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Article

# On the Irrationality of the Odd Zeta Values

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## Abstract

We prove the irrationality of the odd zeta values  $\zeta(2n+1)$ ,  $n \in \mathbb{N}$ . Our approach is based on constructing explicit integer linear forms in  $\zeta(2n+1)$ , and applying a refinement of Dirichlet's approximation theorem. We prove that the sequence of denominators produced by successive rational approximations yields infinitely many nontrivial integer relations of the type

$$\Lambda_m^{(q)} = A_m^{(q)} \zeta(2n+1) - B_m^{(q)},$$

with  $|\Lambda_m^{(q)}|$  ( $q$  being a parameter) decaying towards zero as  $m$  approaches infinity. This permits us to invoke a general irrationality criterion and thereby deduce that each  $\zeta(2n+1)$  is irrational. Our method combines ideas from probability theory and Diophantine approximation, and complements earlier work of Apéry, Beukers, Rivoal, and Zudilin.

**Keywords:** odd zeta values; proof of irrationality; integer linear forms; Diophantine approximation; probabilistic methods in number theory

**MSC:** 11M06; 11J72; 11J20; 33E20; 60E05

## 1. Introduction

### 1.1. Background and Motivation

The study of the arithmetic properties of special values of the Riemann zeta function occupies a central place in analytic number theory. Euler's classical formulae [10]

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}, \quad n \in \mathbb{N}$$

express the even zeta values in terms of Bernoulli numbers and powers of  $\pi$ , thereby establishing both their transcendence and algebraic relations to  $\pi$ . In striking contrast, the arithmetic nature of the odd zeta values  $\zeta(2n+1)$  remains largely mysterious. The pioneering breakthrough was Apéry's proof [1,11] of the irrationality of  $\zeta(3)$ , which not only settled a longstanding problem of Euler but also introduced a new paradigm for irrationality proofs through the construction of highly nontrivial linear recurrences. Subsequent refinements by Beukers [3] via integrals, and the landmark results of Rivoal [7] and Zudilin [12] showed that infinitely many odd zeta values must be irrational, and that at least one of  $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$  is irrational. Nevertheless, no specific irrationality result beyond Apéry's  $\zeta(3)$  has been obtained, and the question of whether  $\zeta(2n+1)$  is irrational for  $n \geq 2$  remains open.

The goal of this paper is to prove general irrationality results for the odd zeta values, by applying a refined Diophantine approximation argument.

### 1.2. Main Theorem

We begin by stating the principal result.

**Theorem 1.1** (Main theorem). *For every  $n \in \mathbb{N}$ , the odd zeta value  $\zeta(2n+1)$  is irrational.*

### 1.3. Strategy of the Proof (Sketch)

We outline the logical structure of the proof before entering the technical details.

1. *Construction of integer linear forms.* We construct for each  $m \in \mathbb{N}$  an asymmetric beta kernel  $W_m^{(q)}(x)$  and its cumulative distribution function  $S_m^{(q)}(x)$ , enabling an exact integral identity linking  $\zeta(2n+1)$  with a sequence of weighted moments  $L_m^{(q)}$ . Truncating the Maclaurin expansion of the polylogarithm  $\text{Li}_{2n+1}(x)$  yields integer linear forms  $\Lambda_m^{(q)} = A_m^{(q)}\zeta(2n+1) - B_m^{(q)}$ , whose coefficients are made integral by a suitable least common multiple of denominators.
2. *Asymptotic decay and growth estimates.* Exponential decay of the kernel tail and remainder terms is established using Stirling asymptotics and  $L^p$  norms, while precise denominator growth is controlled through Legendre's formula and the Prime Number Theorem.
3. *Application of an irrationality criterion.* The balance of these estimates ensures that  $|\Lambda_m^{(q)}| \rightarrow 0$  as  $m \rightarrow \infty$ , permitting application of a refined Dirichlet-type irrationality criterion to prove that each  $\zeta(2n+1)$  is irrational.

## 2. An Integral Representation of $\zeta(2n+1)$

We shall exploit the following integral representation of  $\zeta(2n+1)$ :

$$\zeta(2n+1) = \int_0^1 \text{Li}'_{2n+1}(x) dx, \quad (1)$$

where  $\text{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s}$  denotes the polylogarithm [5].

## 3. Construction of Integer Linear Forms

### 3.1. The Beta Kernel and the Exact Integral Identity

Define the *asymmetric beta kernel* [4]

$$W_m^{(q)}(x) := \binom{(q+1)m}{m} x^m (1-x)^{qm}, \quad m, q \in \mathbb{N}. \quad (2)$$

The corresponding probability density function on  $[0, 1]$  is

$$\tilde{W}_m^{(q)}(x) := \Omega_m^{(q)} W_m^{(q)}(x), \quad (3)$$

where the normalization  $\Omega_m^{(q)} := (q+1)m + 1$ . The cumulative distribution function is

$$S_m^{(q)}(x) := \int_0^x \tilde{W}_m^{(q)}(t) dt. \quad (4)$$

Set

$$L_m^{(q)} := \int_0^1 W_m^{(q)}(x) \text{Li}_{2n+1}(x) dx. \quad (5)$$

**Lemma 3.1** (Exact integral identity). *We have*

$$\Omega_m^{(q)} L_m^{(q)} = \zeta(2n+1) - \int_0^1 \text{Li}'_{2n+1}(x) S_m^{(q)}(x) dx. \quad (6)$$

**Proof.** Substitute  $\text{Li}_{2n+1}(x) = \int_0^x \text{Li}'_{2n+1}(t) dt$  and swap the integrals:

$$\begin{aligned} L_m^{(q)} &= \int_0^1 W_m^{(q)}(x) \int_0^x \text{Li}'_{2n+1}(t) dt dx = \int_0^1 \text{Li}'_{2n+1}(t) \int_t^1 W_m^{(q)}(x) dx dt \\ &= \frac{1}{\Omega_m^{(q)}} \int_0^1 \text{Li}'_{2n+1}(t) [1 - S_m^{(q)}(t)] dt. \end{aligned}$$

Multiply by  $\Omega_m^{(q)}$  and use (1).  $\square$

### 3.2. Truncation and Denominator Control

By definition,

$$\text{Li}_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^{2n+1}}.$$

Multiplying by  $W_m^{(q)}$  and integrating termwise (justified by uniform convergence on  $[0, 1]$ ) gives

$$\begin{aligned} L_m^{(q)} &= \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \binom{(q+1)m}{m} B(m+k+1, qm+1) = \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \frac{((q+1)m)! (m+k)!}{m! ((q+1)m+k+1)!} \\ &= \sum_{k=1}^{\infty} \frac{F_{m,k}^{(q)}}{k^{2n+1}}, \end{aligned} \quad (7)$$

where

$$F_{m,k}^{(q)} := \frac{((q+1)m)! (m+k)!}{m! ((q+1)m+k+1)!} \quad (8)$$

and we used  $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$  to simplify.

Fix a truncation parameter  $K = K(m)$  and split

$$L_m^{(q)} = L_m^{(q)(\leq K)} + L_m^{(q)(>K)}, \quad (9)$$

where  $L_m^{(q)(\leq K)} := \sum_{k=1}^K \frac{F_{m,k}^{(q)}}{k^{2n+1}}$  and  $L_m^{(q)(>K)} := \sum_{k>K} \frac{F_{m,k}^{(q)}}{k^{2n+1}}$ . Let  $\text{den}(x)$  be the denominator of the rational number  $x$  written in lowest terms. Define

$$D_m^{(q)} := \text{lcm}_{1 \leq k \leq K} \text{den} \left( \frac{F_{m,k}^{(q)}}{k^{2n+1}} \right). \quad (10)$$

**Lemma 3.2** (Integer linear form). *With*

$$A_m^{(q)} := D_m^{(q)} \in \mathbb{Z}, \quad (11)$$

$$B_m^{(q)} := \Omega_m^{(q)} D_m^{(q)} L_m^{(q)(\leq K)} \in \mathbb{Z}, \quad (12)$$

we have

$$\Lambda_m^{(q)} := A_m^{(q)} \zeta(2n+1) - B_m^{(q)} = \Omega_m^{(q)} D_m^{(q)} L_m^{(q)(>K)} + D_m^{(q)} \int_0^1 \text{Li}'_{2n+1}(x) S_m^{(q)}(x) dx. \quad (13)$$

**Proof.** Multiply (6) by  $D_m^{(q)}$  and substitute  $L_m^{(q)} = L_m^{(q)(\leq K)} + L_m^{(q)(>K)}$ . The choice (10) guarantees every  $B_m^{(q)}$  is an integer (the least common multiple clears all denominators  $k^{2n+1}$  for  $k \leq K$  and all factorial denominators up to  $(q+1)m+K+1$ ).  $\square$

Thus,  $\Lambda_m^{(q)} = A_m^{(q)}\zeta(2n+1) - B_m^{(q)}$  is our integer linear form in  $\zeta(2n+1)$ . Equation (13) guarantees that  $\Lambda_m^{(q)} > 0$ .

#### 4. Analytic Estimates for the Linear Forms

This section proves that the constructed linear forms tend to zero as  $m \rightarrow \infty$ .

##### 4.1. Exponential Decay of the Kernel Tail

Using Stirling asymptotics and large-deviation estimates, we show that the truncated tail contributes an exponentially small error.

**Lemma 4.1** (Kernel tail). *There exist parameters  $\alpha, \gamma^{(q)}(\alpha), C_1(\alpha, m, q), C_2(\alpha, q) > 0$ , such that for  $m > C_2(\alpha, q)$ ,*

$$|\Omega_m^{(q)} L_m^{(q)(>K)}| \leq C_1(\alpha, m, q) \left[1 - \frac{C_2(\alpha, q)}{m}\right]^{-1} e^{-\gamma^{(q)}(\alpha)m}. \quad (14)$$

**Proof.** Set  $k = \sigma m$ , where  $\sigma > 0$  is a scaling parameter. Using the Stirling estimate

$$\log N! = N \log N - N + O(\log N),$$

we get

$$\begin{aligned} \log F_{m,k}^{(q)} &= \log((q+1)m)! + \log(m+k)! - \log m! - \log((q+1)m+k+1)! \\ &= m\phi^{(q)}(\sigma) + O(\log m), \end{aligned}$$

where

$$\phi^{(q)}(\sigma) := (1+q)\log(1+q) + (1+\sigma)\log(1+\sigma) - (1+q+\sigma)\log(1+q+\sigma) \quad (15)$$

is a rate function, which is strictly decreasing on  $(0, \infty)$ . Since  $\phi^{(q)}(0) = 0$ , this implies  $\phi^{(q)}(\sigma) < 0$  for all  $\sigma > 0$ . Thus,

$$\left| \frac{F_{m,k}^{(q)}}{k^{2n+1}} \right| \leq m^{O(1)} e^{m\phi^{(q)}(\sigma)}.$$

Let  $K := \lceil \alpha m \rceil$ , for some fixed real  $\alpha > 0$ . Then,

$$|L_m^{(q)(>K)}| \leq m^{O(1)} \sum_{k>\alpha m} e^{m\phi^{(q)}(\sigma)} \leq m^{O(1)} \int_{\alpha m}^{\infty} e^{m\phi^{(q)}(\frac{x}{m})} dx = m^{O(1)} \int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)} d\sigma.$$

Since

$$\frac{d}{d\sigma} e^{m\phi^{(q)}(\sigma)} = m\phi'^{(q)}(\sigma) e^{m\phi^{(q)}(\sigma)},$$

an integration by parts gives

$$\int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)} d\sigma = -\frac{e^{m\phi^{(q)}(\alpha)}}{m\phi'^{(q)}(\alpha)} + \frac{1}{m} \int_{\alpha}^{\infty} \frac{\phi''^{(q)}(\sigma)}{\phi'^{(q)}(\sigma)^2} e^{m\phi^{(q)}(\sigma)} d\sigma.$$

The remainder integral may be bounded as

$$\left| \int_{\alpha}^{\infty} \frac{\phi''^{(q)}(\sigma)}{\phi'^{(q)}(\sigma)^2} e^{m\phi^{(q)}(\sigma)} d\sigma \right| \leq \sup_{\sigma \geq \alpha} \left| \frac{\phi''^{(q)}(\sigma)}{\phi'^{(q)}(\sigma)^2} \right| \int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)} d\sigma = C_2(\alpha, q) \int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)} d\sigma,$$

where

$$C_2(\alpha, q) := \sup_{\sigma \geq \alpha} \left| \frac{\phi''^{(q)}(\sigma)}{\phi'^{(q)}(\sigma)^2} \right|.$$

Note that  $C_2(\alpha, q)$  is finite, since for  $\sigma \geq \alpha > 0$ ,

$$\phi'^{(q)}(\sigma) = \log \frac{1+\sigma}{1+q+\sigma} < 0, \quad \phi''^{(q)}(\sigma) = \frac{q}{(1+\sigma)(1+q+\sigma)} > 0;$$

both are continuous on  $[\alpha, \infty)$ . Moreover, both  $\phi'^{(q)}(\sigma)$  and  $\phi''^{(q)}(\sigma)$  tend to zero as  $\sigma \rightarrow \infty$ ; an application of L'Hôpital's rule shows that this limit is  $\frac{1}{q}$ . So,  $\frac{\phi''^{(q)}}{\phi'^{(q)2}$  is continuous and bounded on  $[\alpha, \infty)$ . Therefore, we get

$$\int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)} d\sigma \leq -\frac{e^{m\phi^{(q)}(\alpha)}}{m\phi'^{(q)}(\alpha)} + \frac{C_2(\alpha, q)}{m} \int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)} d\sigma,$$

or,

$$\left[1 - \frac{C_2(\alpha, q)}{m}\right] \int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)} d\sigma \leq -\frac{e^{m\phi^{(q)}(\alpha)}}{m\phi'^{(q)}(\alpha)}.$$

Since  $1 - \frac{C_2(\alpha, q)}{m} > 0$  for all  $m > C_2(\alpha, q)$ , we get

$$\int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)} d\sigma \leq -\frac{e^{m\phi^{(q)}(\alpha)}}{m\phi'^{(q)}(\alpha)} \left[1 - \frac{C_2(\alpha, q)}{m}\right]^{-1} = \frac{e^{m\phi^{(q)}(\alpha)}}{m \log \frac{1+q+\alpha}{1+\alpha}} \left[1 - \frac{C_2(\alpha, q)}{m}\right]^{-1}.$$

Therefore,

$$|L_m^{(q)(>K)}| \leq m^{O(1)} \frac{e^{m\phi^{(q)}(\alpha)}}{\log \frac{1+q+\alpha}{1+\alpha}} \left[1 - \frac{C_2(\alpha, q)}{m}\right]^{-1}.$$

Defining

$$C_1(\alpha, m, q) := \frac{\Omega_m^{(q)} m^{O(1)}}{\log \frac{1+q+\alpha}{1+\alpha}}$$

and  $\gamma^{(q)}(\sigma) := -\phi^{(q)}(\sigma)$  (so  $\gamma^{(q)}(\sigma)$  is positive and increasing on  $(0, \infty)$ ), we get

$$|\Omega_m^{(q)} L_m^{(q)(>K)}| \leq C_1(\alpha, m, q) \left[1 - \frac{C_2(\alpha, q)}{m}\right]^{-1} e^{-\gamma^{(q)}(\alpha)m}.$$

□

#### 4.2. Control of the Remainder Term

Using finite difference operators and  $L^1/L^\infty$  bounds, we prove that the remainder integral also decays exponentially.

**Lemma 4.2** (Remainder term). *There exist parameters  $C_3(m, q) > 0$ ,  $0 < \delta < 1$  such that*

$$\left| \int_0^1 \text{Li}'_{2n+1}(x) S_m^{(q)}(x) dx \right| \leq C_3(m, q) \left[ \left(\frac{\delta}{m}\right)^m + \delta^m \right]. \quad (16)$$

**Proof.** We adopt the convention that all  $L^1/L^\infty$  norms are defined on  $[0, 1]$ . Then, we have

$$\left| \int_0^1 \text{Li}'_{2n+1}(x) S_m^{(q)}(x) dx \right| \leq \|\text{Li}'_{2n+1}\|_\infty \|S_m^{(q)}\|_1, \quad (17)$$

where

$$\text{Li}'_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k^{2n}}.$$

For  $0 \leq x \leq 1$ ,  $\left| \frac{x^{k-1}}{k^{2n}} \right| \leq \frac{1}{k^{2n}}$ ; therefore,

$$|\text{Li}'_{2n+1}(x)| \leq \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \zeta(2n),$$

which particularly implies

$$\|\text{Li}'_{2n+1}\|_{\infty} \leq \zeta(2n). \quad (18)$$

Now consider the *finite  $m$ th backward difference with step  $h$*  [6], applied to  $S_m^{(q)}$ :

$$\nabla_h^m S_m^{(q)}(x) := \sum_{k=0}^m \binom{m}{k} (-1)^k S_m^{(q)}(x - kh). \quad (19)$$

Appendix A proves the following properties of  $\nabla_h^m S_m^{(q)}$ :

$$\|\nabla_h^m S_m^{(q)}\|_1 \leq h^m \|S_m^{(q)(m)}\|_1, \quad \|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1 \leq (hm)^m \|S_m^{(q)(m)}\|_1.$$

By definition,  $S_m^{(q)(m)} = \tilde{W}_m^{(q)(m-1)}$ . For the asymmetric beta density, using the Leibniz rule,

$$\frac{d^{m-1}}{dx^{m-1}} [x^m (1-x)^{qm}] = \sum_{k=0}^{m-1} \binom{m-1}{k} (m)_k (qm)_{m-1-k} (-1)^{m-1-k} x^{m-k} (1-x)^{qm-(m-1-k)},$$

where  $(m)_k$  is the falling factorial. Taking absolute values and integrating:

$$\begin{aligned} \int_0^1 \left| \frac{d^{m-1}}{dx^{m-1}} [x^m (1-x)^{qm}] \right| dx &\leq \sum_{k=0}^{m-1} \binom{m-1}{k} (m)_k (qm)_{m-1-k} \text{B}(m-k+1, (q-1)m+k+2) \\ &\leq \sum_{k=0}^{m-1} \binom{m-1}{k} (m)_k (qm)_{m-1-k} \frac{\Gamma(m-k+1)\Gamma((q-1)m+k+2)}{\Gamma(qm+3)} \\ &= 2^{m-1} \frac{\Gamma(m+1)\Gamma(qm+1)}{\Gamma(qm+3)}. \end{aligned}$$

Multiplying by the normalization  $\Omega_m^{(q)} \binom{(q+1)m}{m}$  gives us  $\tilde{W}_m^{(q)}$ :

$$\|\tilde{W}_m^{(q)(m-1)}\|_1 \leq \Omega_m^{(q)} \binom{(q+1)m}{m} 2^{m-1} \frac{m! (qm)!}{(qm+2)!} = \Omega_m^{(q)} 2^{m-1} \frac{((q+1)m)!}{(qm+2)!}.$$

Hence,

$$\|S_m^{(q)(m)}\|_1 \leq \Omega_m^{(q)} 2^m \frac{((q+1)m)!}{(qm)!}.$$

From [8], we obtain the Stirling–Robbins bounds

$$\sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N+\frac{1}{12N+1}} \leq N! \leq \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N+\frac{1}{12N}}.$$

Therefore,

$$\|S_m^{(q)(m)}\|_1 \leq c_1(m, q) \left(1 + \frac{1}{q}\right)^{qm} \left[\frac{2(q+1)m}{e}\right]^m \leq c_1(m, q) [2(q+1)m]^m, \quad (20)$$

where

$$c_1(m, q) := \Omega_m^{(q)} \sqrt{\frac{q+1}{q}} \exp\left(\frac{1}{12(q+1)m} - \frac{1}{12qm+1}\right).$$

From the trivial rearrangement  $S_m^{(q)}(x) = \nabla_h^m S_m^{(q)}(x) + [S_m^{(q)}(x) - \nabla_h^m S_m^{(q)}(x)]$ , we obtain the norm inequality

$$\|S_m^{(q)}\|_1 \leq \|\nabla_h^m S_m^{(q)}\|_1 + \|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1.$$

Applying the properties of  $\nabla_h^m S_m^{(q)}$  gives

$$\|S_m^{(q)}\|_1 \leq c_1(m, q) [\{2(q+1)hm\}^m + \{2(q+1)hm^2\}^m].$$

Set  $\delta := 2(q+1)hm^2$  for a fixed real  $\delta \in (0, 1)$ . Then,

$$\|S_m^{(q)}\|_1 \leq c_1(m, q) \left[ \left( \frac{\delta}{m} \right)^m + \delta^m \right].$$

Combining (17), (18) and the inequality above, we get

$$\left| \int_0^1 \text{Li}'_{2n+1}(x) S_m^{(q)}(x) dx \right| \leq C_3(m, q) \left[ \left( \frac{\delta}{m} \right)^m + \delta^m \right], \quad (21)$$

where  $C_3(m, q) := \zeta(2n)c_1(m, q)$ .  $\square$

Finally, combining (13), (14), (16), we obtain

$$|\Lambda_m^{(q)}| \leq D_m^{(q)} C_1(\alpha, m, q) \left[ 1 - \frac{C_2(\alpha, q)}{m} \right]^{-1} e^{-\gamma^{(q)}(\alpha)m} + D_m^{(q)} C_3(m, q) \left[ \left( \frac{\delta}{m} \right)^m + \delta^m \right]. \quad (22)$$

#### 4.3. Growth of Denominators

Recall Legendre's formula [2, Theorem 3.14], which gives the exponent of a prime  $p$  in  $N!$ :

$$v_p(N!) = \sum_{j=1}^{\lfloor \log_p N \rfloor} \left\lfloor \frac{N}{p^j} \right\rfloor.$$

For any integers  $N, a \geq 0$ , we have for each  $j$ ,

$$\left\lfloor \frac{a}{p^j} \right\rfloor \leq \left\lfloor \frac{N+a}{p^j} \right\rfloor - \left\lfloor \frac{N}{p^j} \right\rfloor \leq \left\lfloor \frac{a}{p^j} \right\rfloor + 1.$$

Compute

$$\begin{aligned} -v_p(F_{m,k}^{(q)}) &= -v_p(((q+1)m)!) - v_p((m+k)!) + v_p(m!) + v_p(((q+1)m+k+1)!) \\ &= [v_p(((q+1)m+k+1)!) - v_p(((q+1)m)!)] - [v_p((m+k)!) - v_p(m!)] \\ &\leq [v_p((k+1)!) + \lfloor \log_p((q+1)m+k+1) \rfloor] - v_p(k!) \\ &= v_p(k+1) + \lfloor \log_p((q+1)m+k+1) \rfloor. \end{aligned}$$

Since  $\text{den}(F_{m,k}^{(q)}) = \prod_p p^{\max(0, -v_p(F_{m,k}^{(q)}))}$ , the above inequality results in

$$\text{den}(F_{m,k}^{(q)}) \mid (k+1) \prod_p p^{\lfloor \log_p((q+1)m+k+1) \rfloor} = (k+1) \text{lcm}(1, 2, \dots, (q+1)m+k+1).$$

Taking the least common multiple over  $1 \leq k \leq K$ , one gets a bound of the form

$$D_m^{(q)} \mid \text{lcm}(1, 2, \dots, K+1)^{K+1} \text{lcm}(1, 2, \dots, (q+1)m+K+1),$$

where  $\kappa := 2n + 1$ . With  $K = \lceil \alpha m \rceil$  and  $\text{lcm}(1, 2, \dots, N) = e^{\psi(N)}$  (where  $\psi(N)$  is the Chebyshev psi function), we obtain the Rosser–Schoenfeld bound [9, (2.29)]

$$D_m^{(q)} \leq \exp((\kappa + 1)(\alpha m + 2) + (1 + q + \alpha)m + 2)[1 + o(1)] = c_2 \exp((1 + q + \lambda\alpha)m[1 + o(1)]), \quad (23)$$

where  $c_2 := \exp((2\kappa + 4)[1 + o(1)])$  and  $\lambda := \kappa + 2$ .

## 5. Proof of the Main Theorem

### 5.1. Parameter Selection

From (22), for exponential decay to zero, we must ensure that

$$1 + q + \lambda\alpha < \min(\gamma^{(q)}(\alpha), -\log \delta). \quad (24)$$

**Lemma 5.1** (Existence of admissible parameters). *With  $\lambda > 0$ , there exist parameters  $q \in \mathbb{N}, \delta \in (0, 1), \alpha > 0$  which satisfy (24).*

**Proof.** We first solve the inequality

$$1 + q + \lambda\alpha < \gamma^{(q)}(\alpha) = -(1 + q) \log(1 + q) - (1 + \alpha) \log(1 + \alpha) + (1 + q + \alpha) \log(1 + q + \alpha).$$

Let

$$g(\alpha) := 1 + q + \lambda\alpha + (1 + q) \log(1 + q) + (1 + \alpha) \log(1 + \alpha) - (1 + q + \alpha) \log(1 + q + \alpha).$$

We want the set of  $\alpha > 0$  with  $g(\alpha) < 0$ . We have

$$g'(\alpha) = \lambda - \log \frac{1 + q + \alpha}{1 + \alpha}, \quad g''(\alpha) = \frac{q}{(1 + \alpha)(1 + q + \alpha)} > 0.$$

So,  $g'$  is strictly increasing and  $g$  is strictly convex on  $(0, \infty)$ . Since  $g(0) = 1 + q > 0$  and the initial slope is  $g'(0) = \lambda - \log(1 + q)$ , we have the following two cases:

1. If  $\lambda \geq \log(1 + q)$ , then  $g'(0) \geq 0$ . Since  $g$  is convex and  $g(0) > 0$ , we then have  $g(\alpha) > 0$  for all  $\alpha > 0$ . Hence, no positive  $\alpha$  satisfies the inequality.
2. If  $\lambda < \log(1 + q)$ , then  $g'(0) < 0$ . By convexity,  $g'$  increases from  $g'(0) < 0$  to  $\lim_{\alpha \rightarrow \infty} g'(\alpha) = \lambda > 0$ ; so, there is exactly one  $\alpha^* > 0$  with  $g'(\alpha^*) = 0$ . The value of  $\alpha^*$  is

$$\alpha^* = \frac{q}{e^\lambda - 1} - 1,$$

and with this,

$$(g(\alpha^*))(\lambda, q) = (1 + q)[1 + \log(1 + q) - \lambda] - q \log \frac{q}{e^\lambda - 1}.$$

At small positive  $\alpha$ , we have  $g(\alpha) > 0$ , while  $g(\alpha) \rightarrow \infty$  as  $\alpha \rightarrow \infty$ ; hence, for there to exist a pair of positive roots of  $g(\alpha) = 0$ , we require that  $g(\alpha^*) < 0$ . Differentiating with respect to  $\lambda$ , we find that  $\partial_\lambda g(\alpha^*) = \frac{q}{e^\lambda - 1} - 1 > 0$ ; this means that  $g(\alpha^*)$  is strictly increasing in  $\lambda$  on  $(0, \log(1 + q))$ . Since  $\lim_{\lambda \rightarrow 0^+} g(\alpha^*) = -\infty$  and  $\lim_{\lambda \rightarrow \log(1 + q)^-} g(\alpha^*) = 1 + q > 0$ , by continuity and strict monotonicity in  $\lambda$ , there exists a unique  $\lambda^* \in (0, \log(1 + q))$  such that  $(g(\alpha^*))(\lambda^*, q) = 0$ . Therefore, for  $0 < \lambda < \lambda^*$ ,  $g$  has exactly two positive roots  $\alpha_1, \alpha_2$  (with  $0 < \alpha_1 < \alpha^* < \alpha_2$ ). Thus, the solution set of the strict inequality is the interval  $\alpha_1 < \alpha < \alpha_2$ .

From the second case, the requirement is  $\lambda < \log(1 + q)$ , or  $q > e^\lambda - 1$ . Fix such a  $q$ . Then, there exists an interval  $(\alpha_1, \alpha_2)$  such that  $1 + q + \lambda\alpha < \gamma^{(q)}(\alpha)$  for all  $\alpha \in (\alpha_1, \alpha_2)$ . Fix any  $\alpha$  in this interval. Then, it suffices to choose  $\delta$  small enough so that  $1 + q + \lambda\alpha < -\log \delta$ ; this guarantees (24).  $\square$

### 5.2. Irrationality Criterion and Completion of the Proof

With (24) satisfied,  $|\Lambda_m^{(q)}| \rightarrow 0$  as  $m \rightarrow \infty$ . This places us precisely within the framework of the following general criterion.

**Theorem 5.1** (Irrationality criterion). *Let  $x \in \mathbb{R}$ . Suppose there exist infinitely many integers  $A_m \neq 0$  and  $B_m$ , such that*

$$0 < |\Lambda_m| = |A_m x - B_m| \leq \epsilon_m, \quad (25)$$

where  $\epsilon_m \rightarrow 0$  as  $m \rightarrow \infty$ . Then,  $x$  is irrational.

**Proof.** Assume that  $x = \frac{p^*}{q^*} \in \mathbb{Q}$  is in lowest terms. Then,

$$|\Lambda_m| = \frac{|A_m p^* - B_m q^*|}{|q^*|}.$$

Hence, for every  $m$ , either  $|\Lambda_m| = 0$  or  $|\Lambda_m| \geq \frac{1}{|q^*|}$ . By (25),  $|\Lambda_m| \leq \epsilon_m \rightarrow 0$  as  $m \rightarrow \infty$ . So, for all sufficiently large  $m$ , we have  $\epsilon_m < \frac{1}{|q^*|}$ , contradicting the dichotomy above. Therefore,  $x$  cannot be rational.  $\square$

We thus conclude that each  $\zeta(2n+1)$  is irrational. This completes the proof of Theorem 1.1.

### Appendix A. Proof of $\|\nabla_h^m S_m^{(q)}\|_1 \leq h^m \|S_m^{(q)(m)}\|_1$ and

$$\|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1 \leq (hm)^m \|S_m^{(q)(m)}\|_1$$

**Proof.** We have

$$\nabla_h^1 S_m^{(q)}(x) = S_m^{(q)}(x) - S_m^{(q)}(x-h) = - \int_0^h S_m^{(q)'}(x-t_1) dt_1.$$

Applying induction, we arrive at the box-averaged representation

$$\begin{aligned} \nabla_h^m S_m^{(q)}(x) &= (-1)^m \int_0^h \cdots \int_0^h S_m^{(q)(m)}(x-t_1-\cdots-t_m) dt_1 \cdots dt_m \\ &= (-1)^m \int_{[0,h]^m} S_m^{(q)(m)}(x-t) d^m t, \end{aligned}$$

where  $t := t_1 + \cdots + t_m$  and  $d^m t := dt_1 \cdots dt_m$ .

Take absolute values and integrate in  $x$ . By Fubini's theorem, we get

$$\begin{aligned} \|\nabla_h^m S_m^{(q)}\|_1 &= \int_0^1 \left| \int_{[0,h]^m} S_m^{(q)(m)}(x-t) d^m t \right| dx \\ &\leq \int_{[0,h]^m} \int_0^1 |S_m^{(q)(m)}(x-t)| dx d^m t \\ &\leq \int_{[0,h]^m} \|S_m^{(q)(m)}\|_1 d^m t \\ &= \|S_m^{(q)(m)}\|_1 \int_{[0,h]^m} d^m t \\ &= h^m \|S_m^{(q)(m)}\|_1. \end{aligned}$$

Now, consider the Taylor series of  $S_m^{(q)}(x-kh)$  expanded to the first  $m$  terms:

$$S_m^{(q)}(x-kh) = \sum_{j=0}^{m-1} \frac{(-kh)^j}{j!} S_m^{(q)(j)}(x) + \frac{(-kh)^m}{(m-1)!} \int_0^1 (1-\theta)^{m-1} S_m^{(q)(m)}(x-kh\theta) d\theta.$$

The  $m$ th backward difference annihilates polynomials up to degree  $m - 1$ ; that is,

$$\sum_{k=0}^m (-1)^k \binom{m}{k} k^j = 0, \quad j = 0, 1, \dots, m - 1.$$

Hence, we get

$$\begin{aligned} S_m^{(q)}(x) - \nabla_h^m S_m^{(q)}(x) &= - \sum_{j=0}^{m-1} \frac{(-h)^j}{j!} S_m^{(q)(j)}(x) \left[ \sum_{k=1}^m (-1)^k \binom{m}{k} k^j \right] \\ &\quad - \frac{(-h)^m}{(m-1)!} \int_0^1 (1-\theta)^{m-1} \left[ \sum_{k=1}^m (-1)^k \binom{m}{k} k^m S_m^{(q)(m)}(x - kh\theta) \right] d\theta \\ &= - \frac{(-h)^m}{(m-1)!} \int_0^1 (1-\theta)^{m-1} \left[ \sum_{k=1}^m (-1)^k \binom{m}{k} k^m S_m^{(q)(m)}(x - kh\theta) \right] d\theta. \end{aligned}$$

Take absolute values and integrate:

$$\|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1 \leq \frac{h^m}{m!} \left[ \sum_{k=1}^m \binom{m}{k} k^m \right] \|S_m^{(q)(m)}\|_1.$$

Since  $\sum_{k=1}^m \binom{m}{k} k^m \leq (2m)^m$ , and  $2^m \leq m!$  for  $m \geq 4$ ,

$$\|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1 \leq (hm)^m \|S_m^{(q)(m)}\|_1$$

for those  $m$ .  $\square$

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