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Article

Analytical Representations of Heaviside and Ramp Function

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Abstract: In this paper, explicit exact forms of Heaviside Step Function and Ramp Function are presented. These significant functions constitute fundamental concepts of operational calculus and digital signal processing theory and are also involved in many other areas of applied sciences and engineering. In particular, Heaviside and Ramp Function are performed as bi-parametric single - valued functions with only one restriction imposed on each parameter. The novelty of this work when compared to other investigations concerning exact forms of Unit Step Function and/or Ramp Function, is that the proposed analytical formulae are not exhibited in terms of miscellaneous special functions, e.g. Gamma Function, Biexponential Function or any other special functions such as Error Function, Complementary Error Function, Hyperbolic Function, Orthogonal polynomials etc. Hence, these formulae may be much more practical, flexible and useful in the computational procedures which are inserted into operational calculus and digital signal processing techniques as well as other engineering practices.

Keywords: heaviside step function; ramp function; bi-parametric function; exponential quantity; integer part

Mathematical subject classification: 26A06

Introduction

The Ramp Function the notation of which is $\mathrm{R}(x)$, where x denotes the argument, is a discontinuous single - valued function of a real variable with a point discontinuity located at zero. For negative arguments the output of this function is zero, whilst for positive arguments $\mathrm{R}(x)$ is simply x [1,2].

In addition, its first derivative with respect to x is the Heaviside Step Function, also known as the Unit Step Function, the notation of which is $H(x)$. Heaviside Function is commonly used in the mathematics of control theory as well as in signal processing in order to represent a signal that switches on at a specified time and stays switched on indefinitely [2,3]. Heaviside Step Function, represents a sudden change in value at a specific point. Moreover, it is commonly used in engineering and applied physics to model discontinuous events. Indeed, it has many applications in signal processing, control systems, and circuit analysis. For instance, it is used to model events such as turning on or off a switch, sudden changes in voltage or current, and the activation of a system based on a threshold value. Another example of the utility of the Unit Step Function is in electrical circuits, where it can be used to model the behavior of a switch. Moreover, in control systems, it can represent the activation of a system when a certain condition is met. It can also be used in signal processing to model the sharp rise or fall of a signal.

Besides, the second derivative of Ramp Function is the Dirac delta distribution (or δ distribution), also known as the unit impulse [4,5]. Step, Ramp and parabolic functions are called singularity functions [4,5]. Nonetheless, regardless of its name the Dirac delta function does not constitute truly a function and indeed Heaviside Step Function is not stringently differentiable. For a

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rigorous treatment of the Dirac delta distribution elements of Measure Theory are required [5]. In fact, both Heaviside Step Function and Ramp Function have many applications in applied sciences and engineering and are mainly involved in digital signal processing and electrical engineering. Actually, Heaviside Step Function may serve as a test input signal for characterizing the response of linear systems throughout linear system theory and feedback control theory. In addition, it may serve as a multiplicative weighting function to create causal (one-sided) functions [4,5]. For instance, a causal exponential time function may be expressed as follows

$$f(t) = H(t) \cdot EXP(-t) \quad (1)$$

Besides, Ramp Function constitutes a signal the amplitude of which varies linearly with respect to time and can be expressed by several definitions [4]. In digital signal processing, the unit ramp function is a discrete time signal that starts from zero and increases linearly.

Meanwhile, one may remark that a signal is defined as a physical phenomenon which carries some information or data [6]. The signals are usually functions of independent variable time. Nonetheless there are some cases where the signals are not functions of time. For instance, the electrical charge distributed in a body constitutes a signal which is a function of space and not time.

The signal that is specified for every value of time t is called continuous-time signal whilst the signal that is specified at a discrete value of time is called discrete-time signal.

Basic signals play a very important role in signals and systems analysis. The basic continuous time signals are as follows [7,8]: 1. Unit impulse function. 2. Unit step function. 3. Unit ramp function. 4. Unit parabolic function. 5. Unit rectangular pulse (or Gate) function. 6. Unit area triangular function. 7. Unit signum function. 8. Unit Sinc function. 9. Sinusoidal signal. 10. Real exponential signal. 11. Complex exponential signal.

It is worth noting, that the basic Continuous - Time (CT) and Discrete - Time (DT) signals include impulse, step, ramp, parabolic, rectangular pulse, triangular pulse, signum function, sinc function, sinusoid and finally real along with complex exponentials [6–8]. Ramp Function states that the signal will start from time zero and instantly will take a slant shape and depending upon given time characteristics (i.e. either positive or negative, here positive) the signal will follow the straight slant path either towards right or left, here towards right [7–9]. In this context, the ramp function constitutes a type of elementary function which exists only for the positive side and is zero for negative [8–10]. Moreover, the impulse function is obtained by differentiating the ramp function twice [9,10]. On the other hand, it is well - known that an electrical network consists of passive elements like resistors, capacitors and inductors. They are connected in series, parallel and series parallel combinations [10–12]. The currents through and voltages across these elements are obtained by solving integro-differential equations. Alternatively, the elements in the network are transformed from the time domain and an algebraic equation is obtained which is expressed in terms of input and output [10,12,13]. The commonly used inputs are impulse, step, ramp, sinusoids, exponentials etc.

In the past years, there is a lot of research work carried out for the analytical approximation of the Unit Step Function along with Ramp Function and Dirac delta - distribution.

Specifically in Ref. [14] a rigorous representation of Heaviside Step Function was accomplished by a linear combination of exponential functions, whereas for a rigorous study on the numerical approximation of Heaviside Step Function by means of a finite difference method one may refer to Ref. [15]. On the other hand, the limiting form of many sigmoid type functions centered on $x = 0$ may also serve as an approximation to the Heaviside Step Function. For instance, one may remark the following expressions [16]:

$$H(x) = \lim_{k\to\infty} \frac{1}{2}(1 + tanh(kx)) \quad (2)$$

$$H(x) = \lim_{k\to\infty} \left(\frac{1}{2} + \frac{1}{\pi} arctan(kx)\right) \quad (3)$$

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$$H(x) = \lim_{k\to\infty} \frac{1}{2}(1 + erf(kx)) \quad (4)$$

where the notation erf denotes the well - known Error Function

Evidently, by multiplying both sides of the above three equations with the argument x one shall obtain closed form expressions of the Ramp Function.

Besides, there are many explicit forms of Heaviside Step Function and Ramp Function that can be found in the literature.

Specifically, in Ref. [4] an elegant explicit representation of this function was proposed as the summation of two inverse trigonometric functions, by means of the following relationship:

$$H(x) = \frac{1}{2} + \frac{1}{\pi}\cdot\left(arctan(x) + arctan(\frac{1}{x})\right) \quad (5)$$

In this framework, by multiplying both sides of the above equation with the argument x one shall derive the Ramp Function as

$$R(x) = \frac{x}{2} + \frac{x}{\pi}\cdot(arctan(x) + arctan(\frac{1}{x})) \quad (6)$$

In Ref. [17] another analytic exact form of the unit step function, as a summation of two inverse trigonometric functions, was presented

$$H(x) = \frac{3}{4} + \frac{1}{\pi}\cdot\left(arctan(x-1) + arctan(\frac{x-2}{x})\right) \quad (7)$$

and therefore

$$R(x) = \frac{3x}{4} + \frac{x}{\pi}\cdot\left(arctan(x-1) + arctan(\frac{x-2}{x})\right) \quad (8)$$

In Ref. [18] the following elegant analytical form of Heaviside Step Function was performed

$$H(x) = \frac{1}{2} + i\,\frac{ln\,(x) - ln\,(-x)}{2\pi} \quad (9)$$

where i denotes the imaginary unit

In this context, one may also obtain the Ramp Function as

$$R(x) = \frac{x}{2} + i\,\frac{xln\,(x) - xln\,(-x)}{2\pi} \quad (10)$$

In addition, in Ref. [19] a stringent approximation of the Unit Step Function by the use of cumulative distribution functions was carried out, whilst in Refs. [20] and [21] Heaviside Step Function was approached via Raised – Cosine and Laplace Cumulative Distribution Functions and Sigmoid Functions respectively. Besides, in Ref. [22] an analytical exact form the Unit Step Function was exhibited by the use of the floor function, which evidently yields the greatest integer less than or equal to its argument.

Moreover, in Ref. [23] a rigorous approximation of the Unit Step Function was exhibited as the pointwise limit of a sequence of functions, whereas an analogous investigation towards an analytical approach of the Ramp Function was performed in Ref. [24].

Further, in Ref. [25] the Unit Step Function was represented as the summation of six inverse trigonometric functions. In particular, the novel element of this analytical formula is that it contains two arbitrary single valued functions which satisfy only one single condition and does not contain any other special functions such as Gamma function or generalize integrals compared to other representations of the Unit function.

On the other hand, there are many applications of Heaviside and Ramp Function in signal processing, as it can be seen in the literature.

In Ref. [26] a rigorous study on the relations between control signal properties and robustness measurements was carried out. In particular, Heaviside Step Function was used to express the specified output at a load disturbance. In Ref. [27] a single image super-resolution by means of approximated forms of Heaviside Step Function was performed.

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In addition, a valuable investigation concerning the measurement of regularity in discrete time signals was presented in Ref. [28]. Heaviside Step Function was used in the evaluation procedure of the entropy of a regular system with orderly behavior. In Ref. [29] a remarkable study on the use of Generalized Functions in the Study of Signals and Systems was carried out. Specifically, a physical system was simulated as a linear time-invariant (LTI) system. Evidently, such a system is represented mathematically by an ordinary differential equation (ODE), or by a set of coupled ODEs. In this context, Heaviside Step Function was taken into account towards the analytical treatment of this original problem. In Ref. [30] a Green's function analysis of nonlinear thermoacoustic effects under the influence of noise in a combustion chamber was carried out whereas in Ref. [31] an integrated vibration energy harvesting-storage-injection system based on piezoelectric bistable was performed.

Finally, in Ref. [32] a rigorous nonlinear vibration analysis and defect characterization using entropy measure-based learning algorithm in defective rolling element bearings was presented, whilst enhancing convolutional neural network robustness against image noise via an artificial visual system was carried out in Ref. [33].

Now, in the current theoretical investigation Heaviside Step Function together with its antiderivative i.e. Ramp Function are performed as bi-parametric single - valued functions with only one single restriction imposed on each parameter. The novelty of this work when compared to other theoretical investigations concerning analytical exact forms of these seminal functions, is that the proposed explicit formulae are not exhibited in terms of miscellaneous special functions, e.g. Gamma Function, Biexponential Function or any other special functions such as Error Function, Hyperbolic Function, Orthogonal polynomials etc.

2. Towards Explicit Forms of Heaviside and Ramp Function

Let us introduce the following two bi-parametric single – valued functions:

$f: R \to R^{+}$ such that

$$f(a,b,x) = \left[a^{b|x|+x|b|-bx-|bx|} \right] \quad (11)$$

and

$g: R \to R^{+}$ such that

$$g(a,b,x) = x \cdot \left[a^{b|x|+x|b|-bx-|bx|} \right] \quad (12)$$

where a and b are two arbitrarily selected real numbers such that $a > 1$ and $b < 0$.

In addition, the term $\left[a^{b|x|+x|b|-bx-|bx|} \right]$ denotes the integer part of the exponential quantity $a^{b|x|+x|b|-bx-|bx|}$.

Here one may observe that only one constraint has been imposed on each one of the parameters a and b.

3. Claim

The functions f and g coincide with Heaviside Step Function and Ramp Function respectively over the set $(-\infty, 0) \cup [0, +\infty)$.

4. Proof

First, we shall prove that for strictly positive arguments, every output of the bi-parametric function f equals unity, whereas the outputs of the bi-parametric function g coincide with those of the argument itself i.e. the real variable x. Next, we shall prove that for strictly negative arguments

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the outputs of both bi-parametric functions f and g respectively, vanish. Finally, at $x = 0$, we shall prove that the output of the function f yields unity whilst the output of the function g yields zero.

To this end, let us initiate our mathematical analysis by distinguishing the following three cases concerning the domain that the independent variable x belongs.

(i) $x \in (0, +\infty)$

In this context, let us concentrate on the exponential term $a^{b|x|+x|b|-bx-|bx|}$ which appears in both eqns. (1) and (2) and then focus on its exponent i.e. the quantity $b|x| + x|b| - bx - |bx|$. Since the parameter b has been considered beforehand to be a strictly negative number and also the variable x has been currently assumed to lie over the set $(0, +\infty)$ one may infer that the term $|bx|$ is a strictly positive quantity.

Hence the following relationship holds

$$b|x| + x|b| - bx - |bx| = |bx| \cdot \frac{b|x| + x|b| - bx - |bx|}{|bx|} \Leftrightarrow$$

$$b|x| + x|b| - bx - |bx| = |bx| \cdot \left(\frac{b|x|}{|b| \cdot |x|} + \frac{x|b|}{|b| \cdot |x|} - \frac{bx}{|bx|} - \frac{|bx|}{|bx|}\right) \Leftrightarrow$$

$$b|x| + x|b| - bx - |bx| = |bx| \cdot \left(\frac{x}{|x|} + \frac{b}{|b|} - \frac{bx}{|bx|} - 1\right) \Leftrightarrow$$

$$b|x| + x|b| - bx - |bx| = |bx| \cdot (sgn(x) + sgn(b) - sgn(bx) - 1)$$

(13)

Evidently, the notation sgn which appears in eqn. (13) denotes the well known Signum Function.

Since Signum Function yields the sign of any real number, it implies that

$$sgn(x) = 1\,;\ sgn(b) = -1;\ sgn(bx) = -1$$

Thus eqn. (13) can be equivalently written out

$$b|x| + x|b| - bx - |bx| = |bx| \cdot (1 - 1 - (-1) - 1) \Leftrightarrow$$

$$b|x| + x|b| - bx - |bx| = 0 \quad (14)$$

In the sequel, eqn. (11) and eqn. (12) respectively, can be combined with eqn. (14) to yield

$$f(a,b,x) = \left[\, a^0 \right] \Leftrightarrow$$

$$f(a,b,x) = 1 \quad (15)$$

and

$$g(a,b,x) = x \cdot \left[\, a^0 \right] \Leftrightarrow$$

$$g(a,b,x) = x \quad (16)$$

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Thus it was definitely proved that for strictly positive arguments the values of the bi-parametric function $f(a,b,x)$ equal unity, while the values of the bi-parametric function $g(a,b,x)$ coincide with those of the independent variable x.

(ii) $x \in (-\infty, 0)$

According to the same reasoning as before, let us centre on the exponential term $a^{b|x|+x|b|-bx-|bx|}$ and examine the sign of its exponent i.e. the quantity $b|x| + x|b| - bx - |bx|$.

Since the parameter b has been primarily considered to be a strictly negative number and besides the independent variable x is now supposed to lie over the set $(-\infty, 0)$ one may deduce that the term $|bx|$ is again a strictly positive number.

Thus, in the same framework as in the previous case, one may conclude that eqn. (13) still holds.

Hence, one may deduce that

$$sgn\,(x) = -1;\; sgn(b) = -1;\; sgn(bx) = 1$$

Thus one obtains

$$b|x| + x|b| - bx - |bx| = |bx| \cdot (-1 - 1 - 1 - 1) \Leftrightarrow$$

$$b|x| + x|b| - bx - |bx| = (-4) \cdot |bx| \quad (17)$$

Hence the following inequality is evident

$$b|x| + x|b| - bx - |bx| < 0 \quad (18)$$

Then, eqn. (11) and eqn. (12) respectively can be combined with eqn. (17) to yield

$$f(a,b,x) = \left[\, a^{(-4)\cdot|bx|} \right] \Leftrightarrow$$

$$f(a,b,x) = \left[\frac{1}{a^{4|bx|}} \right] \quad (19)$$

and

$$g(a,b,x) = x \cdot \left[\, a^{(-4)\cdot|bx|} \right] \Leftrightarrow$$

$$g(a,b,x) = x \cdot \left[\frac{1}{a^{4|bx|}} \right] \quad (20)$$

Now, since the base of the exponential term $a^{4|bx|}$ which appears in eqn. (20), i.e. the parameter a has been considered beforehand to be strictly greater than 1 one may easily conjecture that the following double inequality holds

$$0 < \frac{1}{a^{4|bx|}} < 1 \quad (21)$$

According to inequality (21), it implies that the values of the strictly positive fraction $\frac{1}{a^{4|bx|}}$ belong to the interval $(0,1)\ \forall x \in (-\infty, 0)$.

In this context, one may infer that for strictly negative arguments, the integer part of this aforementioned fraction vanishes i.e.

$$\left[\frac{1}{a^{4|bx|}} \right] = 0 \quad (22)$$

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or equivalently

$$\left[a^{(-4)\cdot|bx|} \right] = 0 \quad (23)$$

Eqn. (19) and eqn. (20) respectively, can be associated with eqn. (22) to yield

$$f(a,b,x) = 0 \quad (24)$$

and

$$g(a,b,x) = 0 \quad (25)$$

Thus it was definitely proved that the values of both bi-parametric functions f and g vanish for strictly negative arguments.

(iii) $x = 0$

Then one obtains

$$f(a,b,0) = \left[a^{0} \right] \Leftrightarrow$$

$$f(a,b,0) = 1 \quad (26)$$

and

$$g(a,b,0) = 0 \cdot \left[a^{0} \right] \Leftrightarrow$$

$$g(a,b,0) = 0 \quad (27)$$

After all, it was rigorously proved that the bi-parametric functions f and g introduced by eqns. (11) and (12) respectively, are synonymous to Heaviside Step Function and Ramp Function over the set of real numbers.

Discussion

In Section 2, two bi-parametric single - valued functions were proposed with the claim that they are synonymous with the Unit Step Function and the Ramp Function respectively over the set of real numbers. Next, in Section 4 rigorous proofs were given to justify this claim.

According to this viewpoint, one may also derive analytical representations of Signum Function and Absolute Value Function respectively.

Actually, Signum Function and Heaviside Step Function are linked with each other by means of the following linear relationship [34]

$$sgn(x) = 2H(x) - 1 \quad (28)$$

Eqn. (28) can be combined with eqn. (11) to yield

$$sgn(x) = 2 \cdot \left[a^{b|x|+x|b|-bx-|bx|} \right] - 1 \quad (29)$$

where $a > 1$ and $b < 0$

Here, one may observe that Signum Function has also been represented as a bi-parametric function i.e.

$$sgn(x) \equiv h(a,b,x) \quad (30)$$

where $h: R \to \{-1,0,+1\}$ is a bi-parametric bi-parametric single - valued function

Moreover, given an arbitrary real number C one may remark from Calculus [35] that concerning its integer part, notated by $[C]$, the following relationships hold

$2[C] = [2C]$ (31a)

when $C - [C] < \frac{1}{2}$

and

$2[C] = [2C] - 1$ (31b)

when $C - [C] \geq \frac{1}{2}$

Obviously, for non-negative values of the variable x one may deduce that

$$\square^{\square|\square|+\square|\square|-\square\square-|\square\square|} - \left[a^{b|x|+x|b|-bx-|bx|}\right] = 0 \Rightarrow$$

$$a^{b|x|+x|b|-bx-|bx|} - \left[a^{b|x|+x|b|-bx-|bx|}\right] < \frac{1}{2} \quad (32)$$

Thus eqn. (29) can be combined with (31a) to yield

$$sgn(x) = \left[2a^{b|x|+x|b|-bx-|bx|}\right] - 1 \quad (33)$$

However, for strictly negative arguments it was previously shown that $\left[a^{b|x|+x|b|-bx-|bx|}\right]$ vanishes.

and therefore

$$a^{b|x|+x|b|-bx-|bx|} - \left[a^{b|x|+x|b|-bx-|bx|}\right] = a^{b|x|+x|b|-bx-|bx|} \quad (34)$$

In addition, according to inequality (21) one infers

$$-\frac{1}{2} < a^{(-4)\cdot|bx|} - \frac{1}{2} < \frac{1}{2} \quad (35)$$

Hence the quantity $a^{(-4)\cdot|bx|} - \frac{1}{2}$ does not keep the same sign for strictly negative values of the variable x and therefore eqn. (29) cannot be combined with eqn. (31b).

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On the other hand, to calculate Absolute Value Function, the notation of which is ABS, one may proceed as follows:

It is known that Signum Function and ABS Function are linked with each other by means of the following relationship [34]

$$ABS(x) = x \cdot sgn(x) \quad (36)$$

Thus eqn. (36) in association with eqn. (29) yields

$$ABS(x) = 2x \cdot \left[a^{b|x|+x|b|-bx-|bx|} \right] - x \quad (37)$$

Thus ABS function was equivalently expressed in the following bi-parametric form

$$ABS(x) \equiv w(a, b, x) \quad (38)$$

where $w: R \to R^{+}$constitutes a bi-parametric single - valued function

Conclusions

The objective of this analytical investigation was to introduce closed - form representations of Heaviside Step Function and Ramp Function. The novelty of this work when compared to other analytical treatments to these important functions is that the proposed mathematical formulae are not exhibited in terms of miscellaneous special functions or any other special functions such as Error Function, Supplementary Error Function, Hyperbolic Function, Orthogonal polynomials etc.

In closing, as a future work, one may also propose, by taking into consideration this theoretical approach, an analytical representation to the Ceiling Function (also known as the least integer function) which is defined as the smallest integer that is not smaller than the independent real variable x,i.e. it returns the smallest integer which is not smaller than the input decimal.

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