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Linearized Expressions of 3D Rotational Motion

Yinlong Liu *

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Linearized Expressions of 3D Rotational Motion

Yinlong Liu

Independent Researcher; yinlong.liu@tum.de

Abstract: Rotation motion in a three-dimensional physical world refers to an angular displacement of an object around a specific axis in $\mathbb{R}^3$. It is typically formulated as a non-linear and non-convex process due to the nonlinearity and nonconvexity of $\mathbb{SO}(3)$. However, this paper proposes a new perspective that the 3D rotation motion can be expressed by the linear equation without dropping any constraints and increasing any singularities. Moreover, two frequent cases, i.e., $\angle(\mathbf{R}\boldsymbol{x}, \boldsymbol{y}) = 0(\pi)$ and $\angle(\mathbf{R}\boldsymbol{x}, \boldsymbol{y}) = \frac{\pi}{2}$, in computer vision and robotics that can be expressed linearly are deeply discussed in this paper.

Keywords: Rotation motion

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1. Background

Rotational motion [1] in the three-dimensional physical world refers to the circular movement of an object around a fixed axis [2]. It is a basic concept and commonly used in physics, computer graphics, computer vision, robotics, aerospace, and other scientific fields [3–6]. More specifically, in many practical applications, rotational motion is not only an important means of constructing the motion of the target system [7,8] but also an important state parameter to be estimated [9–11]. However, rotation motion is typically formulated as a non-linear and non-convex process [11,12]. It not only makes some linear algorithms, such as Kalman filter [13,14], impossible to directly apply [15], but also makes solving the related problem more sophisticated and difficult [12,16].

In this paper, a new perspective is proposed that **the rotation motion can be linearized without dropping any constraints and increasing any singularities**. It is particularly worth pointing out that there are no special and advanced techniques, such as Lie groups and Lie algebras [17,18], for the methods used in this paper. Only some basic knowledge of linear algebra [19–21] is required to understand the ideas presented in this paper.

2. Rotation Motion

Mathematically, given a point $\boldsymbol{x} \in \mathbb{R}^3$, if it is rotated about a fixed unit vector $\boldsymbol{r} \in \mathbb{S}^2$ through the origin and the rotated angle is θ, then the rotation motion can be formulated by multiplication of a matrix and a vector,

$$\mathbf{R}\boldsymbol{x} \rightarrow \boldsymbol{y} \tag{1}$$

where $\boldsymbol{y} \in \mathbb{R}^3$ is the rotated point by $\boldsymbol{x}$ and $\mathbf{R}$ is a 3×3 rotation matrix. There are some properties regarding the rotation motion [22],

- Shape and Size: Rotation does not alter the shape or size of the object. More Specifically,
    1. $\|\mathbf{R}\boldsymbol{x}\| = \|\mathbf{R}\| \cdot \|\boldsymbol{x}\| = \|\boldsymbol{x}\|$. It means the length of the point $\boldsymbol{x}$ will not be changed by rotation motion. Without loss of generality, the rotation motion in this paper is studied with unit norm vectors, i.e., $\boldsymbol{x} \in \mathbb{S}^2$ on the unit sphere surface, which represents a unit direction vector in the 3D physical world.
    2. $\angle(\mathbf{R}\boldsymbol{x}_1, \mathbf{R}\boldsymbol{x}_2) = \angle(\boldsymbol{x}_1, \boldsymbol{x}_2)$ where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{S}^2$ are any two unit vectors. It means the angle (structure) of the object is unchanged.
- Axis of Rotation: Rotational motion occurs around a fixed axis. Given a vector $\boldsymbol{x} \in \mathbb{S}^2$, if $\mathbf{R}\boldsymbol{x} = \boldsymbol{x}$, it means the after the rotation $\mathbf{R}$, the vector direction $\boldsymbol{x}$ remains unchanged, and the vector $\boldsymbol{x}$ must be parallel to the rotation axis $\boldsymbol{r}$, which is from Euler's rotation theorem and screw theory [23–26]. In addition,

$$\mathbf{R}\boldsymbol{x} = \boldsymbol{x} \Rightarrow (\mathbf{R} - \mathbf{I}_{3\times3})\boldsymbol{x} = 0 \tag{2}$$

where $\mathbf{I}_{3\times3}$ is the 3×3 identity matrix. Algebraically, Eq.(2) means the rotation axis direction $\boldsymbol{r}$ lies in the null space of $\mathbf{R}-\mathbf{I}_{3\times3}$. Alternatively, let $\mathbf{R}\boldsymbol{x}=\lambda\boldsymbol{x}$, then the rotation axis direction $\boldsymbol{r}$ is an eigenvector of $\mathbf{R}$ corresponding to the eigenvalue $\lambda=1$.

At first glance, the rotation motion is a linear process, which might be misled by the matrix representation Eq.(1). Unfortunately, $\mathbf{R}$ is not a regular matrix in $\mathbb{R}^{3\times3}$. A rotation only has three degrees of freedom, since it can be determined by a rotation axis $\boldsymbol{r}\in\mathbb{S}^2$ and a rotation angle $\theta\in[0,\pi]$. Formally, $\mathbf{R}\in\mathbb{SO}(3)$, and a rotation can be represented by a matrix and non-linear constraints,

$$\mathbf{R}=\begin{bmatrix} r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}, \quad s.t. \quad \mathbf{R}^T\mathbf{R}=\mathbf{I}_{3\times3}, \det(\mathbf{R})=+1 \tag{3}$$

According to the famous Rodrigues' rotation formula [27,28], given the rotation axis $\boldsymbol{r}=\begin{bmatrix} r_1 & r_2 & r_3\end{bmatrix}^T\in\mathbb{S}^2$ and rotation angle θ, the matrix can be constructed by

$$\mathbf{R}=\exp(\theta[\boldsymbol{r}]_\times)=\mathbf{I}_{3\times3}+\sin(\theta)[\boldsymbol{r}]_\times+(1-\cos(\theta))[\boldsymbol{r}]_\times^2, \quad [\boldsymbol{r}]_\times=\begin{bmatrix} 0 & -r_3 & r_2\\ r_3 & 0 & -r_1\\ -r_2 & r_1 & 0\end{bmatrix} \tag{4}$$

Moreover, a rotation matrix can be represented by a quaternion vector with unit length [29,30], such that $\boldsymbol{q}=\begin{bmatrix} q_0 & q_1 & q_2 & q_3\end{bmatrix}^T\in\mathbb{S}^3$. Specifically,

$$\mathbf{R}=\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2)\\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1)\\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2\end{bmatrix} \tag{5}$$

It is worth noting that many different forms of parameter representation can represent the same rotation, and each form of representation has its appropriate application scenario [2,31–33]. For example, since a rotation can be determined by a fixed rotation axis $\boldsymbol{r}\in\mathbb{S}^2$ and a rotation angle $\theta\in[0,\pi]$, then the rotation can be represented by axis–angle representation $\theta\cdot\boldsymbol{r}$ [34], which geometrically is a point in a 3D ball of radius π [28] and has been widely used in pose estimation problems [28,35,36]. Nonetheless, throughout most of this paper, quaternions are used to represent rotations [29,30] and thus to illustrate the proposed linearization theory.

> Anyone who has ever used any other parametrization of the rotation group will, within hours of taking up the quaternion parametrization, lament his or her misspent youth [2,31].

3. Linear Expressions for Rotation Motion

Given $\boldsymbol{a}=\begin{bmatrix} a_1 & a_2 & a_3\end{bmatrix}\in\mathbb{S}^2$ and a rotation $\mathbf{R}\in\mathbb{SO}(3)$, the rotation motion can be formulated as

$$\mathbf{R}\boldsymbol{a}=\boldsymbol{b}\Leftrightarrow\angle(\mathbf{R}\boldsymbol{a},\boldsymbol{b})=0\Leftrightarrow\boldsymbol{b}^T\mathbf{R}\boldsymbol{a}=1 \tag{6}$$

where $\boldsymbol{b}=\begin{bmatrix} b_1 & b_2 & b_3\end{bmatrix}\in\mathbb{S}^2$ is rotated by $\boldsymbol{a}$. This formulation is a special case of the next expression for rotational motion. Formally, given $\boldsymbol{a},\boldsymbol{b}\in\mathbb{S}^2$ and $\mathbf{R}\in\mathbb{SO}(3)$ whose quaternion expression is $\boldsymbol{q}\in\mathbb{S}^3$,

$$\angle(\mathbf{R}\boldsymbol{a},\boldsymbol{b})=\tau\Leftrightarrow\boldsymbol{b}^T\mathbf{R}\boldsymbol{a}=\cos(\tau)\Leftrightarrow\boldsymbol{q}^T\mathbf{M}\boldsymbol{q}=\cos(\tau) \tag{7}$$

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Here $\mathbf{M}$ is constructed by $\boldsymbol{a}$ and $\boldsymbol{b}$ [29,37]. Specifically,

$$\mathbf{M}=\begin{bmatrix}0 & -a_1 & -a_2 & -a_3\\ a_1 & 0 & a_3 & -a_2\\ a_2 & -a_3 & 0 & a_1\\ a_3 & a_2 & -a_1 & 0\end{bmatrix}^T\begin{bmatrix}0 & -b_1 & -b_2 & -b_3\\ b_1 & 0 & -b_3 & b_2\\ b_2 & b_3 & 0 & -b_1\\ b_3 & -b_2 & b_1 & 0\end{bmatrix} \tag{8}$$

When $\tau=0$, Eq.(7) degenerates back to Eq.(6).

By conducting eigendecomposition of the real symmetric matrix $\mathbf{M}=\mathbf{U}\Lambda\mathbf{U}^T$ [21,38], where and $\mathbf{U}=\begin{bmatrix}\boldsymbol{u}_0 & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3\end{bmatrix}\in\mathbb{R}^{4\times4}$ is an orthogonal matrix $\mathbf{U}\mathbf{U}^T=\mathbf{I}_{4\times4}$. Observe

$$\det(\mathbf{M}-\lambda\mathbf{I}_{4\times4})=0\Rightarrow\left(\lambda^2-1\right)^2=0 \tag{9}$$

Therefore, the eigenvalue of $\mathbf{M}$ is

$$\Lambda=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix} \tag{10}$$

Accordingly, $\boldsymbol{q}^T\mathbf{U}\Lambda\mathbf{U}^T\boldsymbol{q}=\cos(\tau)$. Let $\boldsymbol{\Gamma}=[\Gamma_0,\Gamma_1,\Gamma_2,\Gamma_3]^T=\mathbf{U}^T\boldsymbol{q}$, then $\boldsymbol{q}=\Gamma_0\boldsymbol{u}_0+\Gamma_1\boldsymbol{u}_1+\Gamma_2\boldsymbol{u}_2+\Gamma_3\boldsymbol{u}_3$. Observe

$$\begin{cases}\boldsymbol{\Gamma}^T\boldsymbol{\Gamma}=\boldsymbol{q}^T\mathbf{U}\mathbf{U}^T\boldsymbol{q}=\Gamma_0^2+\Gamma_1^2+\Gamma_2^2+\Gamma_3^2=1\\ \boldsymbol{q}^T\mathbf{U}\Lambda\mathbf{U}^T\boldsymbol{q}=\boldsymbol{\Gamma}^T\Lambda\boldsymbol{\Gamma}=\Gamma_0^2+\Gamma_1^2-\Gamma_2^2-\Gamma_3^2=\cos(\tau)\end{cases}\Rightarrow\begin{cases}\Gamma_0^2+\Gamma_1^2=\frac{1}{2}(1+\cos(\tau))\\ \Gamma_2^2+\Gamma_3^2=\frac{1}{2}(1-\cos(\tau))\end{cases} \tag{11}$$

Therefore, the possible rotation $\boldsymbol{q}$ can be reformulated linearly

$$\boldsymbol{q}=\Gamma_0\boldsymbol{u}_0+\Gamma_1\boldsymbol{u}_1+\Gamma_2\boldsymbol{u}_2+\Gamma_3\boldsymbol{u}_3 \tag{12}$$

$$=\frac{\sqrt{2(1+\cos(\tau))}}{2}(\cos(\phi)\boldsymbol{u}_0+\sin(\phi)\boldsymbol{u}_1)+\frac{\sqrt{2(1-\cos(\tau))}}{2}(\cos(\psi)\boldsymbol{u}_2+\sin(\psi)\boldsymbol{u}_3) \tag{13}$$

where $\phi,\psi\in[-\pi,\pi]$. This result clearly shows that the rotational motion can be expressed in linear forms.

Geometrically, if $\angle(\mathbf{R}\boldsymbol{a},\boldsymbol{b})=\tau$, then all possible rotation $\boldsymbol{q}$ that satisfy the constraint will be in a torus in $\mathbb{S}^3$. For better understanding, we draw some special cases, i.e., $\tau=\{0,\frac{\pi}{9},\frac{3\pi}{9},\frac{6\pi}{9},\frac{8\pi}{9}\}$ via stereographic projection [39,40], as in Figure 1. It is worth noting that when $\tau=\{0,\pi\}$, i.e., $\mathbf{R}\boldsymbol{a}=\pm\boldsymbol{b}$, the torus in $\mathbb{S}^3$ is degenerated into a one-dimensional circle, more specifically, a great circle [41] in $\mathbb{S}^3$. Formally,

$$\begin{aligned}&①:\quad \text{when}\quad \tau=0\quad\Rightarrow\quad \boldsymbol{q}=\cos(\phi)\boldsymbol{u}_0+\sin(\phi)\boldsymbol{u}_1\\ &②:\quad \text{when}\quad \tau=\pi\quad\Rightarrow\quad \boldsymbol{q}=\cos(\psi)\boldsymbol{u}_2+\sin(\psi)\boldsymbol{u}_3\end{aligned} \tag{14}$$

In addition, when $\tau=\frac{\pi}{2}$, the torus is becoming to the Clifford torus in $\mathbb{S}^3$ [42,43], since it has the equal *diameters* in cross directions. Specifically,

$$\boldsymbol{q}=\frac{1}{2}(\cos(\phi)\boldsymbol{u}_0+\sin(\phi)\boldsymbol{u}_1)+\frac{1}{2}(\cos(\psi)\boldsymbol{u}_2+\sin(\psi)\boldsymbol{u}_3) \tag{15}$$

These special cases will be discussed in depth in subsequent sections.

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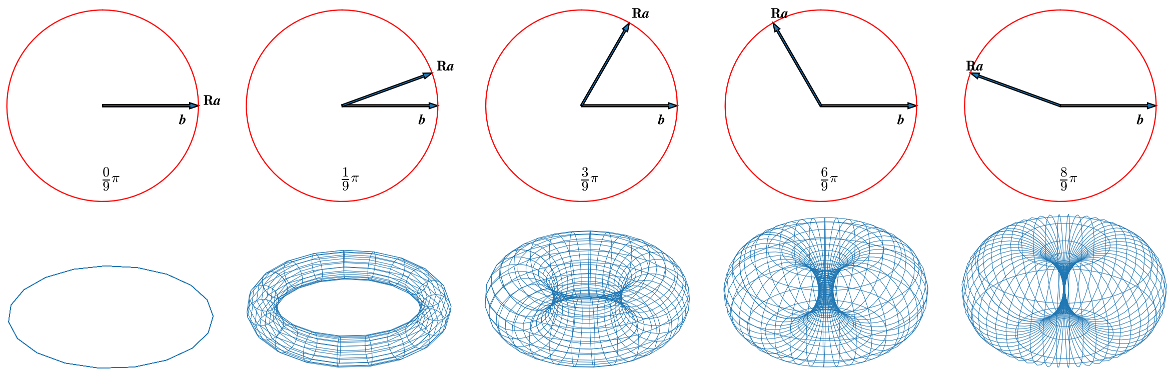

(a) $\tau = 0$ (b) $\tau = \frac{\pi}{9}$ (c) $\tau = \frac{3\pi}{9}$ (d) $\tau = \frac{6\pi}{9}$ (e) $\tau = \frac{8\pi}{9}$

Figure 1. Genera cases of rotation constraints and their corresponding tori in $\mathbb{R}^3$ via stereographic projection.

4. Special Case I: Great Circle in $\mathbb{S}^3$

If $\tau = 0$, which means $\mathbf{R}\boldsymbol{a} = \boldsymbol{b}$, it is one of the most common cases in practical applications. It is worth noting that there are two facts regarding $\mathbf{R}\boldsymbol{a} = \boldsymbol{b}$,

- Not all rotations in $\mathbb{SO}(3)$ can rotate $\boldsymbol{a}$ to $\boldsymbol{b}$.
- There is more than one rotation that satisfies the given rotation motion.

Then according to Eq. (14) all rotations that satisfy $\mathbf{R}\boldsymbol{a} = \boldsymbol{b}$ should be in

$$\boldsymbol{q} = \cos(\phi)\boldsymbol{u}_0 + \sin(\phi)\boldsymbol{u}_1 \tag{16}$$

where $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ are the basis vectors in $\mathbb{S}^3$, and they depend on $\mathbf{M}$, i.e., $\boldsymbol{a}$ and $\boldsymbol{b}$. Geometrically speaking, $\boldsymbol{q}$ is a unit vector in $\mathbb{R}^4$, which is a linear combination of the two vectors $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$. In addition, $\boldsymbol{q}^T\boldsymbol{q} = 1$, which means $\boldsymbol{q}$ is on the surface of the unit quaternion sphere. Therefore, the region where all possible rotations satisfy a specific rotation motion is a great circle in $\mathbb{S}^3$ [41]. Moreover, since $\mathbf{U}^T\mathbf{U} = \mathbf{I}_{4\times 4}$, there will be

$$\begin{cases} \left(\cos(\phi)\boldsymbol{u}_0^T + \sin(\phi)\boldsymbol{u}_1^T\right)\boldsymbol{u}_2 = 0 \\ \left(\cos(\phi)\boldsymbol{u}_0^T + \sin(\phi)\boldsymbol{u}_1^T\right)\boldsymbol{u}_3 = 0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{q}^T\boldsymbol{u}_2 = 0 \\ \boldsymbol{q}^T\boldsymbol{u}_3 = 0 \end{cases} \tag{17}$$

Up until this point, the rotation motion is linearly expressed, which is closely related to $\mathbf{U}$.

In addition,

$$\mathbf{M}\boldsymbol{u}_0 = 1 \cdot \boldsymbol{u}_0; \quad \mathbf{M}\boldsymbol{u}_1 = 1 \cdot \boldsymbol{u}_1; \quad \mathbf{M}\boldsymbol{u}_2 = -1 \cdot \boldsymbol{u}_2; \quad \mathbf{M}\boldsymbol{u}_3 = -1 \cdot \boldsymbol{u}_3 \tag{18}$$

It is easy to confirm [44,45],

$$\mathbf{M}\begin{bmatrix} 0 & -(\boldsymbol{a}-\boldsymbol{b})^T \\ -(\boldsymbol{a}-\boldsymbol{b}) & [\boldsymbol{a}+\boldsymbol{b}]_\times^T \end{bmatrix} = \begin{bmatrix} 0 & -(\boldsymbol{a}-\boldsymbol{b})^T \\ -(\boldsymbol{a}-\boldsymbol{b}) & [\boldsymbol{a}+\boldsymbol{b}]_\times^T \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \doteq -1 \cdot \Omega^- \tag{19}$$

$$\mathbf{M}\begin{bmatrix} 0 & -(\boldsymbol{a}+\boldsymbol{b})^T \\ -(\boldsymbol{a}+\boldsymbol{b}) & [\boldsymbol{a}-\boldsymbol{b}]_\times^T \end{bmatrix} = \begin{bmatrix} 0 & -(\boldsymbol{a}+\boldsymbol{b})^T \\ -(\boldsymbol{a}+\boldsymbol{b}) & [\boldsymbol{a}-\boldsymbol{b}]_\times^T \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \doteq +1 \cdot \Omega^+ \tag{20}$$

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Therefore, any row of Ω^- can be the eigenvector of $\mathbf{M}$ with eigenvalue -1. Similarly, any row of Ω^+ can be the eigenvector of $\mathbf{M}$ with eigenvalue 1.

Accordingly, (1) any row of Ω^- is a linear combination of $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$. (2) any row of Ω^+ is a linear combination of $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$. Consequently, $\text{rank}(\Omega^-) = \text{rank}(\Omega^+) = 2$; Ω^- and Ω^+ are not orthogonal matrices. Therefore, $\Omega^-\boldsymbol{q} = \mathbf{0}$, more specifically,

$$\begin{bmatrix} 0 & a_1 - b_1 & a_2 - b_2 & a_3 - b_3 \\ -(a_1 - b_1) & 0 & -(a_3 + b_3) & a_2 + b_2 \\ -(a_2 - b_2) & a_3 + b_3 & 0 & -(a_1 + b_1) \\ -(a_3 - b_3) & -(a_2 + b_2) & a_1 + b_1 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \mathbf{0} \Leftrightarrow \begin{bmatrix} 0 & (\boldsymbol{a} - \boldsymbol{b})^T \\ -(\boldsymbol{a} - \boldsymbol{b}) & [\boldsymbol{a} + \boldsymbol{b}]_\times \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \mathbf{0} \tag{21}$$

Now the linear expression of $\mathbf{R}\boldsymbol{a} = \boldsymbol{b}$ is constructed.

Furthermore, if more than one rotation motion constraints are given, such that

$$\begin{cases} \mathbf{R}\boldsymbol{a}_1 = \boldsymbol{b}_1 \\ \mathbf{R}\boldsymbol{a}_2 = \boldsymbol{b}_2 \\ \cdots \\ \mathbf{R}\boldsymbol{a}_N = \boldsymbol{b}_N \end{cases} \xrightarrow{\text{linearization}} \begin{cases} \blacksquare_1^-\boldsymbol{q} = \mathbf{0} \\ \blacksquare_2^-\boldsymbol{q} = \mathbf{0} \\ \cdots \\ \blacksquare_N^-\boldsymbol{q} = \mathbf{0} \end{cases} \Longrightarrow \begin{bmatrix} \blacksquare_1^- \\ \blacksquare_2^- \\ \cdots \\ \blacksquare_N^- \end{bmatrix} \boldsymbol{q} = \mathbf{0} \tag{22}$$

Eventually, the corresponding linear system is constructed. It must be acknowledged that there is still a constraint of length 1, i.e., $\boldsymbol{q}^T\boldsymbol{q} = 1$. However, the scale is not particularly important for homogeneous linear equations.

4.1. Alternative Way to Obtain Great Circle in $\mathbb{S}^3$

In this part, a geometrical approach to obtain a great circle in $\mathbb{S}^3$ is presented. Given the rotation motion $\mathbf{R}\boldsymbol{a} = \boldsymbol{b}$, the possible rotation can be formulated as [9,34]

$$\mathbf{R} = \mathbf{R}_{\boldsymbol{b}}(\alpha)\mathbf{R}_{\boldsymbol{a}}^{\boldsymbol{b}} \tag{23}$$

where $\mathbf{R}_{\boldsymbol{b}}(\alpha)$ is a rotation around rotated axis is $\boldsymbol{b}$ and rotated angle is α; $\mathbf{R}_{\boldsymbol{a}}^{\boldsymbol{b}}$ is a rotation that can rotate $\boldsymbol{a}$ to $\boldsymbol{b}$. To use quaternion to represent rotation [30],

$$\mathbf{R}_{\boldsymbol{b}}(\alpha) := \left[\cos\left(\frac{\alpha}{2}\right), \sin\left(\frac{\alpha}{2}\right)\boldsymbol{b}\right] \tag{24}$$

$$\mathbf{R}_{\boldsymbol{a}}^{\boldsymbol{b}} := \left[q_0, [q_1, q_2, q_3]^T\right] = [q_0, \boldsymbol{q}_{123}] \tag{25}$$

The formula for the multiplication of two quaternions is [29]

$$(r_1, \vec{v_1})(r_2, \vec{v_2}) = (r_1 r_2 - \vec{v_1} \cdot \vec{v_2}, r_1\vec{v_2} + r_2\vec{v_1} + \vec{v_1} \times \vec{v_2}) \tag{26}$$

Then,

$$\mathbf{R} := \left[\cos\left(\frac{\alpha}{2}\right)q_0 - \sin\left(\frac{\alpha}{2}\right)\boldsymbol{b}^T\boldsymbol{q}_{123}, q_0\sin\left(\frac{\alpha}{2}\right)\boldsymbol{b} + \cos\left(\frac{\alpha}{2}\right)\boldsymbol{q}_{123} + \sin\left(\frac{\alpha}{2}\right)\boldsymbol{b} \times \boldsymbol{q}_{123}\right] \tag{27}$$

Therefore,

$$\boldsymbol{q} = \begin{bmatrix} q_0 \\ \boldsymbol{q}_{123} \end{bmatrix} \cos\left(\frac{\alpha}{2}\right) + \begin{bmatrix} -\boldsymbol{b}^T\boldsymbol{q}_{123} \\ q_0\boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{q}_{123} \end{bmatrix} \sin\left(\frac{\alpha}{2}\right) \doteq \boldsymbol{q}_{b1}\cos\left(\frac{\alpha}{2}\right) + \boldsymbol{q}_{b2}\sin\left(\frac{\alpha}{2}\right) \tag{28}$$

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Observe

$$q_{b1}^T q_{b1} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{29}$$

$$q_{b2}^T q_{b2} = \left(b^T q_{123}\right)^2 + (q_0 b)^2 + (b \times q_{123})^2 = q_{123}^T q_{123} (\cos \angle(b, q_{123}))^2 + q_0^2 + q_{123}^T q_{123} (\sin \angle(b, q_{123}))^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{30}$$

$$q_{b1}^T q_{b2} = -q_0 b^T q_{123} + q_{123}(q_0 b + b \times q_{123}) = 0 \tag{31}$$

Therefore, q_{b1} and q_{b2} are two orthonormal bases in $\mathbb{R}^4$, which can be considered as u_0 and u_1 in Eq. (16). Consequently, q lies in the great circle in $\mathbb{S}^3$.

5. Special Case II: Clifford Torus in $\mathbb{S}^3$

5.1. Alternative Way to Obtain Clifford Torus in $\mathbb{S}^3$

In the previous text, we provide a general derivation for the connection between rotation constraint, i.e., $b^T \mathbf{R} a = 0$ and Clifford torus. Here we show an alternative way to obtain Clifford torus from rotation constraint.

Given a rotation constraint $b^T \mathbf{R} a = 0$, the possible rotation $\mathbf{R}$ can be (see Figure 2)

$$\mathbf{R}(\alpha, \beta) = \mathbf{R}_{v_2}(\beta) \mathbf{R}_{v_1}(\alpha) \mathbf{R}_0, \quad \alpha, \beta \in [-\pi, \pi] \tag{32}$$

where $v_2 = b$, $v_1^T v_1 = 1$ and $v_1^T v_2 = 0$; $\mathbf{R}_0$ is a rotation that rotates a to v_1; $\mathbf{R}_{v_1}(\alpha) = \exp(\alpha [v_1]_\times)$ and $\mathbf{R}_{v_2}(\beta) = \exp(\beta [v_2]_\times)$ are rotations around directions b_1 and b, respectively. The rotation motion can be as follows,

$$v_1 = \mathbf{R}_0 a, \longrightarrow v_1 = \mathbf{R}_{v_1}(\alpha) \mathbf{R}_0 a, \longrightarrow b^T (\mathbf{R}_{v_2}(\beta) \mathbf{R}_{v_1}(\alpha) \mathbf{R}_0 a) = 0. \tag{33}$$

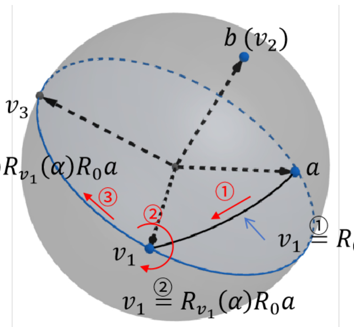

Figure 2. The geometric relationship of one rotation constraint $b^T \mathbf{R} a = 0$ and its solution $\mathbf{R}(\alpha, \beta) = \mathbf{R}_{v_2}(\beta) \mathbf{R}_{v_1}(\alpha) \mathbf{R}_0$.

To use quaternion to represent rotation [30]

$$\mathbf{R}_{v_2}(\beta) := q_{v_2}(\beta) = \begin{bmatrix} \cos\left(\frac{\beta}{2}\right) & v_2^T \sin\left(\frac{\beta}{2}\right) \end{bmatrix}^T \tag{34}$$

$$\mathbf{R}_{v_1}(\alpha) := q_{v_1}(\alpha) = \begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) & v_1^T \sin\left(\frac{\alpha}{2}\right) \end{bmatrix}^T \tag{35}$$

$$\mathbf{R}_0 := \begin{bmatrix} q_0 & q_{123}^T \end{bmatrix}^T \tag{36}$$

The formula for the multiplication of two quaternions is [29]

$$(r_1, \vec{v_1})(r_2, \vec{v_2}) = (r_1 r_2 - \vec{v_1} \cdot \vec{v_2}, r_1 \vec{v_2} + r_2 \vec{v_1} + \vec{v_1} \times \vec{v_2}) \tag{37}$$

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Define $\boldsymbol{v}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2$, then

$$\mathbf{R}_{v_2}(\beta)\mathbf{R}_{v_1}(\alpha) := \begin{bmatrix}\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}\right) & \boldsymbol{v}_2^T\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}\right) + \boldsymbol{v}_1^T\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) - \boldsymbol{v}_3^T\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2}\right)\end{bmatrix}^T \tag{38}$$

In addition, let $w_1 = \frac{1}{2}(\alpha - \beta)$ and $w_2 = \frac{1}{2}(\alpha + \beta)$,

$$\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}\right) = \frac{1}{2}\left(\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\right) = \frac{1}{2}(\cos(w_1) + \cos(w_2)) \tag{39}$$

$$\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}\right) = \frac{1}{2}\left(\sin\left(\frac{\alpha+\beta}{2}\right) - \sin\left(\frac{\alpha-\beta}{2}\right)\right) = \frac{1}{2}(\sin(w_2) - \sin(w_1)) \tag{40}$$

$$\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) = \frac{1}{2}\left(\sin\left(\frac{\alpha+\beta}{2}\right) + \sin\left(\frac{\alpha-\beta}{2}\right)\right) = \frac{1}{2}(\sin(w_2) + \sin(w_1)) \tag{41}$$

$$\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2}\right) = \frac{1}{2}\left(\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)\right) = \frac{1}{2}(\cos(w_1) - \cos(w_2)) \tag{42}$$

Then,

$$\mathbf{R}_{v_2}(\beta)\mathbf{R}_{v_1}(\alpha) := \frac{1}{2}\begin{bmatrix}\cos(w_1) + \cos(w_2) & \sin(w_1)(\boldsymbol{v}_1^T - \boldsymbol{v}_2^T) - \cos(w_1)\boldsymbol{v}_3^T + \sin(w_2)(\boldsymbol{v}_1^T + \boldsymbol{v}_2^T) + \cos(w_2)\boldsymbol{v}_3^T\end{bmatrix}^T \tag{43}$$

Therefore,

$$\mathbf{R}(\alpha,\beta) = \mathbf{R}_{v_2}(\beta)\mathbf{R}_{v_1}(\alpha)\mathbf{R}_0 \tag{44}$$

$$:= \frac{1}{2}\begin{bmatrix}(\cos(w_1) + \cos(w_2))q_0 - (\sin(w_1)(\boldsymbol{v}_1^T - \boldsymbol{v}_2^T) - \cos(w_1)\boldsymbol{v}_3^T + \sin(w_2)(\boldsymbol{v}_1^T + \boldsymbol{v}_2^T) + \cos(w_2)\boldsymbol{v}_3^T)\boldsymbol{q}_{123} \\ (\cos(w_1) + \cos(w_2))\boldsymbol{q}_{123} + (\sin(w_1)(\boldsymbol{v}_1 - \boldsymbol{v}_2) - \cos(w_1)\boldsymbol{v}_3 + \sin(w_2)(\boldsymbol{v}_1 + \boldsymbol{v}_2) + \cos(w_2)\boldsymbol{v}_3)q_0 + \\ (\sin(w_1)(\boldsymbol{v}_1 - \boldsymbol{v}_2) - \cos(w_1)\boldsymbol{v}_3 + \sin(w_2)(\boldsymbol{v}_1 + \boldsymbol{v}_2) + \cos(w_2)\boldsymbol{v}_3) \times \boldsymbol{q}_{123}\end{bmatrix} \tag{45}$$

(46)

$$= \frac{1}{2}\begin{bmatrix}\boldsymbol{k}_0 & \boldsymbol{k}_1 & \boldsymbol{k}_2 & \boldsymbol{k}_3\end{bmatrix}\begin{bmatrix}\cos(w_1) \\ \sin(w_1) \\ \cos(w_2) \\ \sin(w_2)\end{bmatrix}, \quad w_1 = \frac{1}{2}(\alpha - \beta), \text{and}, w_2 = \frac{1}{2}(\alpha + \beta) \tag{47}$$

$$= \frac{1}{2}(\boldsymbol{k}_0\cos(w_1) + \boldsymbol{k}_1\sin(w_1) + \boldsymbol{k}_2\cos(w_2) + \boldsymbol{k}_3\sin(w_2)) \tag{48}$$

where

$$\boldsymbol{k}_0 = \begin{bmatrix}q_0 + \boldsymbol{v}_3^T\boldsymbol{q}_{123} \\ \boldsymbol{q}_{123} - q_0\boldsymbol{v}_3 - \boldsymbol{v}_3 \times \boldsymbol{q}_{123}\end{bmatrix} \quad \boldsymbol{k}_1 = \begin{bmatrix}-(\boldsymbol{v}_1 - \boldsymbol{v}_2)^T\boldsymbol{q}_{123} \\ q_0(\boldsymbol{v}_1 - \boldsymbol{v}_2) + (\boldsymbol{v}_1 - \boldsymbol{v}_2) \times \boldsymbol{q}_{123}\end{bmatrix} \tag{49}$$

$$\boldsymbol{k}_2 = \begin{bmatrix}q_0 - \boldsymbol{v}_3^T\boldsymbol{q}_{123} \\ \boldsymbol{q}_{123} + q_0\boldsymbol{v}_3 + \boldsymbol{v}_3 \times \boldsymbol{q}_{123}\end{bmatrix} \quad \boldsymbol{k}_3 = \begin{bmatrix}-(\boldsymbol{v}_1 + \boldsymbol{v}_2)^T\boldsymbol{q}_{123} \\ q_0(\boldsymbol{v}_1 + \boldsymbol{v}_2) + (\boldsymbol{v}_1 + \boldsymbol{v}_2) \times \boldsymbol{q}_{123}\end{bmatrix} \tag{50}$$

It is easy to verify

$$\begin{bmatrix}\boldsymbol{k}_0 & \boldsymbol{k}_1 & \boldsymbol{k}_2 & \boldsymbol{k}_3\end{bmatrix}^T\begin{bmatrix}\boldsymbol{k}_0 & \boldsymbol{k}_1 & \boldsymbol{k}_2 & \boldsymbol{k}_3\end{bmatrix} = \begin{bmatrix}2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2\end{bmatrix} \tag{51}$$

Geometrically, Eq. (48) describes a Clifford torus [42,46] in $\mathbb{S}^3$.

5.2. Intersections of Two Different Clifford Tori

Given two rotation constraints $\boldsymbol{b}_1^T\mathbf{R}\boldsymbol{a}_1 = 0$ and $\boldsymbol{b}_2^T\mathbf{R}\boldsymbol{a}_2 = 0$, the possible rotation can be considered in the intersections of two Clifford tori (see Figure 3). Here, a solution to solve the intersection line is presented. Specifically,

$$\begin{cases} \boldsymbol{b}_1^T\mathbf{R}\boldsymbol{a}_1 = 0 \\ \boldsymbol{b}_2^T\mathbf{R}\boldsymbol{a}_2 = 0 \end{cases} \Rightarrow \begin{cases} \mathbf{R}(\alpha,\beta) = \mathbf{R}_{v_2}(\beta)\mathbf{R}_{v_1}(\alpha)\mathbf{R}_0 \\ \boldsymbol{b}_2^T\mathbf{R}\boldsymbol{a}_2 = 0 \end{cases} \Rightarrow \boldsymbol{b}_2^T\mathbf{R}_{v_2}(\beta)\mathbf{R}_{v_1}(\alpha)\mathbf{R}_0\boldsymbol{a}_2 = 0 \tag{52}$$

where $v_2 = \boldsymbol{b}_1$ and $v_1 \in \mathbb{S}^2$ is an arbitrary vector and satisfies $v_1^T v_2 = 0$; $\mathbf{R}_0$ is a specific arbitrary rotation that can rotate $\boldsymbol{a}$ to v_1; $\mathbf{R}_{v_1}(\alpha) = \exp(\alpha[v_1]_\times)$ and $\mathbf{R}_{v_2}(\beta) = \exp(\beta[v_2]_\times)$.

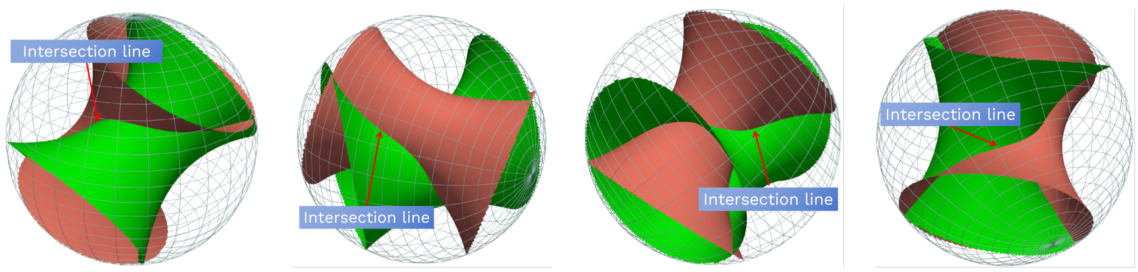

Figure 3. The illustration for intersection curves of two Clifford tori in $\mathbb{R}^3$ by stereographic projection.

According to Rodrigues' rotation formula [3], we have

$$\mathbf{R}_{v1}(\alpha) = \exp(\alpha[v_1]_\times) = \mathbf{I} + \sin(\alpha)[v_1]_\times + (1-\cos(\alpha))[v_1]_\times^2 \tag{53}$$
$$\mathbf{R}_{v2}(\beta) = \exp(\beta[v_2]_\times) = \mathbf{I} + \sin(\beta)[v_2]_\times + (1-\cos(\beta))[v_2]_\times^2 \tag{54}$$

Therefore,

$$\boldsymbol{b}_2^T\left(\mathbf{I} + \sin(\beta)[v_2]_\times + (1-\cos(\beta))[v_2]_\times^2\right)\left(\mathbf{I} + \sin(\alpha)[v_1]_\times + (1-\cos(\alpha))[v_1]_\times^2\right)\mathbf{R}_0\boldsymbol{a}_2 = 0 \tag{55}$$

Clearly, Eq.(55) shows a one-dimensional curve constructed by α and β (see Figure 4).

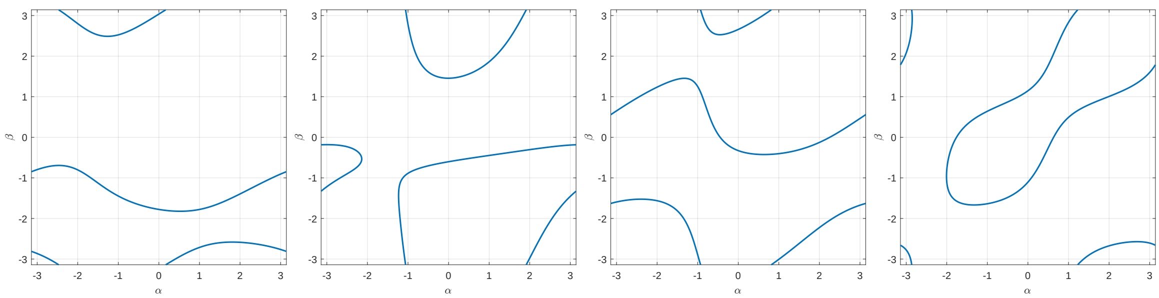

Figure 4. The illustration for intersection curves of two Clifford tori in α and β space.

Specifically, given a $\alpha \in [-\pi, \pi]$, β would be computed by Eq. (55). Intuitively, one should solve the following equation

$$\sin(\beta)A + \cos(\beta)B + C = 0 \tag{56}$$

where $A = \boldsymbol{b}_2^T[v_2]_\times\mathbf{R}_{v_1}(\alpha)\mathbf{R}_0\boldsymbol{a}_2$, $B = -\boldsymbol{b}_2^T[v_2]_\times^2\mathbf{R}_{v_1}(\alpha)\mathbf{R}_0\boldsymbol{a}_2$ and $C = \boldsymbol{b}_2^T v_2 v_2^T\mathbf{R}_{v_1}(\alpha)\mathbf{R}_0\boldsymbol{a}_2$. According to the tangent half-angle formula, when $\beta \neq \pm\pi$,

$$\frac{2\tan\left(\frac{\beta}{2}\right)}{1+\tan^2\left(\frac{\beta}{2}\right)}A + \frac{1-\tan^2\left(\frac{\beta}{2}\right)}{1+\tan^2\left(\frac{\beta}{2}\right)}B + C = 0 \xrightarrow{y \triangleq \tan\left(\frac{\beta}{2}\right)} 2Ay + (1-y^2)B + (1+y^2)C = 0 \Rightarrow (C-B)y^2 + 2Ay + B + C = 0 \tag{57}$$

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Accordingly, β can be computed and rotation $\mathbf{R}(\alpha,\beta)$ can be computed as well. Formally,

$$f : \alpha \in [\pi, \pi] \xrightarrow{\beta \text{ by Eq.(55)}} \mathbf{R}(\alpha,\beta) \tag{58}$$

5.3. Intersections of Three Different Clifford Tori

Given three rotation constraints $\boldsymbol{b}_1^T\mathbf{R}\boldsymbol{a}_1 = 0$, $\boldsymbol{b}_2^T\mathbf{R}\boldsymbol{a}_2 = 0$ and $\boldsymbol{b}_3^T\mathbf{R}\boldsymbol{a}_3 = 0$, the possible rotation can be considered in the intersections of three Clifford tori. Note that it belongs to the famous E3Q3 problem [47], and there are many solutions have been proposed to solve it [48–52]. Here this paper provides a more straight linear way. Specifically,

$$\begin{cases} \boldsymbol{b}_1^T\mathbf{R}\boldsymbol{a}_1 = 0 \\ \boldsymbol{b}_2^T\mathbf{R}\boldsymbol{a}_2 = 0 \\ \boldsymbol{b}_3^T\mathbf{R}\boldsymbol{a}_3 = 0 \end{cases} \Rightarrow \begin{cases} \mathbf{R}(\alpha,\beta) = \mathbf{R}_{v_2}(\beta)\mathbf{R}_{v_1}(\alpha)\mathbf{R}_0 \\ \boldsymbol{b}_2^T\mathbf{R}\boldsymbol{a}_2 = 0 \\ \boldsymbol{b}_3^T\mathbf{R}\boldsymbol{a}_3 = 0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{b}_2^T\mathbf{R}_{v_2}(\beta)\mathbf{R}_{v_1}(\alpha)\mathbf{R}_0\boldsymbol{a}_2 = 0 \\ \boldsymbol{b}_3^T\mathbf{R}_{v_2}(\beta)\mathbf{R}_{v_1}(\alpha)\mathbf{R}_0\boldsymbol{a}_3 = 0 \end{cases} \tag{59}$$

Therefore,

$$\boldsymbol{b}_2^T\left(\mathbf{I} + \sin(\beta)[v_2]_\times + (1-\cos(\beta))[v_2]_\times^2\right)\left(\mathbf{I} + \sin(\alpha)[v_1]_\times + (1-\cos(\alpha))[v_1]_\times^2\right)\mathbf{R}_0\boldsymbol{a}_2 = 0 \tag{60}$$

$$\boldsymbol{b}_3^T\left(\mathbf{I} + \sin(\beta)[v_2]_\times + (1-\cos(\beta))[v_2]_\times^2\right)\left(\mathbf{I} + \sin(\alpha)[v_1]_\times + (1-\cos(\alpha))[v_1]_\times^2\right)\mathbf{R}_0\boldsymbol{a}_3 = 0 \tag{61}$$

There are two unknown variables α and β for two constraints, and the intersections can be computed.

According to tangent half-angle formula (a.k.a. Weierstrass substitution formulas),

$$\sin(\alpha) = \frac{2\tan(\frac{\alpha}{2})}{1+\tan^2(\frac{\alpha}{2})}, \quad \cos(\alpha) = \frac{1-\tan^2(\frac{\alpha}{2})}{1+\tan^2(\frac{\alpha}{2})}, \quad \sin(\beta) = \frac{2\tan\left(\frac{\beta}{2}\right)}{1+\tan^2\left(\frac{\beta}{2}\right)}, \quad \cos(\beta) = \frac{1-\tan^2\left(\frac{\beta}{2}\right)}{1+\tan^2\left(\frac{\beta}{2}\right)} \tag{62}$$

Let $x = \tan(\frac{\alpha}{2})$, and $y = \tan\left(\frac{\beta}{2}\right)$, and when $\alpha \neq \pm\pi, \beta \neq \pm\pi$,

$$\boldsymbol{b}_2^T\left((1+y^2)\mathbf{I} + 2y[v_2]_\times + 2y^2[v_2]_\times^2\right)\left((1+x^2)\mathbf{I} + 2x[v_1]_\times + 2x^2[v_1]_\times^2\right)\mathbf{R}_0\boldsymbol{a}_2 = 0 \tag{63}$$

$$\boldsymbol{b}_3^T\left((1+y^2)\mathbf{I} + 2y[v_2]_\times + 2y^2[v_2]_\times^2\right)\left((1+x^2)\mathbf{I} + 2x[v_1]_\times + 2x^2[v_1]_\times^2\right)\mathbf{R}_0\boldsymbol{a}_3 = 0 \tag{64}$$

The system of equations can be reformulated as

$$\mathcal{F}(x,y) = \begin{bmatrix} y^2 & y & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = 0, \quad \mathcal{G}(x,y) = \begin{bmatrix} y^2 & y & 1 \end{bmatrix} \mathbf{B} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = 0 \tag{65}$$

where $\mathbf{A}$ and $\mathbf{B} \in \mathbb{R}^{3\times3}$, Specifically,

$$\mathbf{A} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} = \boldsymbol{b}_2^T * \begin{bmatrix} (\mathbf{I}+2[v_2]_\times^2)(\mathbf{I}+2[v_1]_\times^2) & (\mathbf{I}+2[v_2]_\times^2)(2[v_1]_\times) & \mathbf{I}+2[v_2]_\times^2 \\ (2[v_2]_\times)(\mathbf{I}+2[v_1]_\times^2) & (2[v_2]_\times)(2[v_1]_\times) & 2[v_2]_\times \\ \mathbf{I}+2[v_1]_\times^2 & 2[v_1]_\times & \mathbf{I} \end{bmatrix} * \mathbf{R}_0\boldsymbol{a}_2 \tag{66}$$

$$\mathbf{B} = \begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix} = \boldsymbol{b}_3^T * \begin{bmatrix} (\mathbf{I}+2[v_2]_\times^2)(\mathbf{I}+2[v_1]_\times^2) & (\mathbf{I}+2[v_2]_\times^2)(2[v_1]_\times) & \mathbf{I}+2[v_2]_\times^2 \\ (2[v_2]_\times)(\mathbf{I}+2[v_1]_\times^2) & (2[v_2]_\times)(2[v_1]_\times) & 2[v_2]_\times \\ \mathbf{I}+2[v_1]_\times^2 & 2[v_1]_\times & \mathbf{I} \end{bmatrix} * \mathbf{R}_0\boldsymbol{a}_3 \tag{67}$$

and $*$ is to calculate the matrix item by item. Geometrically, if only considering real solutions, it describes the intersections of two cures computed by observations in $z = 0$ plane (see Figure 5).

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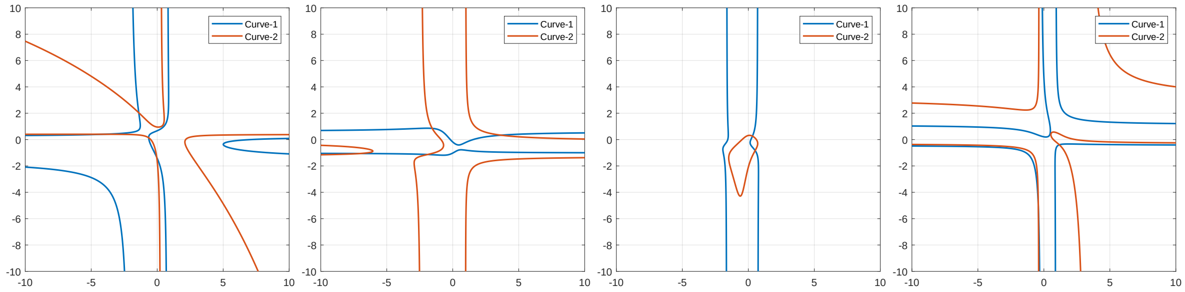

Figure 5. The intersection of two curves that are constructed by rotation constraints in the 2D plane. Specifically, we can consider $z = \mathcal{F}(x, y)$ and $z = \mathcal{G}(x, y)$ as two surfaces in $\mathbb{R}^3$. Let $z = 0$, it means $z = 0$ plane and it cuts the surfaces $\mathcal{F}(x, y)$ and $\mathcal{G}(x, y)$. Consequently, there are two intersection lines, which can be drawn in $z = 0$ plane.

Algebraically, it is a polynomial system containing two equations in two variables, which can be generally solved by resultant-based method [53–55]. We donate

$$\begin{bmatrix} y^2 & y & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \mathcal{A}(y) \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} y^2 & y & 1 \end{bmatrix} \mathbf{B} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \mathcal{B}(y) \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = 0 \tag{68}$$

Both sides of the equations are multiplied by x,

$$\mathcal{A}(y) \begin{bmatrix} x^3 \\ x^2 \\ x \end{bmatrix} = 0, \quad \mathcal{B}(y) \begin{bmatrix} x^3 \\ x^2 \\ x \end{bmatrix} = 0 \tag{69}$$

By stacking equations,

$$\begin{bmatrix} 0 & \mathcal{A}(y) \\ 0 & \mathcal{B}(y) \\ \mathcal{A}(y) & 0 \\ \mathcal{B}(y) & 0 \end{bmatrix} \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \mathcal{H}(y) \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = 0 \tag{70}$$

To solve y, it is to solve $\left|\mathcal{H}(y)\right| = 0$. This determinant is also known as a hidden-variable resultant in computer vision filed [56,57], and it is an univariate polynomial with degree 8 of the hidden variable y.

$$\left|\mathcal{H}(y)\right| = a_8 y^8 + a_7 y^7 + a_6 y^6 + a_5 y^5 + a_4 y^4 + a_3 y^3 + a_2 y^2 + a_1 y + a_0 = 0 \tag{71}$$

In fact, the problem is not very complex, one can directly calculate the explicit expression of each term for the resultant-based method using symbolic math. Practically, one can obtain explicit coefficients $\{a_i\}_{i=0}^{8}$ by the function `resultant` in Matlab [55] or the function `sympy.polys.polytools.resultant` in SymPy[1], and this paper will not list all details to avoid long-and-tedious expressions for readability. Nonetheless, the solution of y can be directly computed by solving roots of a univariate polynomial [58,59]. After solving y, it is easy to calculate x.

1 https://www.sympy.org/en/index.html

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5.3.1. Linear Solution to Solve Intersections of Three Different Clifford Tori

Alternatively, Eq. (70) can be rewritten as a quadratic eigenvalue problem, i.e., $(\lambda^2\mathbf{H}_2 + \lambda\mathbf{H}_1 + \mathbf{H}_0)v = 0$ [56,57,60].

$$\mathcal{H}(y)\begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = 0 \Rightarrow \left(y^2\mathbf{H}_2 + y\mathbf{H}_1 + \mathbf{H}_0\right)\begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = 0 \tag{72}$$

where $\mathbf{H}_0$, $\mathbf{H}_1$ and $\mathbf{H}_2 \in \mathbb{R}^{4\times 4}$. Specifically,

$$\mathbf{H}_2 = \begin{bmatrix} 0 & A_{1,1} & A_{1,2} & A_{1,3} \\ 0 & B_{1,1} & B_{1,2} & B_{1,3} \\ A_{1,1} & A_{1,2} & A_{1,3} & 0 \\ B_{1,1} & B_{1,2} & B_{1,3} & 0 \end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix} 0 & A_{2,1} & A_{2,2} & A_{2,3} \\ 0 & B_{2,1} & B_{2,2} & B_{2,3} \\ A_{2,1} & A_{2,2} & A_{2,3} & 0 \\ B_{2,1} & B_{2,2} & B_{2,3} & 0 \end{bmatrix}, \quad \mathbf{H}_0 = \begin{bmatrix} 0 & A_{3,1} & A_{3,2} & A_{3,3} \\ 0 & B_{3,1} & B_{3,2} & B_{3,3} \\ A_{3,1} & A_{3,2} & A_{3,3} & 0 \\ B_{3,1} & B_{3,2} & B_{3,3} & 0 \end{bmatrix} \tag{73}$$

Notably, there is no x and y in $\mathbf{H}_2$, $\mathbf{H}_1$ and $\mathbf{H}_0$. Accordingly, it is easy to solve the system of polynomial equations by linear way [60].

Specifically, the to-be-solved eigenvector of this problem is $v = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix}^T$ and corresponding eigenvalue is $\lambda = y$. Furthermore, the quadratic eigenvalue problem can be transformed into a generalized eigenvalue problem, i.e., $\mathbf{A}x = \lambda\mathbf{B}x$ [61], which has been extensively studied [57,61–63]. Specifically,

$$\left(\lambda^2\mathbf{H}_2 + \lambda\mathbf{H}_1 + \mathbf{H}_0\right)v = 0 \Rightarrow \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{H}_0 & -\mathbf{H}_1 \end{bmatrix}\begin{bmatrix} v \\ \lambda v \end{bmatrix} = \lambda\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 \end{bmatrix}\begin{bmatrix} v \\ \lambda v \end{bmatrix} \tag{74}$$

Therefore, there will be 8 eigenvalues of this linear problem. The easiest way to solve this problem is using the Matlab function built-in `polyeig`[2]. SciPy also provides a function `scipy.linalg.eig`[3] to solve generalized eigenvalue problems.

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[2] https://www.mathworks.com/help/matlab/ref/polyeig.html
[3] https://docs.scipy.org/doc/scipy/reference/generated/scipy.linalg.eigvals.html#scipy.linalg.eigvals

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