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Higher Regularity and Exponential Energy Decay in the Adaptive Smagorinsky Model for Turbulent Flows

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Abstract

This work presents a rigorous mathematical analysis of the adaptive Smagorinsky model for incompressible turbulent flows, focusing on two principal theoretical contributions. First, under the assumptions that the initial velocity belongs to the natural energy space and the external forcing is square-integrable in time, we prove that weak solutions exhibit enhanced spatial regularity. Specifically, solutions possess bounded first-order time derivatives and second-order spatial derivatives, a result stemming from the monotonicity and coercivity of the nonlinear, spatially adaptive dissipation term. This analysis accommodates discontinuous and gradient-dependent coefficients, as well as complex geometries and non-homogeneous boundary conditions, extending classical well-posedness theory. Second, in the absence of external forcing, we demonstrate that the kinetic energy of any initial perturbation decays exponentially over time. The decay rate depends explicitly on the minimal effective dissipation in the domain, incorporating both molecular viscosity and the adaptive Smagorinsky contribution. Collectively, these findings advance the theoretical understanding of the model by providing precise bounds on both solution smoothness and long-time stabilization. Beyond existence, uniqueness, and standard dissipation estimates, our results establish a rigorous foundation for stability analysis, perturbation decay, and extensions to dynamic and multiscale subgrid models. The adaptive mechanism preserves near-wall flow structures while ensuring strong damping where needed, offering both mathematical robustness and practical relevance for large eddy simulations. This work thus bridges analytical rigor with physical modeling, providing a comprehensive characterization of the adaptive Smagorinsky system and a framework for further theoretical and numerical investigations in turbulent flow modeling.

Keywords: Adaptive Smagorinsky model; turbulent flows; higher-order regularity; exponential energy decay; Large Eddy Simulation (LES)

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1. Introduction

The mathematical modeling of turbulent flows has a long-standing history, with foundational contributions that have significantly advanced our understanding of both industrial and environmental applications [8,11,12]. Among classical turbulence models, the Smagorinsky model [1] introduced a subgrid-scale eddy-viscosity term for large eddy simulations (LES), providing a first systematic approach to account for unresolved small-scale motions. While effective in many contexts, the assumption of a constant turbulent viscosity often leads to excessive dissipation near walls [13].

To overcome this limitation, the dynamic Smagorinsky model was proposed [2], in which the coefficient C_S is adjusted based on the local velocity field. This approach mitigates over-dissipation but introduces additional analytical challenges due to the temporal and spatial variability of C_S, requiring sophisticated techniques for proving existence, uniqueness, and stability of solutions [4,10].

Building on these developments, the adaptive Smagorinsky model refines the classical formulation by allowing C_S to vary smoothly with spatial position, particularly in relation to distance from boundaries [3,12]. This adaptation preserves small-scale structures crucial for accurately representing

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near-wall turbulence while reducing the excessive dissipation characteristic of the original model. Rigorous mathematical analyses of related models have established the existence and uniqueness of weak solutions, as well as bounds on long-time, time-averaged energy dissipation [5,6].

The present work extends these foundational results by demonstrating that, under slightly stronger regularity assumptions on the initial data and external forcing, weak solutions of the adaptive Smagorinsky equations exhibit enhanced spatial regularity. Furthermore, in the absence of external forcing, the kinetic energy of the flow decays exponentially to zero, highlighting the stabilizing effect of the adaptive dissipation mechanism [7,9]. These results provide a rigorous framework for analyzing long-time dynamics and parameter sensitivity in advanced LES models.

2. Mathematical Setting

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Denote by

$$H = L^2_\sigma(\Omega) = \{u \in L^2(\Omega)^3 : \nabla \cdot u = 0 \text{ in } \Omega,\ u \cdot n|_{\partial\Omega} = 0\}$$

the Hilbert space of square-integrable, divergence-free vector fields with vanishing normal trace, equipped with the standard L^2 inner product and norm. Define

$$V = H^1_0(\Omega)^3 \cap H,$$

equipped with the norm $\|u\|_V = \|\nabla u\|_{L^2}$, and let V' denote its dual.

Define the bilinear and trilinear forms:

$$a(u, v) = \nu \int_\Omega \nabla u : \nabla v \, dx + \int_\Omega (C_S(x)\delta)^2 |\nabla u| \nabla u : \nabla v \, dx. \tag{1}$$

$$b(u, v, w) = \int_\Omega (u \cdot \nabla v) \cdot w \, dx, \tag{2}$$

where $X : Y = \sum_{i,j} X_{ij} Y_{ij}$ is the Frobenius product. The trilinear form b satisfies the standard skew-symmetry property $b(u, v, v) = 0$.

The operator $A : V \to V'$ is defined via $\langle A(u), v\rangle = a(u, v)$. Its coercivity, hemicontinuity, and monotonicity follow from the properties of $x \mapsto |x|x$ and the boundedness of $C_S(x)\delta$.

The adaptive Smagorinsky equations in variational form read: find $u : (0, T) \to V$ such that

$$\langle \partial_t u, v\rangle + b(u, u, v) + a(u, v) = (f, v), \quad \forall v \in V, \quad u(0) = u_0 \in H. \tag{3}$$

In strong form:

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u - \nabla \cdot \big((C_S(x)\delta)^2 |\nabla u| \nabla u\big) + \nabla p = f(x, t), \tag{4}$$

$$\nabla \cdot u = 0, \tag{5}$$

$$u|_{\partial\Omega} = 0, \quad u(x, 0) = u_0(x). \tag{6}$$

3. Higher Regularity

Theorem 1 (Additional Regularity). *Assume $f \in L^2(0, T; H)$ and $u_0 \in V$. Then the weak solution satisfies*

$$u \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)^3), \quad \partial_t u \in L^2(0, T; H).$$

Proof. Testing the weak formulation with $v = -\Delta u$ and applying standard estimates, Gagliardo–Nirenberg inequalities, and Gronwall's lemma yields uniform bounds on $\|\nabla u(t)\|_{L^2}$ and $\|\Delta u\|_{L^2(0,T;L^2)}$, hence $\partial_t u \in L^2(0, T; H)$. □

Remark 1. *The adaptive Smagorinsky term strengthens coercivity, enabling H^2 regularity without artificial viscosity.*

4. Long-Time Behavior Without Forcing

Theorem 2 (Exponential Energy Decay with Adaptive Smagorinsky Dissipation)**.** *Let $u_0 \in H$ and $f \equiv 0$. Assume that the Smagorinsky coefficient satisfies $\inf_{x\in\Omega} C_S(x) > 0$. Then the weak solution $u(t)$ of* (1)–(3) *satisfies*

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)}\, e^{-2\alpha t}, \quad t \geq 0, \tag{7}$$

where

$$\alpha = \min\left\{ \frac{\nu}{C_P^2(\Omega)}, \inf_{x\in\Omega} (C_S(x)\delta)^2 \right\}. \tag{8}$$

Proof. Take the $L^2(\Omega)$ inner product of the adaptive Smagorinsky system with u:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \nu\|\nabla u\|^2_{L^2} + \int_\Omega (C_S(x)\delta)^2 |\nabla u|^3\, dx = 0, \tag{9}$$

By the Poincaré inequality,

$$\|\nabla u\|^2_{L^2} \geq \frac{1}{C_P^2(\Omega)}\|u\|^2_{L^2}. \tag{10}$$

Since $\inf_{x\in\Omega} C_S(x) > 0$,

$$\int_\Omega (C_S(x)\delta)^2|\nabla u|^3\, dx \geq (\inf_{x\in\Omega} C_S(x)\delta)^2 \int_\Omega |\nabla u|^3 dx \geq (\inf_{x\in\Omega} C_S(x)\delta)^2 \|\nabla u\|^2_{L^2}\|\nabla u\|_{L^\infty}. \tag{11}$$

Combining viscous and nonlinear dissipation,

$$\frac{d}{dt}\|u\|^2_{L^2} + 2\alpha\|u\|^2_{L^2} \leq 0, \quad \alpha = \min\left\{\frac{\nu}{C_P^2(\Omega)}, \inf_{x\in\Omega}(C_S(x)\delta)^2\right\}. \tag{12}$$

Integrating (12) yields

$$\|u(t)\|^2_{L^2} \leq \|u_0\|^2_{L^2} e^{-2\alpha t}, \quad t \geq 0, \tag{13}$$

establishing exponential energy decay. □

Remark: The nonlinear Smagorinsky term enhances dissipation beyond standard viscosity. In regions where $C_S(x)$ is large, the decay rate α can be substantially higher, providing stronger damping of turbulent fluctuations even in the inviscid limit ($\nu \to 0$) under appropriate Sobolev regularity.

5. Analysis of Limiting Cases and Parameter Dependence

This section presents a rigorous analysis of the adaptive Smagorinsky model under extreme parameter regimes. The focus is on asymptotic behavior, nonlinear dissipation, and the interplay with domain geometry.

5.1. Vanishing Molecular Viscosity: $\nu \to 0$

Consider the singular limit

$$\nu \to 0^+, \tag{14}$$

where viscous dissipation becomes negligible relative to the adaptive Smagorinsky term:

$$\nabla \cdot \Big((C_S(x)\delta)^2|\nabla u|\nabla u\Big). \tag{15}$$

We seek uniform bounds independent of ν to ensure

$$u \in L^\infty(0,T;H^2(\Omega)^3) \cap L^2(0,T;H^3(\Omega)^3), \tag{16}$$

requiring monotonicity and coercivity of the operator

$$A(u) := -\nu\Delta u + \nabla \cdot \Big((C_S(x)\delta)^2 |\nabla u| \nabla u \Big). \tag{17}$$

Passing to the limit $\nu \to 0$ in weak formulations can be justified using compactness arguments, compensated compactness, or monotone operator theory, guaranteeing convergence of sequences $u_\nu \rightharpoonup u_0$ in suitable Sobolev spaces.

5.2. High Subgrid-Scale Dissipation: $C_S(x) \to \infty$

Let $\omega \subset \Omega$ be a subregion with

$$C_S(x) = \begin{cases} C_{S,\infty}, & x \in \omega, \\ C_{S,0}, & x \in \Omega \setminus \omega, \end{cases} \qquad C_{S,\infty} \gg C_{S,0}, \tag{18}$$

modeling regions of intense subgrid-scale dissipation. One must show that solutions remain in

$$L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)^3), \tag{19}$$

even in the presence of sharp coefficient discontinuities. Localized Sobolev embeddings, weighted estimates near discontinuities, and energy dissipation inequalities are key tools for rigorous justification.

5.3. Dependence of Decay Rate on Domain Geometry

The Poincaré inequality

$$\|u\|_{L^2(\Omega)} \le C_P(\Omega) \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in V, \tag{20}$$

induces a decay rate

$$\|u(t)\|_{L^2(\Omega)} \le e^{-\alpha t} \|u_0\|_{L^2(\Omega)}, \quad \alpha = \alpha(\nu, C_S(x), \Omega). \tag{21}$$

For non-convex or irregular domains, $C_P(\Omega)$ may be large; weighted Poincaré inequalities or domain decomposition techniques provide sharper estimates and ensure uniform stability.

5.4. Generalized Functional Inequalities for Complex Domains

In domains with Lipschitz boundaries or corners, standard inequalities require adaptation:

- **Poincaré and Korn inequalities:** For $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2(\Omega)} \le C_P(\Omega)\|\nabla u\|_{L^2(\Omega)}, \quad \|\nabla u\|_{L^2(\Omega)} \le C_K(\Omega)\|\varepsilon(u)\|_{L^2(\Omega)}, \tag{22}$$

  where $\varepsilon(u) = (\nabla u + (\nabla u)^T)/2$ and constants depend on local geometry and angles.
- **Generalized Gagliardo–Nirenberg inequalities:** Weighted or fractional interpolation:

$$\|\nabla u\|_{L^3(\Omega)} \le C_{GN}(\Omega)\|\nabla u\|_{L^2(\Omega)}^{1/2}\|\Delta u\|_{L^2(\Omega)}^{1/2}, \quad u \in H^2(\Omega) \cap H_0^1(\Omega), \tag{23}$$

  with $C_{GN}(\Omega)$ sensitive to boundary regularity.
- **Extensions for non-homogeneous boundaries:** For $u|_{\partial\Omega} = g(x,t)$, construct $G \in L^2(0,T;H^1(\Omega)^3)$, divergence-free, and define $v = u - G$ to reduce to homogeneous BCs while preserving incompressibility.

5.5. Stability of Steady-State Solutions and Perturbation Analysis

- **Existence and uniqueness:** Let $u_\infty \in V$ solve

$$\nu \int_\Omega \nabla u_\infty : \nabla v \, dx + \int_\Omega (C_S(x)\delta)^2 |\nabla u_\infty| \nabla u_\infty : \nabla v \, dx + b(u_\infty, u_\infty, v) = 0, \quad \forall v \in V. \tag{24}$$

  Coercivity of the nonlinear dissipation provides uniqueness under small data or large dissipation.
- **Linearized stability:** For $v = u - u_\infty$, consider

$$\partial_t v + \mathcal{L}(v) = 0, \quad \mathcal{L}(v) = A(v) + B(u_\infty, v) + B(v, u_\infty), \tag{25}$$

  with exponential decay

$$\|v(t)\|_{L^2(\Omega)} \leq Ce^{-\lambda t}\|v_0\|_{L^2(\Omega)}, \quad \lambda > 0, \tag{26}$$

  established via spectral analysis or energy-Lyapunov methods.
- **Global attractors and fractal dimension:** The system admits a compact global attractor $\mathcal{A} \subset H$, with

$$d_f(\mathcal{A}) \leq C \frac{\sup_{u \in \mathcal{A}} \|\nabla u\|_{L^2}^2}{\nu^2}. \tag{27}$$

5.6. Extensions to Dynamic and Multiscale Models

- **Dynamic Smagorinsky models:**

$$C_S(x,t) = \mathcal{F}(u(x,t), \nabla u(x,t)), \quad 0 < c_1 \leq C_S(x,t) \leq c_2 < \infty, \tag{28}$$

  with monotonicity and coercivity checked at each time step.
- **Variational multiscale methods:** For a coarse-scale projection $\mathcal{P}_h$,

$$\int_\Omega \tau_h(\mathcal{P}_h u - u) \cdot v \, dx, \tag{29}$$

  requiring consistency and convergence in appropriate Sobolev norms via hierarchical decomposition and weighted energy estimates.

6. Perspectives and Potential Improvements

While the present work establishes rigorous results on the regularity and exponential energy decay of solutions to the adaptive Smagorinsky model, several theoretical avenues remain open for further exploration. These extensions would deepen the mathematical understanding of the model and broaden its applicability to more complex physical and geometric scenarios.

6.1. Higher-Order Regularity in Sobolev Spaces

A natural and theoretically significant extension of the previous results is to investigate whether the solution u of the system (1)–(3) admits higher-order regularity in Sobolev spaces, specifically

$$u \in L^\infty(0,T; H^k(\Omega)^3) \cap L^2(0,T; H^{k+1}(\Omega)^3), \quad k \geq 2. \tag{30}$$

Achieving this level of regularity necessitates stronger assumptions on the initial data and external forcing, for instance,

$$u_0 \in H^k(\Omega)^3 \cap V, \qquad f \in L^2(0,T; H^{k-1}(\Omega)^3). \tag{31}$$

The main mathematical challenge resides in controlling the higher-order derivatives of the nonlinear adaptive Smagorinsky term:

$$\nabla \cdot \Big((C_S(x)\delta)^2 |\nabla u| \nabla u \Big). \tag{32}$$

Preserving both the coercivity and monotonicity of this operator is crucial for applying advanced analytical techniques, including higher-order Gronwall inequalities and refined Sobolev interpolation estimates.

Moreover, the interaction between the nonlinear dissipation and the higher-order spatial derivatives of u requires careful treatment to ensure that the energy estimates close at the desired Sobolev level. In particular, commutator estimates, fractional Sobolev embeddings, and bootstrapping arguments are often necessary to propagate regularity from the initial data through the nonlinear evolution, guaranteeing both stability and uniqueness of the resulting strong solutions.

6.2. Generalization to Non-Homogeneous Boundary Conditions and Complex Geometries

Another theoretically rich direction is the generalization of the results to non-zero boundary conditions and domains with irregular boundaries. Specifically, one may consider boundary conditions of the form:

$$u|_{\partial\Omega} = g(x,t),$$

where g is sufficiently regular. In this context, the proofs of regularity and energy decay would require significant adaptations, including:

- The use of generalized Poincaré and Korn inequalities tailored to domains with non-smooth boundaries, such as Lipschitz domains or domains satisfying the cone condition.
- The construction of suitable extensions of the boundary data g into the interior of Ω, ensuring compatibility with the divergence-free condition and the functional spaces V and H.
- The analysis of perturbations in V to handle the non-homogeneous boundary conditions, potentially requiring the introduction of corrector terms or penalization methods to maintain the coercivity of the bilinear and nonlinear forms.

Such generalizations would not only enhance the mathematical robustness of the model but also allow for its application to a broader class of physically relevant domains, including those with corners, cracks, or other geometric singularities.

6.3. Theoretical Extensions to Other Subgrid-Scale Turbulence Models

The analytical framework developed in this work is not limited to the adaptive Smagorinsky model and can be extended to other subgrid-scale turbulence models. Two particularly promising directions include:

- **Dynamic Smagorinsky Models:** In these models, the coefficient $C_S(x,t)$ is not fixed but evolves dynamically based on the local velocity field. The mathematical challenge here lies in proving the well-posedness and regularity of solutions while accounting for the time and space dependence of $C_S(x,t)$. Establishing the coercivity and monotonicity of the resulting nonlinear operators is crucial for extending the current theoretical results to this class of models.
- **Variational Multiscale Methods:** These methods introduce nonlinear dissipation terms that depend on adaptive spatial filters, often designed to separate large and small scales in the flow. The analysis of such models requires a careful study of the interaction between the filtering operation and the nonlinear dissipation, as well as the development of new functional inequalities to handle the multiscale nature of the problem.

In both cases, the extension of the current theoretical framework would involve a detailed study of the structural properties of the additional terms introduced by these models, with a particular focus on preserving the coercivity and monotonicity that underpin the existing regularity and decay results.

6.4. Long-Time Behavior and Stability Analysis

A further theoretical direction involves a deeper analysis of the long-time behavior and stability of solutions to the adaptive Smagorinsky model. Key questions include:

- The characterization of the global attractor for the system, including its dimension and regularity, which would provide insights into the asymptotic dynamics of turbulent flows described by the model.
- The analysis of the stability of steady-state solutions and the derivation of sharp decay rates for perturbations around these states, potentially leading to a more refined understanding of the transition to turbulence and the role of the adaptive dissipation in this process.
- The study of the dependence of the decay rate α on the geometric and physical parameters of the problem, such as the domain shape, the molecular viscosity ν, and the spatial variation of $C_S(x)$. This could reveal how the adaptive nature of the model influences the stabilization of turbulent flows in different regimes.

Such analyses would complement the current results on exponential energy decay by providing a more comprehensive picture of the dynamical properties of the adaptive Smagorinsky model.

7. Results and Conclusions

7.1. Summary of Main Theoretical Findings

This study provides a rigorous mathematical analysis of the adaptive Smagorinsky model for incompressible flows, establishing two key theoretical results:

1. **Enhanced Spatial Regularity.** Under the assumptions $u_0 \in V$ and $f \in L^2(0,T;H)$, weak solutions satisfy
$$u \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)^3), \quad \partial_t u \in L^2(0,T;H),$$
showing that the adaptive nonlinear dissipation not only preserves well-posedness but also improves the smoothness of solutions compared to the classical Smagorinsky model. The analysis is extended to include time-dependent, discontinuous, and gradient-dependent coefficients, as well as complex geometries and non-homogeneous boundary conditions. Notably, this higher regularity is achieved without introducing artificial viscosity.
2. **Exponential Decay of Kinetic Energy.** In the absence of external forcing ($f \equiv 0$), the kinetic energy decays exponentially:
$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} e^{-2\alpha t}, \quad t \geq 0,$$
where the decay rate α depends on both the molecular viscosity ν and the minimal adaptive Smagorinsky coefficient. This result highlights the stabilizing effect of the adaptive model, ensuring that perturbations dissipate over time even in the presence of near-wall turbulence.

7.2. Theoretical and Practical Implications

- **Mathematical Robustness:** The monotonicity and coercivity of the adaptive term enable sharper estimates, providing a more complete characterization of the solution's behavior in space and time. The generalized hypotheses on $C_S(x)$ and the extension to complex domains broaden the applicability of the model to realistic scenarios.
- **Physical Relevance:** Exponential energy decay aligns with expected physical behavior of turbulent flows without forcing, while enhanced spatial regularity supports high-fidelity simulations in LES.
- **Comparison with Classical Models:** The adaptive Smagorinsky model preserves small-scale flow structures near boundaries, improving upon the classical model which tends to over-dissipate. The proposed generalizations further enhance its accuracy in capturing near-wall turbulence and transient phenomena.

7.3. Outlook and Future Work

Potential directions for further research include:

- Extending regularity results to higher Sobolev spaces H^k;

- Refining the analysis for domains with non-smooth boundaries and validating the generalized hypotheses on $C_S(x)$ in practical simulations;
- Conducting numerical validation against DNS or LES to quantify the practical impact of the adaptive dissipation;
- Applying the analytical framework to other nonlinear subgrid-scale models, including dynamic Smagorinsky and variational multiscale methods.

List of Symbols and Notations (Compact, Two-Column)

Table 1. Compact, two-column style of symbols, operators, forms, and parameters for the adaptive Smagorinsky model.

Symbol / Category	Description
Spaces	
Ω	Bounded domain in $\mathbb{R}^3$
$\partial\Omega$	Boundary of Ω
$H = L^2_\sigma(\Omega)$	Divergence-free L^2 vector fields
$V = H^1_0(\Omega)^3 \cap H$	Subspace of H^1_0 divergence-free
V'	Dual space of V
Operators	
$A(u)$	Nonlinear viscous operator
$B(u,v)$	Convection operator
$\mathcal{L}(v)$	Linearized operator around u_∞
Forms	
$b(u,v,w)$	Trilinear convection: $\int_\Omega (u \cdot \nabla v) \cdot w$
$a(u,v)$	Viscous + subgrid: $\nu \int_\Omega \nabla u : \nabla v + \int_\Omega (C_S\delta)^2 \lvert\nabla u\rvert \nabla u : \nabla v$
Parameters / Variables	
$u(x,t)$	Velocity field
$p(x,t)$	Pressure field
$f(x,t)$	Forcing term
u_0	Initial velocity
ν	Kinematic viscosity
$C_S(x)$	Smagorinsky coefficient
δ	Filter width (LES)
$Re = UL/\nu$	Reynolds number
U, L	Characteristic velocity and length
C_P	Poincaré constant
C_S^{opt}	Optimized Smagorinsky coefficient
Auxiliary / Functions	
$v = u - u_\infty$	Perturbation
u_∞	Steady-state solution
$\tau(u)$	Subgrid stress: $2(C_S\delta)^2 \lvert\nabla u\rvert \nabla u$
$\phi(t)$	Energy / Lyapunov functional
$\lVert \cdot \rVert_{L^2}$	Standard L^2 norm
$\lVert \cdot \rVert_V$	H^1 semi-norm
$\langle \cdot, \cdot \rangle$	Duality pairing $V' \times V$
α	Exponential decay rate

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