G *accepts w* (*N and* N_G *are equivalent.*)
$\Leftrightarrow w \in L(R)$ $(R = \delta_G(q_{start}, q_{accept}))$

By Lemmas 1.39, 1.40, 1.41, we have the following conclusion:

Lemma 6.
If a language is regular, it is described by a regular expression.

By Lemma 1.37 and Lemma 1.42, we have the following theorem.

Theorem 12.
A language is regular iff some regular expression describes it.

7. Pumping Lemma

Theorem 13. - *Pumping Lemma*
Let A be a language.
Let (S) *denote the following statement:*
$\exists$ *a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into three pieces,* $s = xyz$, *satisfying the following conditions:*
(1) For each $i \geq 0$, $xy^iz \in A$,
(2) $|y| > 0$, *and*
(3) $|xy| \leq p$.
The Pumping Lemma states that A is regular $\Rightarrow (S)$.
$< Proof >$
Since A is regular, there exists a finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ *that recognizes A.*
That is, $A = L(M)$.
Let p be the number of states in M.
Let $s = s_1s_2 \cdots s_n$ *where each* $s_i \in \Sigma$ *and* $0 \leq p \leq n$.
$\exists r_0, r_1, \cdots r_n \in Q$, *such that*
$q_0 = r_0 \xrightarrow{s_1,\delta} r_1 \xrightarrow{s_2,\delta} r_2 \cdots r_{n-1} \xrightarrow{s_n,\delta} r_n, r_n \in F$.
Since $p \leq n, q_0 = r_0 \xrightarrow{s_1,\delta} r_1 \xrightarrow{s_2,\delta} r_2 \cdots r_{p-1} \xrightarrow{s_p,\delta} r_p$ *is a sub path with* $p+1$ *states.*
Since M has only p states, by the pigeonhole principle, $\exists k, l$ *such that* $0 \leq k < l \leq p$ *and* $r_k = r_l$.
Let $x = s_1s_2 \cdots s_k, y = s_{k+1}s_{k+2} \cdots s_l$ *and* $z = s_{l+1}s_{l+2} \cdots s_n$.
Therefore, $r_0 \xrightarrow{x,\hat{\delta}} r_k \xrightarrow{y,\hat{\delta}} r_l \xrightarrow{z,\hat{\delta}} r_n$.
Since $r_k = r_l$, $r_k \xrightarrow{y^i,\hat{\delta}} r_l \ \forall i \geq 0$.

Therefore, $r_0 \xrightarrow{x,\hat{\delta}} r_k \xrightarrow{y^i,\hat{\delta}} r_l \xrightarrow{z,\hat{\delta}} r_n$ *with* $r_n \in F$.
Therefore, M accepts xy^iz.
Therefore, $xy^iz \in A$.
Since $k < l$, $|y| > 0$.
$|xy| = |x| + |y| = k + l - k = l \leq p$.
This completes the proof of the Pumping Lemma.

Theorem 14. *- Pumping Lemma (contra positive form)*
$\neg(S) \Rightarrow$ *A is not regular where*
$\neg(S)$ *is equivalent to:*
$\forall p \geq 1, \exists s \in A$ *with* $|s| \geq p$ *such that whenever* $s = xyz$*, at least one of the conditions* $(1), (2)$*, or* (3) *cannot be satisfied.*
The contra positive form of the Pumping Lemma is used to prove a language is not regular. The general strategy is to find an $s \in A$ *with* $|s| \geq p$ *for any given* $p \geq 1$ *so that whenever s is broken into* $s = xyz$*, at least one of the conditions of* $(1), (2)$*, or* (3) *must be false. This can be usually accomplished by showing one of the following:*
(i) Condition 1 alone is false.
(ii) Condition 3 $\Rightarrow \neg$*(Condition 1)*
(iii) (Condition 2 and Condition 3) $\Rightarrow \neg$*(Condition 1).*

Example 3.
Show that $A = \{0^n1^n \mid n \geq 0\}$ *is not regular.*
The strategy is to create an s that will force y to contain all $0's$ *or all* $1's$ *so that when y is pumped indefinitely,* xy^iz *will contain too many* $0's$ *or* $1's$ *to make it impossible for* xy^iz *to remain in A.*
Since Condition 3 requires $|xy| \leq p$*, a prefix of* 0^p *in s will achieve that purpose.*
Formally, we make the argument as follows.
$\forall p \geq 1$*, let* $s = 0^p1^p$.
$s \in A$ *and* $|s| \geq p$.
If $s = xyz$*, then* $xyz = 0^p1^p$.
Condition 3
$\Rightarrow |xy| \leq p$
$\Rightarrow$ *xy consists of only* $0's$
$\Rightarrow$ *y consists of only* $0's$.
$|xyyz| = |xyz| + |y|$.
Since Condition 2 requires $|y| > 0$*, xyyz adds a positive number of* $0's$ *to xyz.*
Since xyz has equal numbers of $0's$ *and* $1's$*, xyyz must have more* $0's$ *than* $1's$ *and hence is not in A.*
Therefore, (Condition 2 + Condition 3) $\Rightarrow \neg$*(Condition 1) and hence A is not regular.*

Example 4.
Show that $A = \{ww \mid w \in \{0,1\}^*\}$ *is not regular.*
The strategy is to create an s with some leading $0's$ *on the left, say* 0^m *but we also want to make sure that* 0^m *is long enough to force xy to contain all* $0's$ *in it so that when y is pumped up indefinitely, it will create too many* $0's$ *to make it impossible for* $s = ww$.
Since Condition 3 requires $|xy| \leq p$*, we want to make* $m \geq p$.
A natural candidate for s is therefore 0^p10^p1.
To prove that this construction works, however, requires some algebraic manipulation.
Formally, we make the argument as follows.
$\forall p \geq 1$*, take* $s = 0^p10^p1$.
If $s = xyz$*, then* $xyz = 0^p10^p1$.
Condition 3
$\Rightarrow |xy| \leq p$
$\Rightarrow$ *xy consists of only* $0's$

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$\Rightarrow y$ *consists of only* $0's$.
Let $xy^iz = 0^{p'}10^p1$ *where* $p' - p = (i-1)|y|$ *or* $p' = p + (i-1)|y|$.
For $i > 3$, $p' > p + (3-1)$ $\quad (|y| \geq 1$ *by Condition 2)*
Therefore, $p' > p + 2$ *for* $i > 3$.
Assume for contradiction that $\forall i \geq 0$, $xy^iz \in A$.
That is, $xy^iz = 0^{p'}10^p1 = ww$.
For all $i > 3$,
$|w|$
$= \frac{|0^{p'}10^p1|}{2}$
$= \frac{p'+p+2}{2}$
$> \frac{p+2+p+2}{2} \quad (p' > p+2$ *for* $i > 3)$
$= p + 2$
Therefore, $|w| > |10^p1|$.
This implies w *consists of at least two* $1's$.
On the other hand, $p' + p + 2 = 2|w|$.
$p' - |w| = |w| - (p+2) > 0$
$p' > |w|$
This implies w *must consist of all* $0's$.
This leads to a contradiction.
Therefore, (Condition 2 + Condition 3) $\Rightarrow \neg$*(Condition 1) and hence* A *is not regular.*

Example 5.
Show that $A = \{1^{n^2} \mid n \geq 0\}$ *is not regular.*
The idea behind this problem is every time we pump up y*, we increase the length of* s *by an amount of* $|y|$ *which is bounded by* p *and* p *is fixed. On the other hand,* s *has to be the square of a natural number and the difference between two consecutive squares, say* n^2 *and* $(n+1)^2$ *will grow to infinity as* n *goes to infinity. In this case, we don't have to worry about how to create more* $0's$ *in* s *so as to outnumber the* $1's$ *or vice versa. This particular nature of* s *will automatically lead to a contradiction to Condition 1 as* $|s|$ *grows to infinity.*
Proving this to work requires some algebraic manipulation.
The formal argument is made as follows.
$\forall p \geq 1$, *take* $s = 1^{p^2}$
$p \geq 1$
$\Rightarrow p(p-1) \geq 0$
$\Rightarrow p^2 \geq p$
$\Rightarrow |1^{p^2}| \geq |1^p| = p$
Therefore, $|s| \geq p$.
Assume for contradiction that Condition 1 is true.
That is, $\forall i \geq 0$, $xy^iz \in A$.
Both xy^iz *and* $xy^{i+1}z$ *are in* A.
Let $xy^iz = 1^{n^2}$ *and* $xy^{i+1}z = 1^{m^2}$ *where* m *and* n *are positive integers.*
$|xy^iz| = n^2$ *and* $|xy^{i+1}z| = m^2$.
By Condition 2,
$|y| \geq 1$
$\Rightarrow |y^{i+1}| > |y^i|$
$\Rightarrow |xy^{i+1}z| > |xy^iz|$
$\Rightarrow m^2 > n^2$
$\Rightarrow m > n$
$\Rightarrow m \geq n + 1$
By Condition 3, $|xy| \leq p \Rightarrow |y| \leq p$.
Therefore, $|xy^{i+1}z| - |xy^iz| = |y| \leq p$.

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Therefore, $m^2 - n^2 \leq p.$
$(n+1)^2 - n^2 \leq m^2 - n^2 \leq p.$
$2n + 1 \leq p.$
$n \leq \frac{p-1}{2} \cdots\cdots (1)$ *where* (1) *is true for all* $i.$
On the other hand,
Condition 2 $\Rightarrow |y| \geq 1 \Rightarrow |y^i| \geq i.$
$n^2 = |x| + |y^i| + |z| \geq |y^i| \geq i.$
$n \geq \sqrt{i}$ *for all* $i.$
For $i > \frac{(p-1)^2}{4}$, $\sqrt{i} > \frac{p-1}{2}$ *and* $n > \frac{p-1}{2}$
This contradicts (1) *which is true for all* $i.$
Therefore, (Condition 2 + Condition 3) $\Rightarrow \neg$*(Condition 1) and hence* A *is not regular.*

8. Myhill-Nerode Theorem

Definition 13.
$\forall x, y \in \Sigma^*, L \subset \Sigma^*,$
we say that x *and* y *are indistinguishable by L iff* $\forall z \in \Sigma^*, xz \in L \Leftrightarrow yz \in L.$
We say that x *and* y *are distinguishable by L iff there exists* $z \in \Sigma^*$ *such that exactly one of* xz *and* yz *is in* $L.$
If x *and* y *are indistinguishable by L, we write* $x \equiv_L y.$

Proposition 11.
$\equiv_L$ *is an equivalence relation.*
$< Proof >$
$\forall x \in L, xz \in L \Leftrightarrow xz \in L \ \forall z \in \Sigma^*$
$x \equiv_L x$
$\equiv_L$ *is reflexive.*
$\forall x, y \in L,$
$x \equiv_L y$
$\Rightarrow (\forall z \in \Sigma^*, xz \in L \Leftrightarrow yz \in L)$
$\Rightarrow (\forall z \in \Sigma^*, yz \in L \Leftrightarrow xz \in L)$
$\Rightarrow y \equiv_L x$
$\equiv_L$ *is symmetric.*
$\forall x, y, w \in \Sigma^*,$
$(x \equiv_L y) \wedge (y \equiv_L w)$
$\Rightarrow (\forall z \in \Sigma^*, xz \in L \Leftrightarrow yz \in L) \wedge (\forall z \in \Sigma^*, yz \in L \Leftrightarrow wz \in L)$
$\Rightarrow (\forall z \in \Sigma^*, xz \in L \Leftrightarrow wz \in L)$
$\Rightarrow x \equiv_L w$
$\equiv_L$ *is transitive.*

Proposition 12.
$\equiv_L$ *is right congruence. That is* $x \equiv_L y \Rightarrow xa \equiv_L ya \ \forall a \in \Sigma.$
$< Proof >$
$\forall z \in \Sigma^*, a \in \Sigma,$
$xaz \in L \Leftrightarrow yaz \in L$ $\qquad (x \equiv_L y)$
$xa \equiv_L ya$ $\qquad$ *(Definition of* $\equiv_L$*)*

Proposition 13.
$\forall x, y \in \Sigma^*, (x \equiv_L y) \Rightarrow (x \in L \Leftrightarrow y \in L)$
$< Proof >$
Take $z = \epsilon.$
$x\epsilon \in L \Leftrightarrow y\epsilon \in L$
Therefore, $x \in L \Leftrightarrow y \in L.$

Theorem 15. - *Myhill-Nerode Theorem*
Let $L \subset \Sigma^*, X \subset \Sigma^*$.
X *is said to be pairwise distinguishable by* L *iff every two distinct strings in* X *are distinguishable by* L.
The index of L *is defined as*
$Index\, L = max\{|X| \mid X\, is\, pairwise\, distinguishable\, by\, L\}$.
The following statements are true:
(a) *If* L *is recognized by a DFA with* k *states,* L *has an index at most* k.
(b) *If the index of* L *is a finite number* k*, it is recognized by a DFA with* k *states.*
(c) L *is regular iff it has finite index. Moreover, its index is the size of the smallest DFA recognizing it.*
$< Proof >$
(a) Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA with k states that recognizes L.
Assume for contradiction that L has an index greater than k.
$\exists X$ (pairwise distinguishable by L) that has more than k members.
Let $s_1, s_2, s_3 \cdots s_{k+1}$ be $k+1$ distinct and pairwise distinguishable members in X.
$\hat{\delta}(q_0, s_1), \hat{\delta}(q_0, s_2), \hat{\delta}(q_0, s_3), \cdots \hat{\delta}(q_0, s_{k+1})$ are $k+1$ states in Q.
Since $|Q| = k$, by the pigeonhole principle, there are i, j where $1 \le i < j \le k+1$ s.t.
$\hat{\delta}(q_0, s_i) = \hat{\delta}(q_0, s_j)$.
$\forall z \in \Sigma^*$,
$s_i z \in L$
$\Leftrightarrow \hat{\delta}(q_0, s_i z) \in F$ (M recognizes L)
$\Leftrightarrow \hat{\delta}(\hat{\delta}(q_0, s_i), z) \in F$ (Proposition 1.14)
$\Leftrightarrow \hat{\delta}(\hat{\delta}(q_0, s_j), z) \in F$
$\Leftrightarrow \hat{\delta}(q_0, s_j z) \in F$ (Proposition 1.14)
$\Leftrightarrow s_j z \in L$ (M recognizes L)
Therefore, $s_i \equiv_L s_j$ (Definition of $\equiv_L$)
This contradicts the assumption that X is pairwise distinguishable by L.
(b) Let $X = \{s_1, s_2 \cdots, s_k\}$ be pairwise distinguishable by L.
Claim 1. $Index\ L \ge 2 \Rightarrow L \ne \emptyset$ and hence $L = \emptyset \Rightarrow Index\ L = 1$.
<Proof of Claim 1>
$Index\ L \ge 2$
$\Rightarrow \exists X$ (pairwise distinguishable by L) that has at least 2 members.
$\Rightarrow \exists s_i, s_j \in X$ where $s_i \ne s_j$ and s_i, s_j are distinguishable by L.
$\Rightarrow \exists z \in \Sigma^*$ s.t. $s_i z \in L$ and $s_j z \notin L$ or vice versa.
$\Rightarrow L \ne \emptyset$.
Since $L = \emptyset \Rightarrow Index\ L < 2$ or $Index\ L = 1$, $Index\ L$ is defined to be 1 whenever $L = \emptyset$.
Claim 2. $\forall w \in \Sigma^*$, there is one and only one $s_w \in X$ s.t. $w \equiv_L s_w$. Hence by taking $w = \epsilon$, there is one and only one $s_\epsilon \in X$ s.t. $\epsilon \equiv_L s_\epsilon$.
<Proof of Claim 2>
Either $w \in X$ or $w \notin X$.
If $w \in X, \exists s_i \in X$ s.t. $w = s_i$.
Call this s_w so that $w = s_i = s_w$.
Since $\equiv_L$ is reflexive, it follows that $w \equiv_L s_w$.
If $w \notin X$, w must be indistinguishable with a member of X otherwise it will contradict the assumption that $Index\ L = k$.
Therefore, $w \equiv_L s_w$ for some $s_w \in X$.
Either case, $w \equiv_L s_w$ for some $s_w \in X$.
If there is another $s_w' \in X$ s.t. $w \equiv_L s_w'$, then $s_w \equiv_L s_w'$ because $\equiv_L$ is transitive.
This contradicts the assumption that X is pairwise distinguishable by L.

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Therefore, s_w *is unique.*
Claim 3. If $L \neq \emptyset$ *then* $L \cap X \neq \emptyset$
<Proof of Claim 3>
$L \neq \emptyset \Rightarrow \exists w \in L$
By Claim 2, there is one and only one $s_w \in X$ *s.t.* $w \equiv_L s_w$.
By Proposition 1.52, $w \in L \Leftrightarrow s_w \in L$.
Therefore, $s_w \in L \cap X$.
Therefore, $L \cap X \neq \emptyset$.
This completes proof of Claim 3.
If Index $L = k = 1$, $L = \emptyset$ *which is recognized by the one-state DFA,* $M = (\{q\}, \Sigma, \delta, q, \emptyset)$
where $\delta(q, b) = \emptyset \ \forall b \in \Sigma$.
If Index $L = k \geq 2$,
$\exists X = \{s_1, s_2, s_3 \cdots s_k\}$ *where X is pairwise distinguishable by L.*
Let $Q = \{q_1, q_2, q_3 \cdots q_k\}$
Let $f : X \longrightarrow Q$ *such that* $f(s_i) = q_i \ \forall i$ *with* $1 \leq i \leq k$
f *is bijective (one-one and onto).*
$\forall q_i \in Q, \exists$ *a unique* $s_i \in X$ *s.t.* $f(s_i) = q_i$ *since* f *is bijective.*
$\forall a \in \Sigma, \exists$ *a unique* $s_j \in X$ *s.t.* $s_i a \equiv_L s_j$ *by Claim 2.*
Since f *is a bijective mapping, there is a unique* q_j *such that* $f(s_j) = q_j$.
Let $M = (Q, \Sigma, \delta, q_0, F)$ *where*
$\delta : Q \times \Sigma \longrightarrow Q$ *s.t.* $\delta(q_i, a) = q_j$ *where*
$a \in \Sigma, q_i, q_j \in Q$ *s.t.* $f(s_i) = q_i, f(s_j) = q_j$ *where* $s_i \in X, s_j \in X$ *and* $s_i a \equiv_L s_j$.
If there is another $q_k \in Q$ *such that* $\delta(q_i, a) = q_k, \exists s_k \in X$ *such that* $f(s_k) = q_k$ *and*
by definition of $\delta, s_i a \equiv_L s_k$.
Since $\equiv_L$ *is transitive,* $s_j \equiv_L s_k$.
This contradicts that both s_j *and* s_k *are in X and hence must be distinguishable by L.*
Therefore, $\delta(q_i, a) = q_j$ *is uniquely defined.*
$q_0 = q_\epsilon$ *where* $q_\epsilon = f(s_\epsilon)$ *and* s_ϵ *is defined in Claim 2.*
$F = \{f(s) \mid s \in L \cap X\}$
$F \neq \emptyset$ *because of Claim 1 and Claim 3.*
Claim 4. $\forall w \in \Sigma^*, \hat{\delta}(q_\epsilon, w) = q_i \Leftrightarrow w \equiv_L s_i$ *where* $f(s_i) = q_i$.
<Proof of Claim 4>
Claim 4 can be proved by induction on $|w|$.
For $w = \epsilon$, *there exists one and only one* $s_\epsilon \in X$ *s.t.* $\epsilon \equiv_L s_\epsilon$ *by Claim 2.*
$\hat{\delta}(q_\epsilon, w) = q_i$
$\Leftrightarrow \hat{\delta}(q_\epsilon, \epsilon) = q_i$ *($w = \epsilon$)*
$\Leftrightarrow q_\epsilon = q_i$ *(Definition of 1.4(i))*
$\Leftrightarrow f(s_\epsilon) = q_i$ *(Definition of q_ϵ)*
$\Leftrightarrow f(s_\epsilon) = f(s_i)$ *(Definition of q_i)*
$\Leftrightarrow s_\epsilon = s_i$ *(f is bijective)*
$\Leftrightarrow \epsilon \equiv_L s_i$ *($\epsilon \equiv_L s_\epsilon$ by Claim 2)*
$\Leftrightarrow w \equiv_L s_i$ *($w = \epsilon$)*
The statement is true for $w = \epsilon$.
Let $\hat{\delta}(q_\epsilon, wa) = f(s_i) = q_i$.
$\delta(\hat{\delta}(q_\epsilon, w), a) = f(s_i) = q_i$.
$\exists q_j$ *s.t.* $\hat{\delta}(q_\epsilon, w) = q_j$
$\exists s_j$ *s.t.* $f(s_j) = q_j$ *and*
$w \equiv_L s_j$ *(By induction hypothesis)*
$wa \equiv_L s_j a$ *($\equiv_L$ is right congruence by Proposition 1.51)*
$\delta(q_j, a) = q_i$

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$\Rightarrow s_j a \Rightarrow_L s_i$ (Definition of δ)
$\Rightarrow wa \equiv_L s_i$ ($\equiv_L$ is transitive)
Conversely, if $wa \equiv_L s_i$ for some $s_i \in X$,
$\hat{\delta}(q_\epsilon, wa)$
$= \delta(\hat{\delta}(q_\epsilon, w), a)$
$= \delta(q_j, a)$ where $q_j = \hat{\delta}(q_\epsilon, w)$
By induction hypothesis, $w \equiv_L s_j$ because $\hat{\delta}(q_\epsilon, w) = q_j$.
$wa \equiv_L s_j a$ (Right congruence by Proposition 1.51)
Let $\delta(q_j, a) = q_k$
$s_j a \equiv_L s_k$ (By definition of δ)
$wa \equiv_L s_k$ ($\equiv_L$ is transitive)
$wa \equiv_L s_i$ (Assumption)
$s_k = s_i$ (Claim 2)
$f(s_k) = f(s_i)$
$q_k = q_i$
$\hat{\delta}(q_\epsilon, wa)$
$= \delta(q_j, a)$
$= q_k$
$= q_i$
This completes the proof of Claim 4.
It remains to prove $L = L(M)$.
$\forall w \in L, \exists$ one and only one $s_i \in X$ s.t. $w \equiv_L s_i$ (By Claim 2)
$w \in L \Leftrightarrow s_i \in L$ (Proposition 1.52)
Therefore, $s_i \in L$ ($w \in L$)
Since $s_i \in L \cap X$ and $q_i = f(s_i)$, $q_i \in F$ (Definition of F)
$\hat{\delta}(q_\epsilon, w) = q_i$ (Claim 4)
$\hat{\delta}(q_0, w) = q_i$ ($q_0 = q_\epsilon$)
M accepts w ($q_i \in F$)
Conversely, if M accepts w,
$\hat{\delta}(q_\epsilon, w) = q_i$ and $q_i \in F$ ($q_0 = q_\epsilon$)
$w \equiv_L s_i$ where $q_i = f(s_i)$ (Claim 4)
$w \in L \Leftrightarrow s_i \in L$ (Proposition 1.52)
Since $q_i \in F$ and $q_i = f(s_i)$,
$s_i \in L \cap X$ by definition of F.
Therefore, $s_i \in L$.
Therefore, $w \in L$.
$L = L(M)$ and M has k states.

(c) L is regular
$\Rightarrow \exists M$ s.t. $L = L(M)$
$\Rightarrow$ Index $L \leq k$ where $k =$ the number of states in M (by (a))
$\Rightarrow L$ has a finite index
L has a finite index
$\Rightarrow$ Index $L = k$
$\Rightarrow L = L(M)$ for some k-state DFA M (by (b))
$\Rightarrow L$ is regular
Assume for contradiction that there is a k'-state DFA accepting L where $k' < k$.
By (a), Index $L \leq k'$.
This would contradict $k' < k =$ Index L.

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9. An Application of the Myhill-Nerode Theorem

The Myhill-Nerode Theorem can be used to determine whether a language L is regular or non-regular by determining the number of members in X, the set that is pairwise distinguishable by L.

Example 6.
Determine if $L = \{a^n b^n \mid n \geq 0\}$ *is regular.*
Consider $X = \{a, a^2, a^3 \cdots\}$
$\forall$ *distinct* $x, y \in X, x = a^i, y = a^j$ *where* $1 \leq i < j < \infty$
$\exists z = b^i$ *such that*
$xz = a^i b^i \in L$ *and* $yz = a^j b^i \notin L$.
x and y are distinguishable by L. $(x \not\equiv_L y)$
X is pairwise distinguishable by L.
Index $L \geq |X|$
Index L is infinite.
L is not regular.

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