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A Simpson-Type Decomposition of the Euler-Mascheroni Constant and the Irrationality of δ

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Abstract

An elementary and self-contained proof of the existence of the Euler-Mascheroni constant γ is presented, based solely on the Simpson quadrature formula and the convexity of the function $x \mapsto 1/x$. The local logarithmic increments are approximated as follows:

$$\int_{2n-1}^{2n+1} \frac{dx}{x}$$

Using Simpson's rule, a discrete approximation expressed as a finite linear combination of reciprocal integers is constructed. Exploiting the monotonic and convex nature of the function $1/x$, sharp two-sided inequalities relating the numerical approximation to exact logarithmic increments are established. These inequalities imply that the accumulated quadrature errors form a convergent series. Consequently, the following classical limits

$$\gamma = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k} - \log[N]\right)$$

are proven to exist. This approach provides a conceptually simple alternative to traditional proofs based on the Euler-Maclaurin formula, highlighting the direct connection between numerical integration, convexity, and the analytical nature of γ. I further show that γ can be expressed as $(\log[2]+1)/3+\delta$, where both $(\log[2]+1)/3$ and δ are irrational, and where δ arises as the limit of a rational sequence derived from as Simpson-type approximation.

Keywords: The Euler-Mascheroni constant; Harmonic sum; Simson quadrature formula

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1. Introduction

The Euler-Mascheroni constant

$$\gamma = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k} - \log[N]\right)$$

is a classical object of analysis and number theory [1]. Figure 1(a) illustrates the difference of the integration of $1/x$ and the sum of rectangles. Its existence is usually proven via asymptotic expansions derived from the Euler-Maclaurin summation formula or related analytic techniques [2].

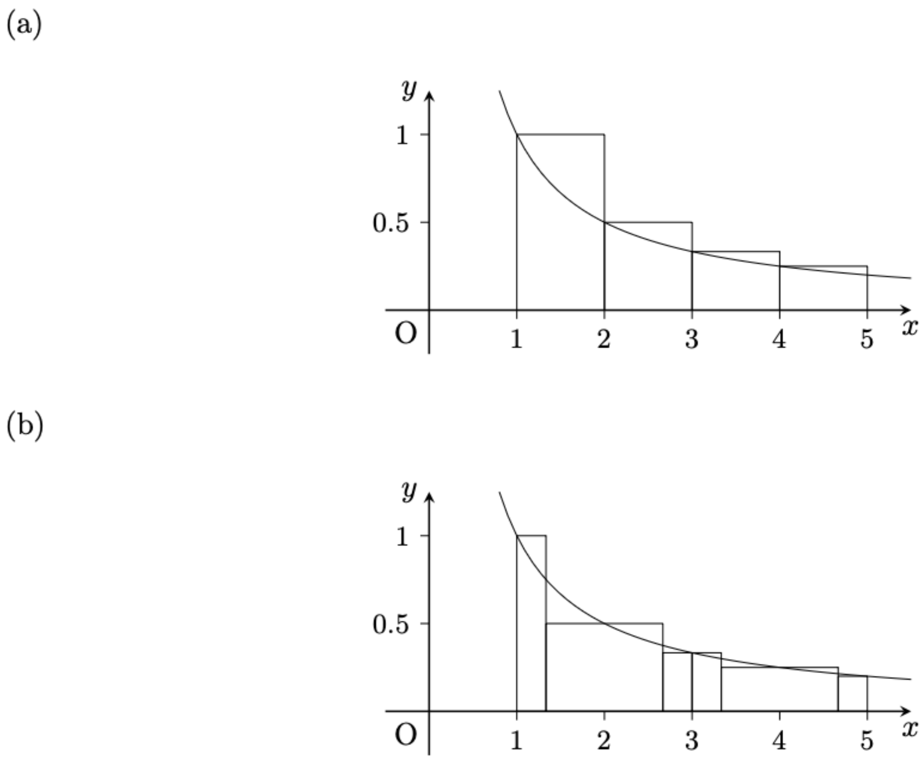

Figure 1. (a) Numerical integration of $1/x$ by the rectangle (left–endpoint) method, illustrating the classical definition of the Euler–Mascheroni constant γ as the limiting difference between the harmonic series and the logarithm. (b) Numerical integration of $1/x$ using Simpson's rule, which produces the Simpson–type weighted sums employed in the present paper.

According to Havil [3], no prove of the irrationality of γ is known, and the problem remains open. In this paper, I show that γ can be represented as the sum of two irrational constants, namely $(\log[2]+1)/3$ and δ, where δ arises as the limit of rational sequence defined via a Simpson-type approximation.

2. Simpson's Approximation of Logarithmic Increments

For integers $n \geq 1$, the following function is defined:

$$f(n) = \int_{2n-1}^{2n+1} \frac{dx}{x} = \log[2n+1] - \log\,[2n-1].$$

This integral is approximated using Simpson's rule with a step size of 1 as illustrated in Figure 1(b).

An additional function is defined:

$$g(n) = \frac{1}{3}\left(\frac{1}{2n-1} + \frac{2}{n} + \frac{1}{2n+1}\right).$$

Notably, $g(n)$ is a finite linear combination of reciprocal integers and depends solely on the values of $1/x$ at the integer points.

The local quadrature error is defined as follows:

$$d(n) = g(n) - f(n).$$

3. Convexity and Comparison Inequalities

The function $1/x$ is strictly decreasing and convex on $(0, \infty)$. Let $p_n(x)$ denote a quadratic polynomial interpolating $1/x$ at the three points.

$$x = 2n - 1, 2n, 2n + 1.$$

By construction,

$$g(n) = \int_{2n-1}^{2n+1} p_n(x) dx.$$

As $1/x$ is convex, the interpolation polynomial $p_n(x)$ lies above $1/x$ on $[2n - 1, 2n + 1]$. Thus, $p_n(x) - f(x) > 0$ except at the midpoint, and integration yields $d(n) > 0$. Consequently,

$$d(n) > 0 \ (n \geq 1).$$

Moreover, convexity implies a comparison across adjacent intervals, namely,

$$f(n) > g(n + 1) \ (n \geq 2).$$

From this the following is obtained:

$$0 < d(n) < g(n) - g(n + 1), (n \geq 2).$$

4. Telescoping Bounds and Error Convergence

Summing the aforementioned inequalities for $n = 2, \dots, N$ yields

$$0 < \sum_{n=2}^{N} d(n) < \sum_{n=2}^{N} \{g(n) - g(n + 1)\}.$$

Both bounds telescope:

$$0 < \sum_{n=2}^{N} d(n) < g(2) - g(N + 1).$$

As $g(n) \to 0$ as $n \to \infty$, the series

$$\sum_{n=2}^{\infty} d(n)$$

converges. The following function is defined:

$$\delta := \sum_{n=2}^{\infty} d(n) < g(2) = \frac{23}{15}$$

5. Recovery of the Euler-Mascheroni Constant

Definition 1 (harmonic sum)

Let

$$H(N) := \sum_{n=1}^{N} \frac{1}{n}$$

where N denotes the n-th harmonic.

Definition 2 (Simpson regularized harmonic sum)

Let n be an odd integer. The interval $[1, n]$ is partitioned into sub-intervals of length two.

$$[1,3], [3,5], \dots, [n - 2, n],$$

As illustrated in Figure 1(b), Simpson's rule is applied to each subinterval to approximate the following:

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$$\int_{x=1}^{\infty} \frac{dx}{x}.$$

The resulting approximation is denoted as $h(N)$, namely,

$$h(N) := \sum_{n=1}^{\frac{N-1}{2}} \frac{1}{3}\left(\frac{1}{2n-1} + \frac{2}{n} + \frac{1}{2n+1}\right).$$

Lemma 1 (representation by integer reciprocal).

The quantity $f(n)$ can be written as a finite linear combination of the reciprocals of the integers, as follows:

$$H(N) = \frac{1}{3}\cdot\frac{1}{1} + \frac{1}{3}\cdot\sum_{n=2}^{N-1}\frac{c_n}{n} + \frac{1}{3}\cdot\frac{1}{n},$$

where

$$c_n = \begin{cases} 4, \text{where } n \text{ is even,} \\ 2, \text{where } n \text{ is odd.} \end{cases}$$

Proof. In Simpson's rule, each subinterval contributes weights $1,4,1$, multiplied by the global factor $1/3$. With a step size of 2,

- each interior odd integer appears twice as an endpoint (weight $2/3$).
- each even integer appears once at the midpoint (weight $4/3$).
- the boundary points 1 and n appear once (weight $1/3$).

Theorem:

$$\lim_{n\to\infty}\{H(n) - h(n)\} = \frac{\log[2]}{3} + \frac{1}{3} = 0.56438239 \ldots$$

Proof:

By comparing the coefficients term-wise, the following were obtained:

for even n,

$$\frac{1}{n} - \frac{4}{3n} = -\frac{1}{3n},$$

for interior odd $n > 3$,

$$\frac{1}{n} - \frac{2}{3n} = \frac{1}{3n}.$$

Therefore,

$$H(n) - f(n) = \frac{1}{3}\left(\sum_{3\le k\le n-2,k \text{ odd}} \frac{1}{k} - \sum_{2\le k\le n-1,k \text{ even}} \frac{1}{k}\right) + \frac{2}{3} + O\left(\frac{1}{n}\right).$$

Alternating harmonic series

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}$$

converges to $\log(2)$. As the boundary terms vanish in the limit, the following was concluded:

$$\lim_{n\to\infty}\{H(n) - f(n)\} = \frac{\log[2]}{3} + \frac{1}{3}.$$

Summing the exact logarithmic increments

$$\sum_{n=1}^{N} f(n) = \log(2N+1) - \log(1) = \log(2N+1).$$

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Combining these expressions and the definition of δ, the following was obtained:

$$\delta = \gamma - \frac{\log[2]}{3} - \frac{1}{3}. \tag{1}$$

Hence, the finiteness of δ is equivalent to

$$\lim_{N\to\infty}\{H(2N+1) - \log[2N+1]\},$$

and, therefore, to the existence of the Euler-Mascheroni constant γ.
The rational sequences are defined as follows:

$$a_n = h(n) - \log(\mathrm{n}).$$

Then,

$$\delta = \lim_{n\to\to\infty} a_n$$

and $a_n \in \mathbb{Q}$ for every n.
Proof.

$$h(n) - \log[\mathrm{n}] = \delta + O\left(\frac{1}{n}\right),$$

$$h(n^2) - \log[n^2] = \delta + O\left(\frac{1}{n^2}\right).$$

Using $\log(n^2) = 2\log n$, rearranging gives

$$\delta = 2h(n) - h(n^2) + O\left(\frac{1}{n}\right).$$

Taking $n \to \infty$ proves $\delta = \lim_{n\to\infty} a_n$. As each $h(n)$ is rational, so is a_n. The numerical values in Table 1 illustrate the rapid convergence of this rational sequence.

$$2h(n) - h(n^2)$$

toward the constant δ. This accelerated convergence is a consequence of cancellation of the dominant $O(1/n)$ error terms in the Simpson approximation.

Table 1. Convergence of $2H_2(n) - H_2(n^2)$.

n	$h(n) - h(n^2)$	$\delta - h(n) + h(n^2)$
3	0.012169312	6.64×10^{-4}
5	0.012735433	9.78×10^{5}
7	0.012806754	2.65×10^{5}
9	0.012823395	9.88×10^{6}
11	0.012828805	4.47×10^{6}
13	0.012830969	2.30×10^{6}

6. Proof That δ Is Irrational

In this section, I prove that δ cannot be a rational number. Note that δ can be obtained by

$$\delta = \lim_{n\to\infty}\{qh(n) - h(n^q)\},$$

For any fixed integer $q \geq 2$.
Let p be an odd prime. From the definition of $h(n)$, we have

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$$h(p) = \frac{1}{3p} + r_p,\ r_p \in \mathbb{Q},$$

containing no factor p in the denominator, because the term $1/p$ appears only in the endpoint weight $1/3$. Thus

$$qh(p) = \frac{q}{3p} + qr_p. \tag{2}$$

On the other hand, in $h(p^q)$, the denominator p never appears: p is neither an endpoint of the summation nor a term selected by the step-size 2.

Hence

$$h(p^q) = s_p,\ s_p \in \mathbb{Q}, \tag{3}$$

with denominator coprime to p. Subtracting (3) from (2) gives

$$qh(p) - h(p^q) = \frac{q}{3p} + t_p, t_p \in \mathbb{Q}, \tag{4}$$

with denominator coprime to p. Therefore, each rational number $qh(p) - h(p^q)$ has a denominator divisible by the prime p. Because p can be chosen to be any prime, the sequence in (1) contains rational numbers whose denominators include arbitrarily large primes.

A property of rational numbers and their approximations

Let $\delta = a/b$ be a rational number in lowest terms. Consider any rational approximation $x = p/q$.

$$\left|\frac{p}{q} - \frac{a}{b}\right| = \frac{|pb-aq|}{bq}, \tag{5}$$

where

$$pb - aq = \det\begin{pmatrix} p & a \\ q & b \end{pmatrix} \in \mathbb{Z}. \tag{5}$$

Thus, if $p/q \neq a/b$, then

$$|pb - aq| \geq 1 \Rightarrow \left|\frac{p}{q} - \frac{a}{b}\right| \geq \frac{1}{bq}. \tag{6}$$

Lemma: If $\left|\frac{p}{q} - \frac{a}{b}\right| < \frac{1}{2b^2}$, then $\frac{p}{q} = \frac{a}{b}$.

A standard consequence is :

Corollary.

Any rational number sufficiently close to a/b must have denominator divisible by b. In particular, the set of prime factors that can appear in the denominators of such approximants is finite.

Assume, for contradiction, that $\delta = a/b \in \mathbb{Q}$. From (1) the sequence

$$x_p = qh(p) - h(p^q)$$

converges to δ.

Thus, for any $\varepsilon < 1/(2b^2)$, all sufficiently large prime p satisfy

$$|x_p - \delta| < \varepsilon. \tag{7}$$

By the Corollary, every such rational x_p must have denominator divisible by b. Hence the set of primes dividing the denominators of the x_p is finite, namely a subset of the finite set of prime divisors of b. However, from (4) we know: each x_p contains the denominator p, and p can be any prime, contradicting the finite-prime requirement above. Since assuming $\delta \in \mathbb{Q}$ leads to a contraction, δ is irrational.

7. Conclusion

This study demonstrated that the convergence of the harmonic series minus the logarithm can be established using solely Simpson's rule and the convexity of the function $1/x$. This provides a simple, purely numerical, and analytic alternative to classical proofs based on the Euler–Maclaurin formula.

The method highlights an unexpected connection between elementary quadrature rules and one of the fundamental constants of analysis and suggests natural extensions to higher-order Newton-Cotes formulas.

In addition, we introduced the constant δ, defined as the limit of rational sequence arising from Simpson-type harmonic approximations, and proved that δ is irrational. Combined with the identity

$$\gamma = \frac{\log[2]}{3} + \frac{1}{3} + \delta,$$

This decomposition expresses γ as the sum of two irrational constants.

For reference, the constants are evaluated as follows:

$$\gamma \approx 0.5772156649015328,$$

$$\delta = \gamma + \frac{4}{3}\log 2 - \frac{3}{2} \approx 0.00141190564812655.$$

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Conflict of interest: The author declares no competing interests.

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