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New Boundary Harnack Principles for Divergence form Elliptic Equations with Right Hand Side

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Abstract: We establish new boundary Harnack principles for divergence form equations with right hand side. In Lipschitz domains, we prove new boundary Harnack principles when the right hand side exhibits polynomial decay. Moreover, in Hölder domains with the Hölder exponent $\alpha \in [1/2, 1]$, we prove new boundary Harnack principles when the right hand side exponential decay.

Keywords: boundary Harnark principles; divergence form; Lipschitz domains; Hölder domains

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1. Introduction

1.1. Background

In this paper, we study boundary Harnack principles for divergence form equations with right hand side. Boundary Harnack principles originally state that, if u and v are two positive harmonic functions in $\Omega \cap B_1$, and they both vanish on $\partial\Omega \cap B_1$, then they are comparable near the boundary $\partial\Omega$.

For elliptic equations without right hand side in Lipschitz domains, we introduce some related research results. Kemper initially proved boundary Harnack principles for harmonic functions by applying the uniqueness of the positive singularity principle and the properties of Kernel functions in [24], while Caffarelli, Fabes, Mortola and Salsa first proved boundary Harnack principles for divergence form operators with bounded measurable coefficients in [8]. In [18], Fabes, Garofalo, Marin-Malave, and Salsa extensively discussed the non-divergence form operators by Fatou theorems. In [23], Jerison and Kenig extended boundary Harnack principles for divergence form operators in non-tangentially accessible domains. Bass and Burdzy proved boundary Harnack principles for non-divergence form elliptic operators with bounded measurable coefficients by applying probabilistic techniques in [5]. More recently, De Silva and Savin discovered a simple yet unified proof of divergence and non-divergence form operators with bounded measurable coefficients in their work published in [13].

For elliptic equations with right hand side in Lipschitz domains, we present some relevant results. In [1], Mark Allen and Shahgolian proved boundary Harnack principles of the Laplace operators by using harmonic functions in spherical polar coordinates under polynomial decaying conditions of the right hand side. In [31], Ros-Oton and Torres-Latorre already proved boundary Harnack principles for divergence form operators with **continuous** coefficients and non-divergence form operators with bounded measurable coefficients. In addition, Ros-Oton and Torres-Latorre gave a counterexample to show that boundary Harnack principles do not hold in $\mathbb{R}^2_+$ for divergence form operators with bounded measurable coefficients, even though the L^q norm of the right hand side is controlled by a sufficiently small constant c_0.

For elliptic equations without right hand side in Hölder domains, we introduce some related background and research results. Boundary Harnack principles for non-divergence form operators in Hölder domains with the exponent $\alpha \in \left(\frac{1}{2}, 1\right]$ were first proven by Banuelos, Bass and Burdzy in [5]Bass1994. Fausto Ferrari first proved boundary Harnack principles for divergence form operators in Hölder domains with exponent $\alpha \in (0, 1]$ in [19]. De Silva and Savin proved boundary Harnack principles for divergence and non-divergence form operators in Hölder domains with exponent $\alpha \in \left(\frac{1}{2}, 1\right]$ by using a simpler method in [13]. Further, in [14], De Silva and Savin, by optimizing the previous approach, generalize this theory in Hölder domains with exponent $\alpha \in (0,1]$.

Based on our current investigation and understanding, boundary Harnack principles for elliptic equations with right hand side in Hölder domains has not been fully studied. In [31], Ros-Oton and Torres-Latorre gave a counterexample to show that boundary Harnack principles do not hold in Lipschitz domains with a large slope, even though the right hand side is bounded and less than a very small constant c_0. Obviously, this example further shows that boundary Harnack principles do not hold in Hölder domains, if without any decaying condition of the right hand side.

In this paper, we apply the positive extension theorem to obtain the decaying estimation of solutions of homogeneous divergence form equations in Lipschitz domains (Hölder domains). Because of the counterexamples mentioned above, by adding a polynomial(exponential) decaying condition to the right hand side, we get boundary Harnack principles in Lipschitz domains(Hölder domains with exponent $\alpha \in \left[\frac{1}{2}, 1\right]$) for divergence form operators with bounded measurable coefficients. At present, for the caes of boundary Harnack principles in Hölder domains with exponent $\alpha < \frac{1}{2}$, we still have not reached a corresponding conclusion.

Finally, the content of this paper is divided into four parts. The first part mainly introduces the main theorem of this paper. In the second part, we introduce some mathematical tools including the A-B-P estimation, the positive extension theorem, the decay theorem, the interior Harnack inequality and an upper bound estimation. The third part is to prove boundary Harnack principles in Lipschitz domains. The content of this part includes a decay lemma of the solution in a Lipschitz cone, a key iteration theorem and a key proposition. The last part is mainly to prove boundary Harnack principles in Hölder domains with exponent $\alpha \in \left[\frac{1}{2}, 1\right]$. The arrangement of the content and the idea of the argument in the last part are basically the same as in the third part.

1.2. The Main Notation

- $x = (x', x_n)$=$(x_1, ..., x_{n-1}, x_n)$ is a point in $\mathbb{R}^n$, $n > 1$.
- $B_\rho(y) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$ is an open ball in $\mathbb{R}^n$ with centre y and radius ρ. $B'_\rho(y)$ is an open ball in $\mathbb{R}^{n-1}$ with centre y' and radius ρ.
- Let $h \in \mathbb{R}_+$. The set $\mathcal{C}_\eta = \{x \in \mathbb{R}^n : \eta|x'| < x_n < h\}$ is called the Lipschitz cone of height h with the Lipschitz constant η. The set $E_\eta = \left\{x \in \mathbb{R}^n : \frac{\eta}{2}|x'|^\alpha < x_n < h\right\}$ is called the Hölder cone of height h with the Hölder constant $\frac{\eta}{2}$ and the range $\alpha \in (0,1)$.
- $d(x) = dist(x, \partial\Omega)$ is the distance between x and $\partial\Omega$.
- Given a function u, we write $u_\pm = \max\{\pm u, 0\}$.
- $|\Omega|$ is the Lebesgue measure of the domain Ω, and $diam\Omega$ is the diameter of the domain Ω.
- $e_i = (\underbrace{0, ...0}_{i-1}, 1, \underbrace{0, ...0}_{n-i}) \in \mathbb{R}^n$.

1.3. Setting

In what follows, let $\mathcal{L}$ be a divergence form elliptic operator with bounded measurable coefficients, i.e.

$$\mathcal{L}u = Div(A(x)\nabla u) \qquad with \qquad \lambda\mathcal{I} \leqslant A(x) \leqslant \Lambda\mathcal{I}, \tag{1.1}$$

where $0 < \lambda \leqslant \Lambda$.

Definition 1. *Let L be a positive constant. Let $\omega : B'_1 \to \mathbb{R}$ be a Lipschitz function satisfied $\omega(0) = 0$ and $|\omega(x') - \omega(y')| \leq L|x' - y'|$, for any x' and y' in B'_1. We say Ω is a Lipschitz domain with the Lipschitz constant L if $\Omega = \left\{(x', x_n) \in B'_1 \times \mathbb{R} : x_n > \omega(x')\right\}$.*

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Definition 2. *Let constants $H > 0$ and $\alpha \in (0, 1]$. Let $\omega : B_1' \to \mathbb{R}$ be a Hölder function satisfied $\omega(0) = 0$ and $|\omega(x') - \omega(y')| \leq H|x' - y'|^\alpha$, for any x' and y' in B_1'. We say Ω is a Hölder domain with exponent α and the Hölder constant H if $\Omega = \left\{(x', x_n) \in B_1' \times \mathbb{R} : x_n > \omega(x')\right\}$.*

Definition 3. *(De Giorgi super-class) Let $E \subset \mathbb{R}^n$ be an open set. We say $u \in DG_-(E, C)$ if $u \in W^{1,2}_{loc}(E)$ satisfies*

$$\int_{B_r(y)} |\nabla(u-k)_-|^2 dx \leq \frac{C}{(R-r)^2} \int_{B_R(y)} |(u-k)_-|^2 dx,$$

for all $k \in \mathbb{R}$, $B_r(y) \subset B_R(y) \subset E$, where C is a positive constant depending on λ, Λ and the dimension n.

1.4. Main Results

Theorem 1. *There exists a large positive constant q^* and small positive constants η and c_0 such that the following holds.*

Let Ω be a Lipschitz domain with the Lipschitz constant $L < \frac{\eta}{2}$. Let $u > 0$ and $v > 0$ be weak solutions of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v = g & \text{in } \Omega \cap B_1 \\ v = 0 & \text{on } \partial\Omega \cap B_1, \end{cases} \tag{1.2}$$

with $\|f\|_{L^\infty(B_r(y))} \leq c_0 r^{q^*-1}$ and $\|g\|_{L^\infty(B_r(y))} \leq c_0 r^{q^*-1}$, for any $r \in (0, 1)$, $y \in \partial\Omega \cap B_1$. Suppose further that $v(e_n/2) \geq 1$ and $u(e_n/2) \leq 1$.

Then we have

$$u \leq Cv \quad \text{in } B_{1/2} \cap \Omega, \tag{1.3}$$

where C, c_0, η and q^* depend on λ, Λ and the dimension n.

With Theorem 1, it is straightforward to obtain the following Remark. If necessary, we can take a smaller constant c_0.

Remark 1. *Let $u > 0$ and $v > 0$ satisfy (1.2) with f and g satisfying*

$$\begin{cases} |f(x)| \leq c_0 d(x)^{q^*-1} \\ |g(x)| \leq c_0 d(x)^{q^*-1} \end{cases} \quad \text{or} \quad \begin{cases} |f(x)| \leq c_0 [\Gamma_h(x)]^{q^*-1} \\ |g(x)| \leq c_0 [\Gamma_h(x)]^{q^*-1}, \end{cases}$$

where $\Gamma_h(x) = x_n - \omega(x')$.

Then the boundary Harnack principle (1.3) also holds.

Theorem 2. *Let Ω be a Hölder domain with $\alpha \in \left[\frac{1}{2}, 1\right]$ and the Hölder constant $H < \frac{4-\epsilon}{4\sqrt{3}}$. There exists a large positive constant C_2 and a small positive constant $c_0 > 0$ such that the following holds.*

Let $u > 0$ and $v > 0$ be weak solutions of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v = g & \text{in } \Omega \cap B_1 \\ v = 0 & \text{on } \partial\Omega \cap B_1, \end{cases} \tag{1.4}$$

with $\|f\|_{L^\infty(B_r(y))} \leq c_0 r^{-2} \exp(-4C_2 r^{-1})$ and $\|g\|_{L^\infty(B_r(y))} \leq c_0 r^{-2} \exp(-4C_2 r^{-1})$, for any $r \in (0, 1/2)$, $y \in \partial\Omega \cap B_1$. Suppose further that $v(e_n/2) \geq 1$ and $u(e_n/2) \leq 1$.

Then we have

$$u \leq Cv \quad \text{in } B_{1/2} \cap \Omega, \tag{1.5}$$

where C, c_0, C_2 depend on α, λ, Λ and the dimension n.

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2. Preliminaries

In this section, we introduce five common theorems to prepare for later proofs.

The following theorem is called the A-B-P estimation. In the proof of our main theorem, we only need to consider the case of $p = n$.

Theorem 3. *([22, Theorem 8.16]) Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain. Let $u \in C(\overline{\Omega})$ and satisfy $\mathcal{L}u \geqslant f$ with $f \in L^p(\Omega)$, $p > n/2$.*

Then

$$\sup_{\Omega} u \leqslant \sup_{\partial\Omega} u + C\|f\|_{L^p(\Omega)},$$

where C depend on diamΩ, p, λ, Λ and the dimension n.

The following theorem is called the positive expansion. This theorem plays a crucial role in the section 3.

Theorem 4. *([15, Theorem 9.1]) Let constants $M > 0$, $\epsilon \in (0,1)$ and $r_2 \in (\frac{1}{2}, 1)$. Let $E \subset \mathbb{R}^n$ be an open set. For any non-negative function $u \in DG_-(E, C)$. Assume that $u \geq M$ in $B_{\epsilon R}(y)$ and $B_{2R} \subset E$.*

Then we have

$$u \geq \sigma\epsilon^{q_1} M \quad in \;\; B_R(y),$$

where $q_1 = C'(\epsilon r_2)^{-2(n-1)}(1 - r_2)^{-(2n+3)}$ and $\sigma = \exp\left\{-C'(\epsilon r_2)^{-2(n-1)}(1 - r_2)^{-(2n+2)}\right\}$ with C' is a positive constant depending on λ, Λ and the dimension n.

Moreover, if $\epsilon \leqslant e^{-1}$, we have

$$u \geqslant \epsilon^q M \quad in \; B_R(y),$$

where the positive constant $q := 2q_1$.

The following theorem is the decay theorem of the weak subsolution in the ball.

Theorem 5. *([31, Corollary 2.5]) Let $R \in (0,1]$ and u satisfy*

$$\begin{cases} \mathcal{L}u \geqslant f & in \; B_R \\ u \leqslant 1 & in \; B_R, \end{cases}$$

where $f \in L^n(B_R)$. Suppose that there exists $\theta \in (0,1]$ such that $|\{u \leqslant 0\} \cap B_R| \geqslant \theta|B_R| > 0$ and $\|f\|_{L^n(B_R)} \leqslant \xi(\theta)$.

Then we have

$$\sup_{B_{R/2}} u \leqslant 1 - \gamma(\theta),$$

where $\xi(\theta) > 0$ and $\gamma(\theta) \in (0,1)$ depend on θ, λ, Λ and the dimension n.

The following theorem is the interior Harnack inequality.

Theorem 6. *([31, Lemma 3.6 and Lemma 3.7]) Let Ω be a Lipschitz domain with the Lipschitz constant $L < 1/16$. Let $u > 0$ be a weak solution of*

$$\begin{cases} \mathcal{L}u = f & in \; \Omega \cap B_1 \\ u = 0 & on \; \partial\Omega \cap B_1, \end{cases}$$

where $f \in L^n(B_1)$. We define $A := \{x \in \Omega \cap \overline{B_{1-\delta}} : \Gamma_h(x) \geq \delta\}$, where $\Gamma_h(x) = x_n - \omega(x')$, $\delta \in (0, 1/3)$.

Then we have

$$\sup_A u \leq C\left(\inf_A u + \|f\|_{L^n(B_1)}\right),$$

where C depend on δ, λ, Λ *and the dimension* n.

Moreover, if $u(e_n/2) \leq 1$, *then there exist positive constants* p *and* C_p *depending on* λ, Λ, $\|f\|_{L^n(B_1)}$ *and the dimension* n, *such that*

$$\|u\|_{L^p(B_1)} \leq C_p\ .$$

The following proposition is an upper bound estimate of the weak solution.

Proposition 1. *([31, Proposition 3.8]) Let* Ω *be a Lipschitz domain and* u *be a weak solution of*

$$\begin{cases} \mathcal{L}u = f & in\ \Omega \cap B_1 \\ u = 0 & on\ \partial\Omega \cap B_1, \end{cases}$$

where $f \in L^n(B_1)$.

If $u \in L^p(\Omega \cap B_1)$ *for any* $p > 0$, *then*

$$\sup_{B_r} u \leq M(n, p, \lambda, \Lambda, r)\left(\|u\|_{L^p(B_1)} + \|f\|_{L^n(B_1)}\right),$$

where $r \in (0, 1)$.

3. Proof of Theorem 1

The following lemma is the decay estimate of the weak supersolution in the Lipschitz cone. This lemma plays a crucial role in the proof of the following theorem.

Lemma 1. *There exists a small constant* $\eta > 0$ *and a large positive constant* q^* *depending on* λ, Λ *and the dimension* n, *such that the following holds.*

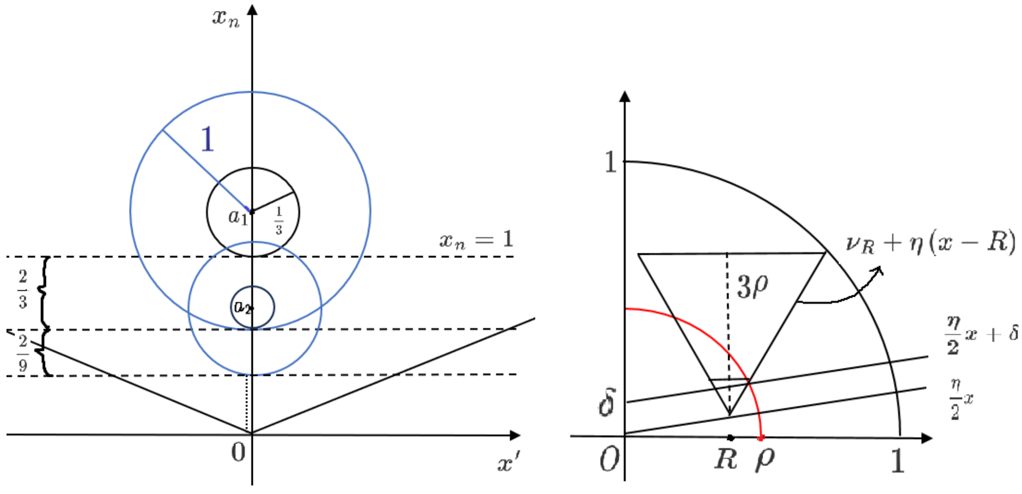
Let u *satisfy*

$$\begin{cases} \mathcal{L}u \leq 0 & in\ \mathcal{C}_\eta \\ u \geq 1 & in \{x_n \geq 1\} \cap \mathcal{C}_\eta \\ u \geq 0 & on\ \partial\mathcal{C}_\eta, \end{cases} \tag{3.1}$$

where the cone $\mathcal{C}_\eta = \{x \in \mathbb{R}^n : \eta|x'| < x_n < 3\}$.

Then for any $t \in (0, \frac{1}{3})$, *we have*

$$u(te_n) \geq t^{q^*}.$$



(a) (b)

Proof. Let $\epsilon = \frac{1}{3}$ and $r_2 = \frac{6}{7}$ in Theorem 4. We take $a_i = \frac{4}{3}\left(\frac{1}{3}\right)^{i-1}$ and construct two series of balls and denote them as $\left\{B_{\left(\frac{1}{3}\right)^i}(0,a_i)\right\}$, $\left\{B_{\left(\frac{1}{3}\right)^{i-1}}(0,a_i)\right\}$ for all $i \in \mathbb{N}^+$, where $(0,a_i)$ represents a point in $\mathbb{R}^n$ with $x' = 0$ and $x_n = a_i$ (see figure(a)). In addition, we know that $\sum_{i=1}^{\infty} \frac{2}{3^i} = 1$.

Obviously, $u(x) \geqslant 1$ in $B_{\frac{1}{3}}(0,a_1)$ and by Theorem 4, we have $u(x) \geqslant \left(\frac{1}{3}\right)^q$ in $B_1(0,a_1)$. Thus, $u(x) \geqslant \left(\frac{1}{3}\right)^q$ in $B_{\frac{1}{3^2}}(0,a_2)$ and by Theorem 4 again, we have $u(x) \geqslant \left(\frac{1}{3}\right)^{2q}$ in $B_{\frac{1}{3}}(0,a_2)$. Repeating the above process, we have $u(x) \geqslant \left(\frac{1}{3}\right)^{(k+1)q}$ in $B_{\frac{1}{3^k}}(0,a_{k+1})$, for all $k \in \mathbb{N}$.

For any $t \in \left(0,\frac{1}{3}\right)$, taking $k_0 \in \mathbb{N}^+$, such that $1 - 2\sum_{i=1}^{k_0+1}\left(\frac{1}{3}\right)^i \leqslant t \leqslant 1 - 2\sum_{i=1}^{k_0}\left(\frac{1}{3}\right)^i$, then we have $te_n \in B_{\left(\frac{1}{3}\right)^{k_0}}(0,a_{k_0+1})$ and $\left(\frac{1}{3}\right)^{k_0+1} \leqslant t \leqslant \left(\frac{1}{3}\right)^{k_0}$. Hence, we have $u(te_n) \geqslant \left(\frac{1}{3}\right)^{(k_0+1)q}$. Then $\frac{k_0+1}{k_0} \leq 2$ implies

$$\left(\frac{1}{3}\right)^{(k_0+1)q} \geqslant \left[\left(\frac{1}{3}\right)^{k_0}\right]^{2q} \geqslant t^{2q}.$$

Taking $q^* := 2q$, then we have

$$u(te_n) \geqslant t^{2q} = t^{q^*},\ \textit{for all}\ t \in \left(0,\frac{1}{3}\right).$$

Finally, if we take $\eta = \frac{1}{7\sqrt{n-1}}$, then all the balls will be located in cone $\mathcal{C}_\eta$. □

The following theorem shows that, by constructing an iteration, the solution of $\mathcal{L}u = f$ is positive at some distance from the boundary.

Theorem 7. *Let Ω be a Lipschitz domain with the Lipschitz constant $L < \frac{\eta}{2}$. The positive constants η and q^* are determined by Lemma 1. Let u satisfy*

$$\begin{cases} \mathcal{L}u = f & \textit{in } \Omega \cap B_1 \\ u = 0 & \textit{on } \partial\Omega \\ u \geq 1 & \textit{in } \Omega \cap \{x \in B_1 : \Gamma_h(x) > \delta\} \\ u \geq -\epsilon & \textit{in } \Omega \cap B_1, \end{cases} \tag{3.2}$$

where $\|f\|_{L^\infty(B_r(y))} \leq c_0 r^{q^*-1}$, *for any* $r \in (0,1)$, $y \in \partial\Omega \cap B_1$.

Then we have

$$\begin{cases} u \geq \rho^{q^*+1} & in\ \Omega \cap \{x \in B_\rho : \Gamma_h(x) > \rho\delta\} \\ u \geq -\rho^{q^*+1}\epsilon & in\ \Omega \cap B_\rho, \end{cases} \tag{3.3}$$

for some small enough constants $\rho, \epsilon, \delta, c_0 \in (0,1)$ *depending on* λ, Λ *and the dimension* n.

Proof. Now that, we prove the first inequality in (3.3). For any point $(x_0{}', x_{0n}) \in \{x \in B_\rho : \rho\delta < \Gamma_h(x) < \delta\}$, we write $\widetilde{x_0} := (x_0', \omega(x_0'))$ and the cone

$$\mathcal{C}_{\rho,\eta} := \widetilde{x_0} + \rho\mathcal{C}_\eta = \{x \in \mathbb{R}^n : \eta|x' - x_0{}'| < x_n - \omega(x_0{}') < 3\rho\}$$

and the positive upper cone

$$\mathcal{C}^+_{\rho,\eta} := \mathcal{C}_{\rho,\eta} \cap \{x_n > \omega(x_0{}') + \rho\},$$

where $\mathcal{C}_\eta = \{x \in \mathbb{R}^n : \eta|x'| < x_n < 3\}$.

Let $\rho = 3\delta$, $R \in [0,\rho]$, $\nu_R := \frac{\eta}{2}R$ and $x_\rho = \frac{\rho}{\eta} + R$ (see Figure 2). We have

$$\frac{\eta}{2}x_\rho + \delta = \frac{\eta}{2}\left(\frac{\rho}{\eta} + R\right) + \frac{\rho}{3} = \frac{5}{6}\rho + \nu_R.$$

Obviously, $\nu_R + \rho > \nu_R + \frac{5}{6}\rho$ implies that $C^+_{\rho,\eta} \subset \Omega \cap \{x \in B_1 : \Gamma_h(x) > \delta\}$. Now we take δ small enough, then we have $\mathcal{C}_{\rho,\eta} \subset \Omega \cap B_1$. Because $(x_0{}', x_{0n}) \in \{x \in B_\rho : \rho\delta < \Gamma_h(x) < \delta\}$, we have

$$\rho\delta < x_{0n} - \omega(x_0') < \delta \quad and\ then \quad \frac{\rho}{3} < \frac{x_{0n} - \omega(x_0')}{\rho} < \frac{1}{3}.$$

We define

$$\widetilde{u}(x) := u(\widetilde{x_0} + \rho x) + \epsilon \quad and \quad \Phi(x) := \mathcal{L}\widetilde{u}(x) = \rho^2 f(\widetilde{x_0} + \rho x).$$

Let $\widetilde{u} = u_1 + u_2$ satisfy

$$\begin{cases} \mathcal{L}u_1 = 0 & in\ \mathcal{C}_\eta \\ u_1 = \widetilde{u} & on\ \partial\mathcal{C}_\eta \end{cases} \quad and \quad \begin{cases} \mathcal{L}u_2 = \Phi & in\ \mathcal{C}_\eta \\ u_2 = 0 & on\ \partial\mathcal{C}_\eta. \end{cases}$$

By Theorem 3, we have

$$\begin{aligned} \|u_2\|_{L^\infty(\mathcal{C}_\eta)} &\leqslant C\|\Phi\|_{L^n(\mathcal{C}_\eta)} \leqslant C\rho^2\left(\int_{\mathcal{C}_\eta} |f(\widetilde{x_0} + \rho x)|^n dx\right)^{\frac{1}{n}} \\ &= C\rho\left(\int_{\mathcal{C}_{\rho,\eta}} |f(y)|^n dy\right)^{\frac{1}{n}} \leqslant C'\rho\|f\|_{L^\infty(B_\rho(\widetilde{x_0}))}|B_\rho(\widetilde{x_0})|^{\frac{1}{n}} \\ &\leqslant C'C(n)c_0\rho^{q^*+1}. \end{aligned}$$

On the other hand, we apply Lemma 1 to u_1 to result that $u_1(te_n) \geq t^{q^*}$, for any $t \in (0, \frac{1}{3})$. We have

$$u\big((x_0{}', \omega(x_0{}')) + \rho te_n\big) \geq t^{q^*} - C'C(n)c_0\rho^{q^*+1} - \epsilon, \quad for\ any\ t \in (0, \frac{1}{3}).$$

Therefore, let $\rho \leqslant \left(\frac{1}{3}\right)^{q^*+1}$, $\epsilon \leqslant (\rho)^{q^*+1}$, $c_0 \leqslant \frac{1}{C'C(n)}$, then

$$\begin{aligned} u(x_0) &\geqslant u\left((x_0', \omega(x_0')) + \rho\frac{x_{0n} - \omega(x_0')}{\rho} e_n\right) \\ &\geqslant \left(\frac{x_{0n} - \omega(x_0')}{\rho}\right)^{q^*} - C'C(n)c_0\rho^{q^*+1} - \epsilon \\ &\geqslant \left(\frac{\rho}{3}\right)^{q^*} - C'C(n)c_0\rho^{q^*+1} - (\rho)^{q^*+1} \\ &\geqslant \rho^{q^*+1}. \end{aligned}$$

Next, we prove the second inequality in (3.3). Let $x_0 \in B_{1-2\delta}$ and $\Gamma_h(x) \leqslant \delta$ (recalling $\delta = \rho/3$). We define $v(x) := \frac{u(x)^-}{\epsilon} \geqslant 0$. Extending v by 0 below $\partial\Omega$, then we have $\mathcal{L}v \geqslant -\frac{f^+}{\epsilon}$ in $B_{2\delta}(x_0)$. Let $\mathcal{C}_z$ be the downwards cone with slope η and vertex in $z := (x_0', \omega(x_0'))$. Then we have $v = 0$ in $\mathcal{C}_z \cap B_{2\delta}(x_0)$ and $|\mathcal{C}_z \cap B_{2\delta}(x_0)| \geqslant C(\eta)|B_{2\delta}(x_0)|$. Taking c_0 small enough, we have

$$\begin{aligned} \left(\int_{B_{2\delta}(x_0)} \left|\frac{f^+}{\epsilon}\right|^n\right)^{\frac{1}{n}} &\leqslant \left(\int_{B_{3\delta}(z)} \left|\frac{f^+}{\epsilon}\right|^n\right)^{\frac{1}{n}} \leqslant \frac{1}{\epsilon}\|f\|_{L^n(B_{3\delta}(z))} \\ &\leqslant \frac{3\delta C(n)}{\epsilon}\|f\|_{L^\infty(B_{3\delta}(z))} \leqslant c_0 C(n)\frac{(3\delta)^{q^*}}{\epsilon} \\ &\leqslant \xi(\eta), \end{aligned} \tag{3.4}$$

where $\xi(\eta)$ come from Theorem 5. By Theorem 5, we have $v \leqslant 1-\gamma$ in $B_\delta(x_0)$.

Now we define $v_j(x) := \frac{v}{(1-\gamma)^j}(x)$ in $B_\delta(x_0)$ with $x_0 \in B_{1-2j\delta}$ and $\Gamma_h(x_0) \leqslant \delta$. By mathematical induction, it is easy to show that $v \leqslant (1-\gamma)^j$ in $B_{1-2j\delta}$, i.e. $u \geqslant -(1-\gamma)^j\epsilon$ in $B_{1-2j\delta}$. We take $j_0 \in \mathbb{N}^+$ satisfied $1 - 2(j_0+1)\delta \leqslant \rho \leqslant 1 - 2j_0\delta$ (recalling $\rho = 3\delta$). Hence, we have

$$\frac{1}{\frac{2}{3}j_0 + \frac{5}{3}} \leqslant \rho \leqslant \frac{1}{\frac{2}{3}j_0 + 1}.$$

We take ρ small enough, such that

$$(1-\gamma)^{j_0} \leqslant \left(\frac{1}{\frac{2}{3}j_0 + \frac{5}{3}}\right)^{q^*+1},$$

then $(1-\gamma)^{j_0} \leqslant \rho^{q^*+1}$.

As same as (3.4), we have

$$\begin{aligned} \left(\int_{B_{2\delta}(x_0)} \left|\frac{f^+}{(1-\gamma)^{j_0}\epsilon}\right|^n\right)^{\frac{1}{n}} &\leqslant \frac{1}{(1-\gamma)^{j_0}\epsilon}\|f\|_{L^n(B_{3\delta}(z))} \\ &\leqslant c_0 C(n)\frac{(3\delta)^{q^*}}{(1-\gamma)^{j_0}\epsilon} \\ &\leqslant \xi(\eta). \end{aligned}$$

By $1 - 2j_0\delta \geqslant \rho$ with $(1-\gamma)^{j_0} \leqslant \rho^{q^*+1}$ and $u \geqslant -(1-\gamma)^{j_0}\epsilon$ in $B_{1-2j_0\delta}$, then we have $u \geqslant -\rho^{q^*+1}\epsilon$ in B_ρ. □

Now, we iterate Theorem 7 to obtain the following proposition.

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Proposition 2. *Let Ω be a Lipschitz domain with the Lipschitz constant $L < \frac{\eta}{2}$. The positive constants η and q^* are determined by Lemma 1. Let u be a weak solution of (3.2) with $\|f\|_{L^\infty(B_r(y))} \leq c_0 r^{q^*-1}$, for any $r \in (0,1)$, $y \in \partial\Omega \cap B_1$.*

Then we have

$$\begin{cases} u > 0 & in\ \Omega \cap B_{2/3} \\ u(te_n) \geqslant t^{q^*+1} & for\ all\ t \in (0,1). \end{cases}$$

Proof. Let $u_0(x) := u(x)$ and $f_0(x) := f(x)$. We define

$$u_{j+1}(x) := \frac{u_j(\rho x)}{\rho^{q^*+1}}, \quad f_{j+1}(x) := \frac{f_j(\rho x)}{\rho^{q^*-1}}.$$

where $x \in \Omega_{j+1}$ and $\rho x \in \rho\Omega_{j+1} := \Omega_j$. Obviously, $L_j \geq L_{j+1}$, where L_j is the Lipschitz constant of Ω_j.

First, we already have $\|f_0\|_{L^\infty(B_r(y))} \leq c_0 r^{q^*-1}$, for any $r \in (0,1)$, $y \in \partial\Omega \cap B_1$. By mathematical induction, we have

$$\begin{cases} \mathcal{L}u_j = f_j & in\ \Omega_j \cap B_1 \\ u_j = 0 & on\ \partial\Omega_j \end{cases} with \begin{cases} u_j \geq 1 & in\ \Omega_j \cap \{x \in B_1 : \Gamma_h(x) > \delta\} \\ u_j \geq -\epsilon & in\ \Omega_j \cap B_1, \end{cases}$$

where $\|f_j\|_{L^\infty(B_r(y))} \leq c_0 r^{q^*-1}$, for any $r \in (0,1)$, $y \in \partial\Omega \cap B_1$. By Theorem 7, we have

$$\begin{cases} u_j \geq \rho^{q^*+1} & in\ \Omega_j \cap \{x \in B_\rho : \Gamma_h(x) > \rho\delta\} \\ u_j \geq -\rho^{q^*+1}\epsilon & in\ \Omega_j \cap B_\rho, \end{cases}$$

then

$$\begin{cases} u_{j+1} \geq 1 & in\ \Omega_{j+1} \cap \{x \in B_1 : \Gamma_h(x) > \delta\} \\ u_{j+1} \geq -\epsilon & in\ \Omega_{j+1} \cap B_1. \end{cases}$$

Moreover, we have

$$\begin{aligned} \|f_{j+1}\|_{L^\infty(B_r(y))} &= \frac{1}{\rho^{q^*-1}} \|f_j(\rho x)\|_{L^\infty(B_r(y))} \\ &\leqslant \frac{c_0}{\rho^{q^*-1}} (\rho r)^{q^*-1} \\ &= c_0 r^{q^*-1}. \end{aligned}$$

So for all $j \in \mathbb{N}$, we have $u_j(te_n) \geqslant 1$, for every $t \in (\delta, 1)$.

For any $t \in (0,1)$, there exists $j_0 \in \mathbb{N}$ such that $\delta < \frac{t}{\rho^{j_0}} < 1$. Thus, we have $t < \rho^{j_0}$ and

$$u(te_n) = u_0(te_n) = \rho^{q^*+1} u_1\left(\frac{te_n}{\rho}\right) = \ldots = \rho^{j_0(q^*+1)} u_{j_0}\left(\frac{te_n}{\rho^{j_0}}\right) \geqslant t^{q^*+1}.$$

Finally, taking ϵ, ρ and c_0 smaller, then $u(x_0 + \frac{x}{3})$ also satisfy (3.2) for any $x_0 \in B_{2/3} \cap \partial\Omega$. By translating the coordinates, we get

$$u\left(x_0 + t\frac{e_n}{3}\right) > 0, \quad for\ all\ t \in (0,1).$$

Then this implies

$$u > 0 \quad in \quad \Omega \cap B_{2/3}.$$

□

3.1. Proof of Theorem 1

Proof. Let Ω be a Lipschitz domain with the Lipschitz constant $L < \eta/2$. The positive constants η and q^* are determined by Lemma 1. By Theorem 6 and Proposition 1, we have $u \leqslant M$ in $B_{3/4}$. We consider v in the set $E := \{x \in \bar{B}_{3/4} : \Gamma_h(x) \geqslant (3\delta)/4\}$. Obviously, $E \subset \left\{x \in \bar{B}_{1-(3\delta)/4} : \Gamma_h(x) \geqslant (3\delta)/4\right\}$. By Theorem 6 and $v(e_n/2) \geqslant 1$, we have

$$v \geqslant C^{-1} - \|f\|_{L^n(B_1)} \geqslant C^{-1} - C(n)\|f\|_{L^\infty(B_1)} \geqslant C^{-1} - C(n)c_0 \geqslant (2C)^{-1} := m \quad \textit{in } E,$$

where the constant c_0 is small enough such that $C^{-1} - C(n)c_0 \geqslant (2C)^{-1}$.

We define

$$\varphi(x) := \frac{1+\epsilon}{m}v(x) - \frac{\epsilon}{M}u(x)$$

where $\epsilon > 0$ is chosen later. Once we prove that $\varphi > 0$ in $B_{1/2}$, by choosing $C = M(1+\epsilon)/(m\epsilon)$, then $Cv - u > 0$ follows.

Obviously, $\varphi \geqslant v/m \geqslant 1$ in E, and $\varphi \geqslant -\epsilon$ in $B_{3/4}$. We have

$$\begin{aligned}\|\mathcal{L}\varphi\|_{L^\infty(B_r(y))} &\leqslant \frac{1+\epsilon}{m}\|\mathcal{L}v\|_{L^\infty(B_r(y))} + \frac{\epsilon}{M}\|\mathcal{L}u\|_{L^\infty(B_r(y))} \\ &\leqslant \left(\frac{1+\epsilon}{m} + \frac{\epsilon}{M}\right)c_0 r^{q^*-1}, \quad \textit{for any } r \in (0,1)\,,\ y \in \partial\Omega.\end{aligned}$$

Let $\widetilde{\varphi}(x) := \varphi(3x/4)$. Then $\widetilde{\varphi} \geqslant 1$ in $\Omega \cap \{x \in B_1 : \Gamma_h(x) \geqslant \delta\}$ and $\widetilde{\varphi} \geqslant -\epsilon$ in $\Omega \cap B_1$. Taking $\epsilon, c_0 > 0$ small enough and applying Proposition 2, then we get $\widetilde{\varphi} > 0$ in $B_{2/3}$. Thus, $\varphi > 0$ in $B_{1/2}$. □

4. Proof of Theorem 2

A simple lemma is given here to prepare for the following iteration of the ball in the Hölder cone.

Lemma 2. *Let constants $\epsilon \in (0,1), \eta > 0$ and $\alpha \in \left[\frac{1}{2}, 1\right]$. Then there exists a decreasing and positive sequence $\{R_k\}_{k=1}^\infty$ satisfied $R_k \to 0$ as $k \to +\infty$ and*

$$\epsilon R_k^\alpha = 1 - \eta \sum_{j=1}^{k} R_j, \tag{4.1}$$

for any $k \in \mathbb{N}^+$. Moreover, there exists a constant $C_1 > 0$ such that

$$\epsilon R_k^{\frac{1}{2}} \leqslant \frac{C_1\epsilon^2}{\eta k}, \quad \textit{for any } k \in \mathbb{N}^+. \tag{4.2}$$

Proof. We apply mathematical induction to prove (4.1). First, we show that there exists R_1 satisfying $\epsilon R_1^\alpha = 1 - \eta R_1$. Let $\varphi(t) := 1 - \eta t - \epsilon t^\alpha$ and note that $\varphi(0) = 1 > 0$, $\varphi\left(\frac{1}{\eta}\right) = -\epsilon\left(\frac{1}{\eta}\right)^\alpha < 0$ with $\varphi'(t) < 0$ in $\left(0, \frac{1}{\eta}\right)$. By the zero point theorem, there exists R_1 such that $1 - \eta R_1 - \epsilon R_1^\alpha = 0$. By mathematical induction, we suppose that $\epsilon R_k^\alpha = 1 - \eta \sum_{j=1}^{k} R_j$.

Next, we will show (4.1) for $k+1$. Denoting $a_k = 1 - \eta \sum_{j=1}^{k-1} R_j$, then we have $1 - \eta \sum_{j=1}^{k} R_j = 1 - \eta \sum_{j=1}^{k-1} R_j - \eta R_k = a_k - \eta R_k$. We define

$$\begin{aligned}\Phi(t) &:= a_k - \eta R_k - \eta t - \epsilon t^\alpha \\ &= \epsilon R_k^\alpha - \epsilon t^\alpha - \eta t.\end{aligned}$$

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Then we have $\Phi(0) = \epsilon R_k^\alpha$ and $\Phi(R_k) = -\eta R_k^\alpha < 0$ with $\Phi'(t) < 0$ in $(0, R_k)$. By the zero point theorem, there exists R_{k+1} satisfied $0 < R_{k+1} < R_k$ and $a_k - \eta R_k - \eta R_{k+1} - \epsilon R_{k+1}^\alpha = 0$, i.e. $\epsilon R_{k+1}^\alpha = a_k - \eta(R_k + R_{k+1})$.

We deserve that $0 < R_{k+1} < R_k$ and

$$\epsilon R_k - \epsilon R_{k+1} = \eta R_{k+1}. \tag{4.3}$$

We have $R_\infty = \lim_{k\to\infty} R_k = 0$. By (4.3) and letting $k \to \infty$, we have $R_\infty = 0$. By (4.1), we get $\eta \sum_{j=1}^{\infty} R_j = 1$.

Finally, we show (4.2). Obviously, if $\alpha = \frac{1}{2}$ holds true, then $\alpha \in [\frac{1}{2}, 1]$ holds even more. So, we consider $\alpha = \frac{1}{2}$. We have $\epsilon R_k^{\frac{1}{2}} - \epsilon R_{k+1}^{\frac{1}{2}} = \eta R_{k+1}$, i.e. $R_k^{\frac{1}{2}} - R_{k+1}^{\frac{1}{2}} = \frac{\eta}{\epsilon} R_{k+1} := C_0 R_{k+1}$ with C_0 depending only on η and ϵ. Now, we have $R_k = R_{k+1} + 2C_0 R_{k+1}^{\frac{3}{2}} + C_0^2 R_{k+1}^2$. For k large enough, we have $(R_k - R_{k+1})/(2C_0 R_{k+1}^{\frac{3}{2}}) = 1 + \frac{C_0}{2} R_{k+1}^{\frac{1}{2}} \to 1$. Hence, we get $R_k - R_{k+1} \backsim 2C_0 R_{k+1}^{\frac{3}{2}}$ and

$$\begin{aligned}
\lim_{k\to\infty} k^2 R_k &= \lim_{k\to\infty} k^2/(1/R_k) \\
&= \lim_{k\to\infty} [(k+1)^2 - k^2]/(1/R_{k+1} - 1/R_k) \\
&= \lim_{k\to\infty} (2k+1) R_{k+1}^2/(2C_0 R_{k+1}^{\frac{3}{2}}) \\
&= \lim_{k\to\infty} [(2k+1) - (2k-1)]/[2C_0(1/\sqrt{R_{k+1}} - 1/\sqrt{R_k})] \\
&= \lim_{k\to\infty} \sqrt{R_k R_{k+1}}/[C_0(\sqrt{R_k} - \sqrt{R_{k+1}})] \\
&= \lim_{k\to\infty} R_{k+1}/(C_0^2 R_{k+1}) \\
&= 1/C_0^2.
\end{aligned}$$

Hence, we have $R_k^{\frac{1}{2}} \backsim 1/(C_0 k) = \epsilon/(\eta k)$.

By $\epsilon R_k^{\frac{1}{2}} \backsim \epsilon^2/(\eta k)$, there exists a constant C_1 such that $(C_1 \epsilon^2)/(\eta k) \geqslant \epsilon R_k^{\frac{1}{2}}$, for any $k \in \mathbb{N}^+$.
□

The following lemma is the decay lemma of the weak supersolution in the Hölder cone. This lemma plays a crucial role in the proof of the following theorem.

Lemma 3. *Let $\alpha \in \left[\frac{1}{2}, 1\right]$. There exist constants $C_2 > 0$ and $\eta > 0$ depending on α, λ, Λ and the dimension n, such that the following holds.*

Let u be a solution of

$$\begin{cases} \mathcal{L}u \le 0 & \text{in } E_\eta \\ u \ge 1 & \text{in} \{x_n \ge 1\} \cap E_\eta \\ u \ge 0 & \text{on } \partial E_\eta, \end{cases} \tag{4.4}$$

where the $E_\eta = \{x \in \mathbb{R}^n : \frac{\eta}{2}|x'|^\alpha < x_n < 3\}$.

Then for any $t \in (0, \frac{1}{3})$, we have $u(te_n) \geqslant e^{-\frac{C_2}{t}}$.

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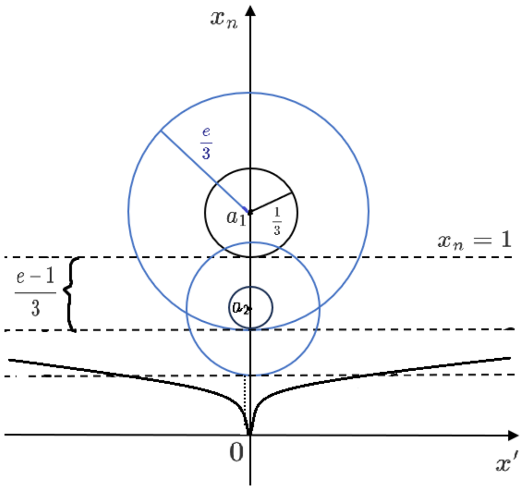

(c)

Proof. We take $R_1 = \frac{1}{3}$ and $\eta = \left(1 - \frac{e-1}{3}\right)3^{\alpha}$. By Lemma 2, we have a decreasing and positive sequence $\{R_k\}$ satisfied $R_k \to 0$ as $k \to +\infty$ and $\eta R_k{}^{\alpha} = 1 - (e-1)\sum_{j=1}^{k} R_j$ for any $k \in \mathbb{N}^+$. Moreover, there exists a constant C_1 such that $\frac{C_1 \epsilon^2}{\eta k} \geqslant \epsilon R_k^{\frac{1}{2}}$, for any $k \in \mathbb{N}^+$.

We constructed two series of balls and denoted them as $\{B_{R_i}(0, a_i)\}$ and $\{B_{eR_i}(0, a_i)\}$, for all $i \in \mathbb{N}^+$ (see figure (c)). Now, there exists $k_0 \in \mathbb{N}^+$ such that $1 - (e-1)\sum_{j=1}^{k_0+1} R_j \leqslant t \leqslant 1 - (e-1)\sum_{j=1}^{k_0} R_j$, i.e. $\eta R^{\alpha}_{k_0+1} \leqslant t \leqslant \eta R^{\alpha}_{k_0}$ and $te_n \in B_{2eR_{k_0+1}}(x_{k_0+1})$.

As $u(x) \geqslant 1$ in $B_{R_1}(x_1)$, Theorem 4 implies that $u(x) \geqslant e^{-q}$ in $B_{eR_1}(x_1)$. Hence, $u(\tau e^n) \geqslant e^{-q}$ in $B_{R_2}(x_2)$. By Theorem 4 again, we have $u(x) \geqslant e^{-2q}$ in $B_{2eR_2}(x_2)$. Now keep doing it this way, we get $u(x) \geqslant e^{-kq}$ in $B_{eR_k}(x_k)$ for any $k \in \mathbb{N}^+$. Hence, $u(te^n) \geqslant e^{-(k_0+1)q} = e^{-q}(e^{-q})^{k_0}$ and $\eta R_{k_0}^{\frac{1}{2}} \leqslant \frac{C_1 \eta^2}{(e-1)k_0}$. As $\eta R_{k_0}^{\frac{1}{2}} \geqslant \eta R^{\alpha}_{k_0} \geqslant t$, we have $t \leqslant \frac{C_1\eta^2}{(e-1)k_0}$, i.e. $k_0 \leqslant \frac{C_1\eta^2}{(e-1)t}$. Because $t < 1$ and $q > 1$, we have $e^{-q}(e^{-q})^{k_0} \geqslant (e^{-q})(e^{-q})^{\frac{C_1\eta^2}{(e-1)t}} \geqslant e^{-\frac{qC_1\eta^2+q}{t}} := e^{-\frac{C_2}{t}}$ for any $t \in (0, \frac{1}{3})$. Thus, we have $u(te_n) \geqslant e^{-\frac{C_2}{t}}$, for any $t \in (0, \frac{1}{3})$.

□

The following theorem works similarly to Theorem 7.

Theorem 8. *Let Ω be a Hölder domain with $\alpha \in [1/2, 1)$ and the Hölder constant $H < \frac{4-e}{4\sqrt{3}}$. Let u satisfy*

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \\ u \geq 1 & \text{in } \Omega \cap \{x \in B_1 : \Gamma_h(x) > \delta\} \\ u \geq -\epsilon & \text{in } \Omega \cap B_1, \end{cases} \tag{4.5}$$

. $\|f\|_{L^\infty(B_r(y))} \leq c_0 r^{-2}\exp(-4C_2 r^{-1})$ *for any* $r \in (0, 1/2)$, $y \in \partial\Omega \cap B_1$.

Then we have

$$\begin{cases} u \geq \exp(-4C_2\rho^{-1}) & \text{in } \Omega \cap \{x \in B_\rho : \Gamma_h(x) > \rho\delta\} \\ u \geq -\exp(-4C_2\rho^{-1})\epsilon & \text{in } \Omega \cap B_\rho, \end{cases} \tag{4.6}$$

where the positive constant C_2 is determined by Lemma 3 and some small enough constants $\rho, \epsilon, \delta, c_0 \in (0, 1)$ depend on α, λ, Λ and the dimension n.

Proof. Now that, we prove the first inequality in (4.6). For any $(x_0', x_{0n}) \in \{x \in B_\rho : \rho\delta < \Gamma_h(x) < \delta\}$, we write $\tilde{x_0} := (x_0', \omega(x_0'))$ and the Hölder cone

$$E_{\rho,\eta} = \tilde{x_0} + \rho E_\eta = \left\{x \in \mathbb{R}^n : \frac{\eta}{2}|x' - x_0'|^\alpha < x_n - \omega(x_0') < 3\rho\right\}$$

and the positive upper Hölder cone

$$E_\rho^+ = E_{\rho,\eta} \cap \{x_n > \omega(x_0') + \rho\},$$

where $E_\eta = \left\{x \in \mathbb{R}^n : \frac{\eta}{2}|x'|^\alpha < x_n < 3\right\}$.

We take $\rho = 3\delta$, it's easy to get $E_\rho^+ \subset \Omega \cap \{x \in B_1 : \Gamma_h(x) > \delta\}$. If δ small enough, we have $E_{\rho,\eta} \subset \Omega \cap B_1$. Because $x_0 = (x_0', x_{0n})$ in $\{x \in B_\rho : \rho\delta < \Gamma_h(x) < \delta\}$, we have

$$\rho\delta < x_{0n} - \omega(x_0') < \delta \quad and \quad then \quad \frac{\rho}{3} < \frac{x_{0n} - \omega(x_0')}{\rho} < \frac{1}{3}.$$

We define

$$\tilde{u}(x) := u(\tilde{x_0} + \rho x) + \epsilon \quad and \quad \Phi(x) := \mathcal{L}\tilde{u}(x) = \rho^2 f(\tilde{x_0} + \rho x).$$

Let $\tilde{u} = u_1 + u_2$ satisfy

$$\begin{cases} \mathcal{L}u_1 = 0 & in\ E_\eta \\ u_1 = \tilde{u} & on\ \partial E_\eta \end{cases} \quad and \quad \begin{cases} \mathcal{L}u_2 = \Phi & in\ E_\eta \\ u_2 = 0 & on\ \partial E_\eta. \end{cases}$$

By Theorem 3, we have

$$\begin{aligned} \|u_2\|_{L^\infty(E_\eta)} &\leqslant C\|\Phi\|_{L^n(E_\eta)} \leqslant C\rho^2\left(\int_{E_\eta} |f(\tilde{x_0} + \rho x)|^n dx\right)^{\frac{1}{n}} \\ &= C\rho\left(\int_{E_{\rho,\eta}} |f(y)|^n dy\right)^{\frac{1}{n}} \leqslant C'\rho\|f\|_{L^\infty(B_\rho(\tilde{x_0}))}|B_\rho(\tilde{x_0})|^{\frac{1}{n}} \\ &\leqslant C'C(n)c_0 \exp\left(-4C_2\rho^{-1}\right). \end{aligned}$$

On the other hand, we apply Lemma 3 to u_1 to result that $u_1(te_n) \geq \exp(-C_2 t^{-1})$, for any $t \in (0, \frac{1}{3})$. Then we get

$$u((x_0', \omega(x_0')) + \rho t e_n) \geq \exp\left(-C_2 t^{-1}\right) - C'C(n)c_0 \exp\left(-4C_2\rho^{-1}\right) - \epsilon, \quad for\ any \quad t \in (0, \frac{1}{3}).$$

Therefore,

$$\begin{aligned} u(x_0) &\geqslant u\left((x_0', \omega(x_0')) + \rho\frac{x_{0n} - \omega(x_0')}{\rho}e_n\right) \\ &\geqslant \exp\left(-C_2\left(\frac{x_{0n} - \omega(x_0')}{\rho}\right)^{-1}\right) - C'C(n)c_0 \exp\left(-4C_2\rho^{-1}\right) - \epsilon \\ &\geqslant \exp\left(-C_2\left(\frac{\rho}{3}\right)^{-1}\right) - C'C(n)c_0 \exp\left(-4C_2\rho^{-1}\right) - \epsilon \\ &\geqslant 3\exp\left(-4C_2\rho^{-1}\right) - C'C(n)c_0 \exp\left(-4C_2\rho^{-1}\right) - \epsilon \\ &\geqslant \exp\left(-4C_2\rho^{-1}\right). \end{aligned}$$

Next, we prove the second inequality in (4.6). Let $x_0 \in B_{1-2\delta}$ and $\Gamma_h(x) \leqslant \delta$. We define $v(x) := \frac{u(x)^-}{\epsilon} \geqslant 0$ and extend v by 0 below $\partial\Omega$. Thus, $\mathcal{L}v \geqslant -\frac{f^+}{\epsilon}$ in $B_{2\delta}(x_0)$. We write $z := (x_0', \omega(x_0'))$.

Let E_z be the downwards Hölder cone with the Hölder constant η and vertex in z. Then we have $v = 0$ in $E_z \cap B_{2\delta}(x_0)$ and $|E_z \cap B_{2\delta}(x_0)| \geqslant C(\eta)|B_{2\delta}(x_0)|$. Hence, we have

$$\begin{aligned}\left(\int_{B_{2\delta}(x_0)} \left|\frac{f^+}{\epsilon}\right|^n\right)^{\frac{1}{n}} &\leqslant \left(\int_{B_{3\delta}(z)} \left|\frac{f^+}{\epsilon}\right|^n\right)^{\frac{1}{n}} \\ &\leqslant \frac{3\delta C(n)}{\epsilon}\|f\|_{L^\infty(B_{3\delta}(z))} \\ &\leqslant c_0 C(n)\frac{\rho^{-2}\exp(-4C_2\rho^{-1})}{\epsilon} \\ &\leqslant \xi(\eta),\end{aligned}$$

where $\xi(\eta)$ come from Theorem 5 and make c_0 small enough such that the last inequality holds. By Theorem 5, we have $v \leqslant 1-\gamma$ in $B_\delta(x_0)$.

Now we define $v_j(x) := \frac{v}{(1-\gamma)^j}(x)$ in $B_\delta(x_0)$ with $x_0 \in B_{1-2j\delta}$ and $\Gamma_h(x_0) \leqslant \delta$. By mathematical induction, it is easy to show that $v \leqslant (1-\gamma)^j$ in $B_{1-2j\delta}$, i.e. $u \geqslant -(1-\gamma)^j\epsilon$ in $B_{1-2j\delta}$. We take $j_0 \in \mathbb{N}^+$ satisfied $1-2(j_0+1)\delta \leqslant \rho \leqslant 1-2j_0\delta$, with $\rho = 3\delta$. Hence, we have

$$\frac{1}{\frac{2}{3}j_0+\frac{5}{3}} \leqslant \rho \leqslant \frac{1}{\frac{2}{3}j_0+1}.$$

We take ρ small enough, such that

$$\exp(j_0\ln(1-\gamma)) \leqslant \exp\left(-4C_2\left(\frac{2}{3}j_0+\frac{5}{3}\right)\right) \quad \textit{and then} \quad (1-\gamma)^{j_0} \leqslant \exp\left(-4C_2\left(\frac{2}{3}j_0+\frac{5}{3}\right)\right).$$

Hence, we have $(1-\gamma)^{j_0} \leqslant \exp(-4C_2\rho^{-1})$.

Finally, we have

$$\begin{aligned}\left(\int_{B_{2\delta}(x_0)} \left|\frac{f^+}{(1-\gamma)^{j_0}\epsilon}\right|^n\right)^{\frac{1}{n}} &\leqslant \left(\int_{B_{3\delta}(z)} \left|\frac{f^+}{(1-\gamma)^{j_0}\epsilon}\right|^n\right)^{\frac{1}{n}} \leqslant \frac{1}{(1-\gamma)^{j_0}\epsilon}\|f\|_{L^n(B_{3\delta}(z))} \\ &\leqslant \frac{3\delta C(n)}{(1-\gamma)^{j_0}\epsilon}\|f\|_{L^\infty(B_{3\delta}(z))} \leqslant c_0 C(n)\frac{\rho^{-2}\exp(-4C_2\rho^{-1})}{(1-\gamma)^{j_0}\epsilon} \\ &\leqslant \xi(\eta),\end{aligned}$$

where $\xi(\eta)$ come from Theorem 5 and make c_0 small enough such that the last inequality holds.

By $1-2j_0\delta \geqslant \rho$ with $(1-\gamma)^{j_0} \leqslant \exp(-4C_2\rho^{-1})$, and $u \geqslant -(1-\gamma)^{j_0}\epsilon$ in $B_{1-2j_0\delta}$.

Hence, we have $u \geqslant -\exp(-4C_2\rho^{-1})\epsilon$ in B_ρ. □

Now, by iterating Theorem 8, we get the following proposition.

Proposition 3. *Let Ω be a Hölder domain with $\alpha \in [1/2, 1)$ and the Hölder constant $H < \frac{4-e}{4\sqrt{3}}$. The positive constant C_2 is determined by Lemma 3. Let u be a weak solution of (4.5) and $\|f\|_{L^\infty(B_r(y))} \leq c_0 r^{-2}\exp(-4C_2 r^{-1})$, for any $r \in (0, 1/2)$, $y \in \partial\Omega \cap B_1$.*

Then we have

$$\begin{cases} u > 0 & \textit{in } \Omega \cap B_{2/3} \\ u(te_n) \geqslant \exp(-4C_2 t^{-1}) & \textit{for all } t \in (0,1). \end{cases}$$

Proof. Let $u_0(x) := u(x)$ and $f_0(x) := f(x)$. We define

$$u_{j+1}(x) := \frac{u_j(\rho x)}{\exp(-4C_2\rho^{-1})}, \quad f_{j+1}(x) := \frac{\rho^2 f_j(\rho x)}{\exp(-4C_2\rho^{-1})}$$

with $x \in \Omega_{j+1}$ and $\rho x \in \rho\Omega_{j+1} := \Omega_j$, where the Hölder constant of Ω_{j+1} is the same as or smaller than Ω_j.

First we already have $\|f_0\|_{L^\infty(B_r(y))} \leq c_0 r^{-2}\exp(-4C_2 r^{-1})$, for any $r \in (0, 1/2)$, $y \in \partial\Omega \cap B_1$. By mathematical induction, we have

$$\begin{cases} \mathcal{L}u_j = f_j & in\ \Omega_j \cap B_1 \\ u_j = 0 & on\ \partial\Omega_j \end{cases} with \begin{cases} u_j \geq 1 & in\ \Omega_j \cap \{x \in B_1 : \Gamma_h(x) > \delta\} \\ u_j \geq -\epsilon & in\ \Omega_j \cap B_1, \end{cases}$$

where $\|f_j\|_{L^\infty(B_r(y))} \leq c_0 r^{-2}\exp(-4C_2 r^{-1})$, for any $r \in (0, 1/2)$, $y \in \partial\Omega \cap B_1$. By Theorem 8, we have

$$\begin{cases} u_j \geq \exp(-4C_2\rho^{-1}) & in\ \Omega_j \cap \{x \in B_\rho : \Gamma_h(x) > \rho\delta\} \\ u_j \geq -\exp(-4C_2\rho^{-1})\epsilon & in\ \Omega_j \cap B_\rho. \end{cases}$$

So, we have

$$\begin{cases} u_{j+1} \geq 1 & in\ \Omega_{j+1} \cap \{x \in B_1 : \Gamma_h(x) > \delta\} \\ u_{j+1} \geq -\epsilon & in\ \Omega_{j+1} \cap B_1. \end{cases}$$

Moreover, we have

$$\begin{aligned} \|f_{j+1}\|_{L^\infty(B_r(y))} &= \frac{\rho^2}{\exp(-4C_2\rho^{-1})}\|f_j(\rho x)\|_{L^\infty(B_r(y))} \\ &\leqslant \frac{c_0\rho^2}{\exp(-4C_2\rho^{-1})}(\rho r)^{-2}\exp\left(-4C_2(\rho r)^{-1}\right) \\ &\leqslant c_0 r^{-2}\exp\left(-4C_2 r^{-1}\right). \end{aligned}$$

Then for every $j \in \mathbb{N}$, we have $u_j(te_n) \geqslant 1$, for any $t \in (\delta, 1)$. For any $t \in (0,1)$, there exists $j_0 \in \mathbb{N}$ such that $\delta < \frac{t}{\rho^{j_0}} < 1$. Thus, $t < \rho^{j_0} < \frac{\rho}{j_0}$ and

$$u(te_n) = u_0(te_n) = \exp\left(-4C_2\rho^{-1}\right)u_1\left(\frac{te_n}{\rho}\right) = ... = \exp\left(-4j_0C_2\rho^{-1}\right)u_{j_0}\left(\frac{te_n}{\rho^{j_0}}\right) \geqslant \exp\left(-4C_2t^{-1}\right).$$

Finally, taking ϵ, δ and c_0 smaller, then $u(x_0 + \frac{x}{3})$ also satisfy (4.5) for any $x_0 \in B_{2/3} \cap \partial\Omega$. By translating the coordinates, we get

$$u\left(x_0 + t\frac{e_n}{3}\right) > 0, \quad for\ all\ t \in (0,1).$$

Then this implies

$$u > 0 \quad in \quad \Omega \cap B_{2/3}.$$

□

4.1. Proof of Theorem 2

Proof. By Theorem 6 and Proposition 1, we have $u \leqslant M$ in $B_{3/4}$. We consider the function v in the set $E := \{x \in \bar{B}_{3/4} : \Gamma_h(x) \geqslant (3\delta)/4\}$. Obviously, $E \subset \left\{x \in \bar{B}_{1-(3\delta)/4} : \Gamma_h(x) \geqslant (3\delta)/4\right\}$. Thus, by Theorem 6 and $v(e_n/2) \geqslant 1$, we have

$$v \geqslant C^{-1} - \|f\|_{L^n(B_1)} \geqslant C^{-1} - C(n)\|f\|_{L^\infty(B_1)} \geqslant C^{-1} - C(n)c_0 \geqslant (2C)^{-1} := m \quad in\ \ E,$$

where the constant c_0 is small enough such that $C^{-1} - C(n)c_0 \geqslant (2C)^{-1}$.

We define

$$\varphi(x) := \frac{1+\epsilon}{m}v(x) - \frac{\epsilon}{M}u(x),$$

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where $\epsilon > 0$ to be chosen later. We will prove that $\varphi > 0$ in $B_{1/2}$ and therefore, choosing $C = M(1+\epsilon)/(m\epsilon)$, we have $Cv - u > 0$.

Obviously, $\varphi \geqslant v/m \geqslant 1$ in E, and $\varphi \geqslant -\epsilon$ in $B_{3/4}$. We have

$$\begin{aligned}\|\mathcal{L}\varphi\|_{L^\infty(B_r(y))} &\leqslant \frac{1+\epsilon}{m}\|\mathcal{L}v\|_{L^\infty(B_r(y))} + \frac{\epsilon}{M}\|\mathcal{L}u\|_{L^\infty(B_r(y))} \\ &\leqslant \left(\frac{1+\epsilon}{m} + \frac{\epsilon}{M}\right)c_0 r^{-2}\exp\left(-4C_2 r^{-1}\right), \quad \textit{for any } r \in (0,1/2)\,,\ y \in \partial\Omega.\end{aligned}$$

Where the positive constant C_2 is determined by Lemma 3.

Let $\widetilde{\varphi}(x) := \varphi(3x/4)$. Then $\widetilde{\varphi} \geqslant 1$ in $\Omega \cap \{x \in B_1 : \Gamma_h(x) \geqslant \delta\}$ and $\widetilde{\varphi} \geqslant -\epsilon$ in $\Omega \cap B_1$. Taking small enough ϵ, $c_0 > 0$ to apply Proposition 3. We get $\widetilde{\varphi} > 0$ in $B_{2/3}$, thus $\varphi > 0$ in $B_{1/2}$. □

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