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A Note on the Hermite-Hadamard Inequalities via Fourier Transform

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Abstract

The objective of this note is to establish new versions of the Hermite-Hadamard inequalities, for (h,m)-convex functions of the second type, by means of a completely new approach: the Fourier Transform.

Keywords: Hermite-Hadamard Inequality; Fourier transform; (h,m)-convex modified functions of second type

MSC: 26A33; 26D10; 26D15

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1. Introduction

Convex functions have played a fundamental and transversal role in many branches of mathematics, and their importance has grown significantly in the 20*th* and 21*st* centuries. From convex optimization to Measure and Probability Theory, through various fields such as: Real and Functional Analysis, Economics and Game Theory, Machine Learning and Data Science, Information Theory and Entropy, among others, it has been acquiring marked relevance, which has translated into more and more researchers becoming interested in this concept, which has brought an increase in research and results, causing various generalizations and extensions of the classical concept (interested readers can consult [16]).

Definition 1. *A set $\mathcal{I} \subset \mathbb{R}$ is said to be convex function if*

$$\tau\xi + (1-\tau)\varsigma \in \mathcal{I}$$

for each $\xi, \varsigma \in \mathcal{I}$ and $\tau \in [0,1]$.

Definition 2. *The function $\psi : \mathcal{I} \to \mathbb{R}$, is said to be convex function if the following inequality holds:*

$$\psi(\tau\xi + (1-\tau)\varsigma) \leq \tau\psi(\xi) + (1-\tau)\psi(\varsigma)$$

for all $\xi, \varsigma \in \mathcal{I}$ and $\tau \in [0,1]$.

The above inequality holds in opposite direction for concave function. Convex functions are at the heart of several classical integral inequalities, which are key tools in real analysis, probability, measure theory and other areas: Jensen's inequality, Dragomir-Agarwal inequality, Prekopa-Leindler inequality and, by far the best known, the Hermite-Hadamard inequality.

These inequalities are of central importance. Below is a statement of this double inequality:

Suppose that ψ is a convex function on the closed real interval $[\xi, \varsigma]$ where $\xi \neq \varsigma$. Therefore (see[1,8,9,13,15])

$$\psi\left(\frac{\xi+\varsigma}{2}\right) \leq \frac{1}{\varsigma-\xi}\int_{\xi}^{\varsigma}\psi(\varkappa)d\varkappa \leq \frac{\psi(\xi)+\psi(\varsigma)}{2}.$$

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For more recent developments related to the Hermite–Hadamard inequality, the reader may consult references [2,5–7,11,12,18].

Several important inequalities have been established using different types of convexity. One such type is the modified (h, m)–convexity.

This class was defined in [3,4] as follows:

Definition 3. *Let $\psi : \mathcal{I} \subseteq [0, \infty) \to [0, \infty)$ and $h : [0, 1] \to (0, 1]$. If inequality*

$$\psi(\xi\tau + m\varsigma(1-\tau)) \leq \psi(\xi)h^s(\tau) + m\psi(\varsigma)(1 - h^s(\tau)) \tag{1}$$

is fulfilled $\forall \varsigma, \xi \in \mathcal{I}$ and $\tau \in [0, 1]$, where $s \in [0, 1]$ and $m \in [0, 1]$. The function ψ is then referred to as a modified (h, m)–convex function of the first type on $\mathcal{I}$.

Definition 4. *Let $h : [0, 1] \to (0, 1]$ and $\psi : \mathcal{I} \subseteq [0, \infty) \to [0, \infty)$. If inequality*

$$\psi(\xi\tau + m\varsigma(1-\tau)) \leq \psi(\xi)h^s(\tau) + m\psi(\varsigma)(1 - h(\tau))^s \tag{2}$$

is fulfilled $\forall \varsigma, \xi \in \mathcal{I}$ and $\tau \in [0, 1]$, where $s \in [-1, 1]$ and $m \in [0, 1]$.The function ψ is then referred to as a modified (h, m)–convex function of the second type on $\mathcal{I}$.

Remark 1. *Those interested can check that Definitions 3 and 4 encompass many of the known notions of convexity: classic convex, s-convex, $s - (\alpha, m)$-convex, (s, m)-convex, (α, s)-convex, (α, m)-convex and others. So we have*

Example 1: Power Function Let $f(x) = x^p$, with $p \geq 1$. This function is convex in $[0, +\infty)$ and, under certain conditions on h, m, and s, it can satisfy the above inequalities.

Example 2: Exponential Function Let $f(x) = e^{kx}$ with $k > 0$. This function is convex and, as in the previous case, can satisfy the conditions of (h, m)-modified convexity for appropriate values of the parameters.

Example 3: Logarithmic Function Let $f(x) = log(1 + x)$. This function is concave in $[0, +\infty)$, but considering the definitions of (h, m)-modified concavity, it can be a valid example.

It can also be extended to the generalized logarithmic family, for example:

$f(x) = log^q(1 + x)$, with $q \in (0, 1]$ Or even to functions like: $f(x) = arctan(x)$, $f(x) = \sqrt{x+1} - 1$ as long as m are chosen small enough and $s \in (0, 1)$ are chosen to smooth the inequality sufficiently.

In the following we present some basic concepts of the Fourier transform of a function (see page 75 of [17] and page 580 of [10]).

Definition 5. *If a function $g : \mathbb{R} \to F$ is piecewise continuous in each finite interval and is absolutely integrable in $\mathbb{R}$, then the Fourier transform of $g \in L(\mathbb{R})$ denoted by $\mathcal{F}g$ is given by the integral*

$$\mathcal{F}g(x) = \int_{-\infty}^{+\infty} g(z)e^{ixz}\,dz \tag{3}$$

and

$$\mathcal{F}g(x) = \int_{-\infty}^{+\infty} g(z)e^{-ixz}\,dz. \tag{4}$$

The inverse Fourier transform is given by

$$g(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}g(x)e^{-ixz}\,dx \tag{5}$$

and

$$g(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}g(x)e^{ixz}\,dx. \tag{6}$$

The properties of the Fourier Transform and its inverse, defined above, can be consulted in the texts cited above.

In this paper, we present various new forms of the Hermite–Hadamard inequality, within the context of (h,m)-convex modified functions of second type, using Fourier Transform.

Main outcomes

The following result is a version of the classic Hermite-Hadamard inequality.

Theorem 1. *Let $g : I \to \mathbb{R}$ be a convex function, I a real interval, with $a < b$ and $a, b \in I$. Then, the following inequalities:*

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &\leq \frac{i\sigma}{1-e^{-i\sigma(b-a)}}\left[\left(h^s\left(\frac{1}{2}\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s\right)(\mathcal{F}g(\sigma-b)+\mathcal{F}g(\sigma+a))\right] \\ \leq & \frac{i\sigma(b-a)}{1-e^{-i\sigma(b-a)}}\left[h^s\left(\frac{1}{2}\right)\left(g(a)J_1+mg\left(\frac{b}{m}\right)J_2\right)\right. \\ + & \left.\left(1-h\left(\frac{1}{2}\right)\right)^s\left(g(b)J_1+mg\left(\frac{a}{m}\right)J_2\right)\right] \end{aligned} \tag{7}$$

for Fourier integral transform are fulfilled, where $\mathcal{F}g$ is the Fourier Transform of function $g(u)$, with $u \in [a,b]$.

Proof. From (h,m)-convexity of g we have (with $m = 1$ and $t = \frac{1}{2}$ we have

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &= g\left(\frac{(ta+(1-t)b)+(tb+(1-t)a)}{2}\right) \\ \leq\ & h^s\left(\frac{1}{2}\right)g(ta+(1-t)b)+\left(1-h\left(\frac{1}{2}\right)\right)^s g(tb+(1-t)a). \end{aligned}$$

Let $r = \sigma(b-a)$, let's multiply the previous inequality by e^{-irt} and integrate the result between 0 and 1 with respect to t, so we have

$$\begin{aligned} & g\left(\frac{a+b}{2}\right)\int_0^1 e^{-irt}dt \leq h^s\left(\frac{1}{2}\right)\int_0^1 e^{-irt}g(ta+(1-t)b)dtg(ta+(1-t)b) \\ + & \left(1-h\left(\frac{1}{2}\right)\right)^s\int_0^1 e^{-irt}g(tb+(1-t)a)dt \\ & g\left(\frac{a+b}{2}\right)\frac{1-e^{-i\sigma(b-a)}}{i\sigma(b-a)} \leq h^s\left(\frac{1}{2}\right)\frac{1}{b-a}\int_a^b e^{-\sigma(b-u)}f(u)du \\ + & \left(1-h\left(\frac{1}{2}\right)\right)^s\frac{1}{b-a}\int_a^b e^{-\sigma(u-a)}f(u)du \\ & g\left(\frac{a+b}{2}\right)\frac{1-e^{-i\sigma(b-a)}}{i\sigma(b-a)} \leq \frac{1}{b-a}\left[\left(h^s\left(\frac{1}{2}\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s\right)(\mathcal{F}g(\sigma-b)+\mathcal{F}g(\sigma+a))\right] \\ & g\left(\frac{a+b}{2}\right) \leq \frac{i\sigma}{1-e^{-i\sigma(b-a)}}\left[\left(h^s\left(\frac{1}{2}\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s\right)(\mathcal{F}g(\sigma-b)+\mathcal{F}g(\sigma+a))\right]. \end{aligned} \tag{8}$$

Thus, we have the first part of the inequality (7).

For the second part, let's use the (h,m)-convexity of g, so we have

$$\begin{aligned} g(ta+(1-t)b) &\leq g(a)h^s(t)+mg\left(\frac{b}{m}\right)(1-h(t))^s \\ g(tb+(1-t)a) &\leq g(b)h^s(t)+mg\left(\frac{a}{m}\right)(1-h(t))^s. \end{aligned}$$

Let's multiply both inequalities by e^{-irt} and integrating witgh respect to t over $[0,1]$ leads us to

$$\int_0^1 g(ta+(1-t)b)e^{-irt}dt \le g(a)\int_0^1 h^s(t)e^{-irt}dt + mg\left(\frac{b}{m}\right)\int_0^1 (1-h(t))^s e^{-irt}dt$$
$$\int_0^1 g(tb+(1-t)a)e^{-irt}dt \le g(b)\int_0^1 h^s(t)e^{-irt}dt + mg\left(\frac{a}{m}\right)\int_0^1 (1-h(t))^s e^{-irt}dt.$$

From here, using the definitions of J_1 and J_2 we have

$$\frac{1}{b-a}\int_a^b e^{-\sigma(b-u)}f(u)du \le g(a)J_1 + mg\left(\frac{b}{m}\right)J_2$$
$$\frac{1}{b-a}\int_a^b e^{-\sigma(u-a)}f(u)du \le g(b)J_1 + mg\left(\frac{a}{m}\right)J_2.$$

After changing variables in the integrals of both left members, multiplying the first inequality by $h^s\left(\frac{1}{2}\right)$, the second by $\left(1-h\left(\frac{1}{2}\right)\right)^s$ and adding the results obtained, we get

$$\frac{1}{b-a}\left[\left(h^s\left(\frac{1}{2}\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s\right)(\mathcal{F}g(\sigma-b)+\mathcal{F}g(\sigma+a))\right]$$
$$\le \quad h^s\left(\frac{1}{2}\right)\left(g(a)J_1(t)+mg\left(\frac{b}{m}\right)J_2\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s\left(g(b)J_1+mg\left(\frac{a}{m}\right)J_2\right).$$

After multiplying both sides by $\frac{i\sigma(b-a)}{1-e^{-i\sigma(b-a)}}$, we obtain the second part of the desired inequality. This completes the proof. □

Remark 2. *For convex functions, this result becomes Theorem 3.1 of [14].*

To prove our main Theorems, we need the following equality:

Lemma 1. *Let g be a real-valued function defined on a closed real interval $[a,b]$, differentiable on (a,b), and let $w' \in L_1[a,b]$. If $g' \in L_1[a,b]$, then the following equality holds:*

$$-\frac{(g(a)+g(b))(1-e^{-i\sigma(b-a)})}{i\sigma(b-a)^2}+\frac{1}{(b-a)^2}[\mathcal{F}g(\sigma-b)+\mathcal{F}g(\sigma+a)]$$
$$= \int_0^1\left[\int_0^t e^{-irz}dz\right](g'(ta+(1-t)b)-g'(tb+(1-t)a))dt. \quad (9)$$

Proof. Let $r=\sigma(b-a)$ as before and let us denote for simplicity

$$I_1 = \int_0^1\left[\int_0^t e^{-irz}dz\right]g'(ta+(1-t)b)dt,$$
$$I_2 = \int_0^1\left[\int_0^t e^{-irz}dz\right]g'(tb+(1-t)a)dt.$$

So

$$I = \int_0^1\left[\int_0^t e^{-irz}dz\right](g'(ta+(1-t)b)+g'(tb+(1-t)a))dt$$
$$= \int_0^1\left[\int_0^t e^{-irz}dz\right]g'(ta+(1-t)b)dt+\int_0^1\left[\int_0^t e^{-irz}dz\right]g'(tb+(1-t)a)dt = I_1 - I_2.$$

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Integrating by parts in I_1, we get

$$I_1 = -\frac{g(a)}{b-a}\int_0^1 e^{-irz}dz + \frac{1}{b-a}\int_0^1 e^{-irt}g(ta+(1-t)b)dt$$
$$= -\frac{g(a)}{b-a}\frac{(1-e^{-i\sigma(b-a)})}{i\sigma(b-a)} + \frac{1}{(b-a)^2}\mathcal{F}g(\sigma-b)$$
$$= -\frac{g(a)(1-e^{-i\sigma(b-a)})}{i\sigma(b-a)^2} + \frac{1}{(b-a)^2}\mathcal{F}g(\sigma-b), \tag{10}$$

result obtained after making the change of variable $z = ta+(1-t)b$ in the integral and using the definition of Fourier Transform and the shift property.

Analogously, making $z = tb+(1-t)a$

$$I_2 = \frac{g(b)(1-e^{-i\sigma(b-a)})}{i\sigma(b-a)^2} - \frac{1}{(b-a)^2}\mathcal{F}g(\sigma+a), \tag{11}$$

Subtracting (11) from (10) gives the desired result. □

Hereafter, we use the following notation:

$$\mathcal{L} = -\frac{(g(a)+g(b))(1-e^{-i\sigma(b-a)})}{i\sigma(b-a)^2} + \frac{1}{(b-a)^2}[\mathcal{F}g(\sigma-b)+\mathcal{F}g(\sigma+a)].$$

Theorem 2. *Let $g:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) such that $g'\in L_1[a,b]$. If $|g'|$ is a (h,m)–convex modified function of the second type on $[a,b]$ for some fixed $m\in[0,1]$ and $\frac{a}{m},\frac{b}{m}\in Dom(|g'|)$, then the following inequality holds:*

$$|\mathcal{L}| \le \frac{1}{i\sigma(b-a)}\left[(|g'(a)|+|g'(b)|)\mathcal{H}_1 + m\left(\left|g'\left(\frac{b}{m}\right)\right| + \left|g'\left(\frac{a}{m}\right)\right|\right)\mathcal{H}_2\right], \tag{12}$$

where

$$\mathcal{H}_1 = \int_0^1\left(1-e^{-i\sigma(b-a)t}\right)h^s(t)dt,\ \mathcal{H}_2 = \int_0^1\left(1-e^{-i\sigma(b-a)t}\right)(1-h(t))^s dt.$$

Proof. Applying modulus in Lemma 1, we have

$$|\mathcal{L}| \le |I_1| + |I_2|. \tag{13}$$

From the definition of I_1 and I_2 we easily have

$$|I_1| \le \int_0^1\left[\int_0^t e^{-irz}dz\right]|g'(ta+(1-t)b)|dt$$

and

$$|I_2| \le \int_0^1\left[\int_0^t e^{-irz}dz\right]|g'(tb+(1-t)a)|dt.$$

Using the (h,m) convexity of $|g'|$ we derive

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$$\begin{aligned}
&\int_0^1 \left[\int_0^t e^{-irz}dz\right] |g'(ta+(1-t)b)|dt \\
\leq\ & \int_0^1 \left[\int_0^t e^{-irz}dz\right] \left(|g'(a)|h^s(t) + m\left|g'\left(\frac{b}{m}\right)\right|(1-h(t))^s\right)dt \\
=\ & \frac{1}{i\sigma(b-a)}\left(|g'(a)|\int_0^1\left(1-e^{-i\sigma(b-a)t}\right)h^s(t)dt + m\left|g'\left(\frac{b}{m}\right)\right|\int_0^1\left(1-e^{-i\sigma(b-a)t}\right)(1-h(t))^s dt\right),
\end{aligned}$$

and

$$|I_2| \leq \frac{1}{i\sigma(b-a)}\left(|g'(b)|\int_0^1\left(1-e^{-i\sigma(b-a)t}\right)h^s(t)dt + m\left|g'\left(\frac{a}{m}\right)\right|\int_0^1\left(1-e^{-i\sigma(b-a)t}\right)(1-h(t))^s dt\right).$$

Using these last two inequalities we obtain the desired result. □

Corollary 1. *Under the assumptions of Theorem 2,*

1. *If we choose $m=1$, then we derive the following inequality*

$$|\mathcal{L}| \leq \frac{1}{i\sigma(b-a)}(|g'(a)| + |g'(b)|)(\mathcal{H}_1 + \mathcal{H}_2).$$

$\mathcal{H}_1$ and $\mathcal{H}_2$ are as before.

2. *If $s=m=1$, then*

$$|\mathcal{L}| \leq \frac{1}{i\sigma(b-a)}(|g'(a)| + |g'(b)|)(\mathcal{H}_3(t) + \mathcal{H}_4(t)).$$

where

$$\mathcal{H}_3 = \int_0^1\left(1-e^{-i\sigma(b-a)t}\right)h(t)dt,\ \mathcal{H}_4 = \int_0^1\left(1-e^{-i\sigma(b-a)t}\right)(1-h(t))dt.$$

Next we refine equation 12 by imposing additional conditions on $|g'|$.

Theorem 3. *Let $g:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) such that $g'\in L_1[a,b]$. If $|g'|^q$, with $\frac{1}{p}+\frac{1}{q}=1$, is a (h,m)–convex modified function of the second type on $[a,b]$ for some fixed $m\in[0,1]$ and $\frac{a}{m}, \frac{b}{m}\in Dom(|g'|)$, then the following inequality holds:*

$$|\mathcal{L}| \leq \frac{1}{i\sigma(b-a)}\mathcal{H}_5\left\{\left(|g'(a)|^q\mathcal{H}_6 + m\left|g'\left(\frac{b}{m}\right)\right|^q\mathcal{H}_7\right)^{\frac{1}{q}} + \left(|g'(b)|^q\mathcal{H}_6 + m\left|g'\left(\frac{a}{m}\right)\right|^q\mathcal{H}_7\right)^{\frac{1}{q}}\right\} \tag{14}$$

where $\mathcal{H}_5 = \left(\int_0^1\left(1-e^{-i\sigma(b-a)t}\right)^p dt\right)^{\frac{1}{p}}$, $\mathcal{H}_6 = \int_0^1 h^s(t)dt$ and $\mathcal{H}_7 = \int_0^1(1-h(t))^s dt$.

Proof. By using Hölder inequality in view of the fact that $|g'|^q$ is a (h,m)–convex modified of the second type, for $|I_1|$ and $|I_2|$, we get

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$$
\begin{aligned}
|I_1| &\leq \int_0^1 \left[\int_0^t e^{-irz}dz\right] |g'(ta+(1-t)b)| dt \\
&\leq \left(\int_0^1 \left[\int_0^t e^{-irz}dz\right]^p dt\right)^{\frac{1}{p}} \left(\int_0^1 |g'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 \left[\frac{\left(1-e^{-i\sigma(b-a)t}\right)}{i\sigma(b-a)}\right]^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left(|g'(a)|^q h^s(t) + m\left|g'\left(\frac{b}{m}\right)\right|^q (1-h(t))^s\right) dt\right)^{\frac{1}{q}} \\
&= \frac{1}{i\sigma(b-a)} \left(\int_0^1 \left(1-e^{-i\sigma(b-a)t}\right)^p dt\right)^{\frac{1}{p}} \left(|g'(a)|^q \int_0^1 h^s(t)dt + m\left|g'\left(\frac{b}{m}\right)\right|^q \int_0^1 (1-h(t))^s dt\right)^{\frac{1}{q}},
\end{aligned}
$$

and

$$
|I_2| \leq \frac{1}{i\sigma(b-a)} \left(\int_0^1 \left(1-e^{-i\sigma(b-a)t}\right)^p dt\right)^{\frac{1}{p}} \left(|g'(b)|^q \int_0^1 h^s(t)dt + m\left|g'\left(\frac{a}{m}\right)\right|^q \int_0^1 (1-h(t))^s dt\right)^{\frac{1}{q}}
$$

Thus, from $|I_1| + |I_2|$ we have (14). The proof is complete. □

Corollary 2. *Under the assumptions of Theorem 3,*

1. *Choosing $m = 1$, then we obtain the following inequality*

$$
|\mathcal{L}| \leq \frac{1}{i\sigma(b-a)} \mathcal{H}_5 \left\{ \left(|g'(a)|^q \mathcal{H}_6 + |g'(b)|^q \mathcal{H}_7\right)^{\frac{1}{q}} + \left(|g'(b)|^q \mathcal{H}_6 + |g'(a)|^q \mathcal{H}_7\right)^{\frac{1}{q}} \right\}.
$$

2. *If $s = m = 1$ then*

$$
\begin{aligned}
|\mathcal{L}| \leq \frac{1}{i\sigma(b-a)} \mathcal{H}_5 \Bigg\{ &\left(|g'(a)|^q \int_0^1 h(t)dt + m|g'(b)|^q \int_0^1 (1-h(t))dt\right)^{\frac{1}{q}} \\
&+ \left(|g'(b)|^q \int_0^1 h(t)dt + m|g'(a)|^q \int_0^1 (1-h(t))dt\right)^{\frac{1}{q}} \Bigg\}.
\end{aligned}
$$

Theorem 4. *Let $g : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) such that $g' \in L_1[a,b]$. If $|g'|^q$, with $q > 1$, is a (h,m)–convex modified function of the second type on $[a,b]$ for some fixed $m \in [0,1]$ and $\frac{a}{m}, \frac{b}{m} \in Dom(|g'|)$, then the following inequality holds:*

$$
\begin{aligned}
|\mathcal{L}| \leq\ & 2\frac{\left(e^{-i\sigma(b-a)} + i\sigma(b-a) - 1\right)^{1-\frac{1}{q}}}{(i\sigma(b-a))^{2(1-\frac{1}{q})}} \\
\times\ & \left\{ \left(|g'(a)|^q \mathcal{H}_1 + m\left|g'\left(\frac{b}{m}\right)\right|^q \mathcal{H}_2\right)^{\frac{1}{q}} + \left(|g'(b)|^q \mathcal{H}_1 + m\left|g'\left(\frac{a}{m}\right)\right|^q \mathcal{H}_2\right)^{\frac{1}{q}} \right\},
\end{aligned} \tag{15}
$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ as in the Theorem 2.

Proof. By using power–mean inequality in view of the fact that $|g'|^q$ is a (h,m)–convex modified of the second type, for $|I_1|$ and $|I_2|$, we get

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$$
\begin{aligned}
|I_1| &\leq \int_0^1 \left[\int_0^t e^{-irz}dz\right] |g'(ta+(1-t)b)|dt \\
&\leq \left(\int_0^1 \left[\int_0^t e^{-irz}dz\right]dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left[\int_0^t e^{-irz}dz\right] |g'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 \left[\frac{\left(1-e^{-i\sigma(b-a)t}\right)}{i\sigma(b-a)}\right]dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left[\int_0^t e^{-irz}dz\right]\left(|g'(a)|^q h^s(t) + m\left|g'\left(\frac{b}{m}\right)\right|^q (1-h(t))^s\right)dt\right)^{\frac{1}{q}} \\
&= \frac{\left(e^{-i\sigma(b-a)} + i\sigma(b-a) - 1\right)^{1-\frac{1}{q}}}{(i\sigma(b-a))^{2(1-\frac{1}{q})}} \left(|g'(a)|^q \int_0^1 \left[\int_0^t e^{-irz}dz\right] h^s(t)dt + m\left|g'\left(\frac{b}{m}\right)\right|^q \int_0^1 \left[\int_0^t e^{-irz}dz\right](1-h(t))^s dt\right)^{\frac{1}{q}},
\end{aligned}
$$

Or, taking into account the accepted notations, we can write:

$$
|I_1| \leq \frac{\left(e^{-i\sigma(b-a)} + i\sigma(b-a) - 1\right)^{1-\frac{1}{q}}}{(i\sigma(b-a))^{2(1-\frac{1}{q})}} \left(|g'(a)|^q \mathcal{H}_1 + m\left|g'\left(\frac{b}{m}\right)\right|^q \mathcal{H}_2\right)^{\frac{1}{q}},
$$

Similarly, for I_2 we get

$$
|I_2| \leq \frac{\left(e^{-i\sigma(b-a)} + i\sigma(b-a) - 1\right)^{1-\frac{1}{q}}}{(i\sigma(b-a))^{2(1-\frac{1}{q})}} \left(|g'(b)|^q \mathcal{H}_1 + m\left|g'\left(\frac{a}{m}\right)\right|^q \mathcal{H}_2\right)^{\frac{1}{q}},
$$

Thus, for $|I_1| + |I_2|$ we have

$$
\begin{aligned}
&|I_1| + |I_2| \\
\leq\ & 2\frac{\left(e^{-i\sigma(b-a)} + i\sigma(b-a) - 1\right)^{1-\frac{1}{q}}}{(i\sigma(b-a))^{2(1-\frac{1}{q})}} \\
\times\ & \left\{\left(|g'(a)|^q \mathcal{H}_1 + m\left|g'\left(\frac{b}{m}\right)\right|^q \mathcal{H}_2\right)^{\frac{1}{q}} + \left(|g'(b)|^q \mathcal{H}_1 + m\left|g'\left(\frac{a}{m}\right)\right|^q \mathcal{H}_2\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

This inequality completes the proof. □

Theorem 5. *Let $g : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) such that $g' \in L_1[a,b]$. If $|g'|^q$, with $\frac{1}{p} + \frac{1}{q} = 1$, is a (h,m)–convex modified function of the second type on $[a,b]$ for some fixed $m \in [0,1]$ and $\frac{a}{m}, \frac{b}{m} \in Dom(|g'|)$, then the following inequality holds:*

$$
|\mathcal{L}| \leq \frac{2\mathcal{H}_8}{p(i\sigma(b-a))^p} + \frac{1}{q}\left\{\mathcal{H}_6\left(|g'(a)|^q + |g'(b)|^q\right) + m\mathcal{H}_7\left(\left|g'(\frac{a}{m})\right|^q + \left|g'(\frac{b}{m})\right|^q\right)\right\},
$$

where $\mathcal{H}_8 = (\mathcal{H}_5)^p$ and $\mathcal{H}_5$, $\mathcal{H}_6$ and $\mathcal{H}_7$ as in the Theorem 3.

Proof. By using Young inequality view and of the fact that $|g'|^q$ is a (h,m)–convex modified of the second type, for $|I_1|$ and $|I_2|$ as before, we get

$$
\begin{aligned}
&|I_1| \leq \int_0^1 \left[\int_0^t e^{-irz}dz\right] |g'(ta+(1-t)b)|dt \\
\leq\ & \frac{1}{p}\int_0^1 \left[\int_0^t e^{-irz}dz\right]^p dt + \frac{1}{q}\int_0^1 |g'(ta+(1-t)b)|^q dt \\
\leq\ & \frac{1}{p}\int_0^1 \left[\frac{\left(1-e^{-i\sigma(b-a)t}\right)}{i\sigma(b-a)}\right]^p dt + \frac{1}{q}\int_0^1 \left(|g'(a)|^q h^s(t) + m\left|g'\left(\frac{b}{m}\right)\right|^q (1-h(t))^s\right)dt \\
=\ & \frac{1}{p(i\sigma(b-a))^p}\int_0^1 \left(1-e^{-i\sigma(b-a)t}\right)^p dt + \frac{1}{q}\left(|g'(a)|^q \int_0^1 h^s(t)dt + m\left|g'\left(\frac{b}{m}\right)\right|^q \int_0^1 (1-h(t))^s dt\right),
\end{aligned}
$$

In the same way

$$
\begin{aligned}
&|I_2| \\
\leq\ & \frac{1}{p(i\sigma(b-a))^p}\int_0^1 \left(1-e^{-i\sigma(b-a)t}\right)^p dt + \frac{1}{q}\left(|g'(b)|^q \int_0^1 h^s(t)dt + m\left|g'\left(\frac{a}{m}\right)\right|^q \int_0^1 (1-h(t))^s dt\right).
\end{aligned}
$$

Thus, Adding these last two inequalities together, we obtain the desired result. This completes the proof of the Theorem. □

Remark 3. *The reader will have no difficulty in stating the Corollaries corresponding to these last results, for (h,m)-convex and h-convex functions.*

2. Other Results

Remark 4. *Obviously for different cases of h we can derive new inequalities of the Hermite-Hadamard type, in particular if $h(t) = t$, that is, if we consider convex functions, new results will be obtained that complete [14].*

So we have, for example,

Theorem 6. *Let $g : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) such that $g' \in L_1[a,b]$. If $|g'|$ is a convex function on $[a,b]$, then the following inequality holds:*

$$
|\mathcal{L}| \leq \frac{1}{i\sigma(b-a)}\left[(|g'(a)| + |g'(b)|)\mathcal{H}_9 + m\left(\left|g'\left(\frac{b}{m}\right)\right| + \left|g'\left(\frac{a}{m}\right)\right|\right)\mathcal{H}_{10}\right], \tag{16}
$$

where

$$
\mathcal{H}_9 = \frac{(\sigma(b-a))^2 + 2 - 2(1+i\sigma(b-a))e^{-i\sigma(b-a)}}{(\sigma(b-a))^2},\ \mathcal{H}_{10} = \frac{2\left(1-i\sigma(b-a)-e^{-i\sigma(b-a)}\right)-(\sigma(b-a))^2}{(\sigma(b-a))^2}.
$$

3. Conclusions

In this work, we establish new versions of the Hermite-Hadamard inequalities for (h,m)-convex functions of the second kind using an innovative approach based on the Fourier Transform.

The main results include new Hermite-Hadamard inequalities for (h,m)-convex functions of the second kind, as well as the generalization of these inequalities to broader contexts through the use of new integral operators. Furthermore, conditions under which these inequalities hold are established, thus broadening their applicability. Particular cases are also presented as examples to illustrate the effectiveness of the results obtained.

The approach presented here, the use of the Fourier Transform, opens up new possibilities for working with other inequalities, which predicts a new development of the Theory of Inequalities.

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