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Schrödinger-Dirac Formalism in Finite Ring Continuum

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Abstract

We extend the algebraic framework of the Finite Ring Continuum (FRC) to the domain of quantum and relativistic dynamics. Building on the previously established results on symmetry-complete finite fields $\mathbb{F}_p$ and their quadratic extensions $\mathbb{F}_{p^2}$, we show that both the Schrödinger and Dirac equations admit exact finite-field realizations. Within a single Euclidean shell $\mathbb{F}_p$, the discrete Schrödinger equation governs reversible, scale-periodic evolution of framed wavefunctions under finite-difference operators. Upon extension to the Lorentzian shell $\mathbb{F}_{p^2}$, the finite-field Clifford algebra $\mathrm{Cl}(1,3;\mathbb{F}_{p^2})$ and the corresponding orthogonal group $O(Q_\nu, \mathbb{F}_{p^2})$ of split type (Witt index 1) support a fully covariant finite-field Dirac equation, in which the Minkowski metric and causal structure emerge algebraically from the separation of square classes. The two regimes—Euclidean and Lorentzian—together form a consolidation-innovation cycle: powering operations within $\mathbb{F}_p$ compress information by merging residue classes, while the quadratic extension $\mathbb{F}_{p^2}$ restores completeness by generating new algebraic degrees of freedom. This alternation of compression and expansion provides an intrinsic algebraic origin for the coexistence of reversible quantum evolution and irreversible causal propagation. The resulting formalism reproduces the essential dynamical structure of relativistic quantum mechanics within a purely finite and relational arithmetic, eliminating any dependence on the continuum hypothesis.

Keywords: finite fields; quadratic forms; finite ring cosmology; algebraic causality; relativistic algebra; symmetry classes; discrete spacetime; relational physics

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1. Introduction

The Finite Ring Continuum (FRC) program [1,2] seeks to reconstruct the foundational structures of mathematics and physics within a finite algebraic framework. By replacing the assumption of an actual continuum with relational finitude, it establishes a complete hierarchy of framed number classes —integers, rationals, reals and complex numbers—constructed entirely over finite fields $\mathbb{F}_p$ of prime order $p = 4t + 1$. This framework eliminates ontological infinity while preserving all essential algebraic properties of conventional arithmetic. In this setting, the discrete geometry of each symmetry-complete shell $\mathbb{F}_p$ provides a natural analogue of Euclidean space, whereas its quadratic extension $\mathbb{F}_{p^2}$ introduces a Lorentzian split and thereby a notion of causality.

The purpose of the present note is to extend this algebraic foundation from static structure to *dynamics*. We show that both the Schrödinger and Dirac equations—the central equations of quantum and relativistic wave mechanics—arise naturally within the FRC framework. The Schrödinger equation governs reversible evolution within a single Euclidean shell $\mathbb{F}_p$, while the Dirac equation emerges in the quadratic extension $\mathbb{F}_{p^2}$, where the Lorentzian metric and causal structure are algebraically realized. Together they reveal the dual aspects of the FRC universe: the reversible, scale-periodic internal dynamics of a fixed shell, and the irreversible, information-expanding dynamics across shells.

1.1. Conceptual Background

The idea that physical law may have a finite arithmetic basis has been proposed in various forms since the early development of modern mathematical physics. The most direct algebraic precedent is the theory of finite fields and modular arithmetic, whose structure and applications have been comprehensively treated in standard works such as Lidl and Niederreiter [3] and Lam [4]. These foundations support the construction of quadratic forms, orthogonal groups, and Clifford algebras over finite fields—tools that are indispensable for defining Lorentz-like symmetry in a discrete setting.

The conceptual motivation for FRC resonates with a broad tradition of finitist and relational thinking. Early intuitions can be traced to Weyl's constructive critique of the continuum [5], to Brouwer's intuitionism [6], and to the later ultrafinitist program of Yessenin–Volpin [7], Parikh [8], and Sazonov [9], which emphasized feasibility and constructive bounds as foundational constraints. From a physical viewpoint, the FRC approach is closely aligned with the relational perspective of Smolin [10,11], who argues that time, space, and law are not absolute but emergent from relations among finite informational systems. Lev's comprehensive work on finitist foundations of quantum theory [12] and Zeilberger's advocacy for discrete analysis [13] further demonstrate that classical mathematical structures can be regarded as limiting or degenerate cases of finite ones.

Within mathematical physics, several discrete or arithmetic models have attempted to capture relativistic and quantum features without continuum assumptions. D'Ariano and collaborators [14] derived quantum theory from finite informational postulates; Benci and Di Nasso [15,16] developed finitist frameworks based on numerosities; and Lloyd [17] analyzed physical limits of computation as bounds on the information capacity of the universe. All these lines of research suggest that a coherent finite reconstruction of physics is both conceptually possible and computationally natural.

The FRC formalism builds upon these insights in a mathematically explicit manner. In the foundational paper [1], finite arithmetic operations are reinterpreted as symmetry transformations: addition as translation, multiplication as scaling, and powering as expansion. The subsequent Lorentzian note [2] demonstrated that the essential algebraic features of special relativity—the invariant interval and Lorentz symmetry—emerge within finite-field arithmetic only after a quadratic extension $\mathbb{F}_{\mathrm{p}^2}$ is introduced. The present work completes this triad by constructing dynamical equations consistent with these symmetries.

1.2. Objectives and Outline

The specific objectives of this study are threefold:

1. To formulate the discrete Schrödinger equation within a symmetry-complete Euclidean shell $\mathbb{F}_{\mathrm{p}}$ and demonstrate exact unitary evolution under finite-difference operators;
2. To derive the finite-field Dirac equation over the quadratic extension $\mathbb{F}_{\mathrm{p}^2}$ and establish its covariance under the finite orthogonal group $O(Q_\nu, \mathbb{F}_{\mathrm{p}^2})$ of split type (Witt index 1);
3. To interpret the transition $\mathbb{F}_{\mathrm{p}} \to \mathbb{F}_{\mathrm{p}^2}$ as an algebraic consolidation-innovation cycle, linking reversible quantum dynamics with the irreversible emergence of causal structure.

Section 2 reviews the construction of symmetry-complete shells and the Lorentzian extension. Section 3 defines discrete derivative operators and framed wavefunctions over finite fields. Section 4 formulates the finite Schrödinger equation and demonstrates its scale-periodicity, while Section 5 extends the framework to the Dirac equation over $\mathbb{F}_{\mathrm{p}^2}$. Section 6 interprets both equations within the consolidation-innovation cycle, and Section 7 discusses conceptual implications and future directions.

The overall aim is to demonstrate that finite arithmetic is sufficient to reproduce the full kinematic and dynamical structure of relativistic quantum mechanics, thereby strengthening the case for finite algebra as a universal language of physical law.

2. Background: FRC Shells and Lorentzian Extension

2.1. Symmetry-Complete Shells over $\mathbb{F}_\mathrm{p}$

Let $p = 4t + 1$ be a prime and let $\mathbb{F}_\mathrm{p}$ denote the finite field of cardinality p. Following [1], the multiplicative group $\mathbb{F}_\mathrm{p}^\times$ is cyclic of order $4t$ and hence contains an element $i \in \mathbb{F}_\mathrm{p}$ satisfying $i^2 = -1$. This ensures the existence of the structural class

$$\{1,\, i,\, -1,\, -i\} = \{i^k \mid k = 0, 1, 2, 3\},$$

which provides the minimal two-dimensional algebraic structure supporting rotational symmetry. A field $\mathbb{F}_\mathrm{p}$ with this property is called *symmetry-complete* of radius t.

The additive, multiplicative, and powering operations

$$T_a : \; x \mapsto x + a, \qquad S_m : \; x \mapsto mx, \qquad P_\varepsilon : \; x \mapsto x^\varepsilon, \tag{1}$$

generate the symmetry group $G = \langle T_a, S_m, P_\varepsilon \rangle$ acting on $\mathbb{F}_\mathrm{p}$. The additive and multiplicative actions define, respectively, the *meridians* and *latitudes* of a discrete spheroid $\mathcal{S}_\mathrm{p}$ embedded in a symbolic $(1,3)$-dimensional symmetry space

$$U = \bigcup_t U_t, \qquad U_t = \mathbb{F}_\mathrm{p}^4 / G.$$

Each point $(t; a, m, \varepsilon) \in U$ represents a local arithmetic frame characterized by the parameters of translation a, scaling m, and powering ε. The subspace U_t of fixed radius t is referred to as a *space-like shell*, or simply an *FRC shell*, as illustrated in Figure 1. Within a single shell all three operations in (1) are bijective, and the field $\mathbb{F}_\mathrm{p}$ exhibits an intrinsic Euclidean symmetry where no causal (time-like) distinction exists.

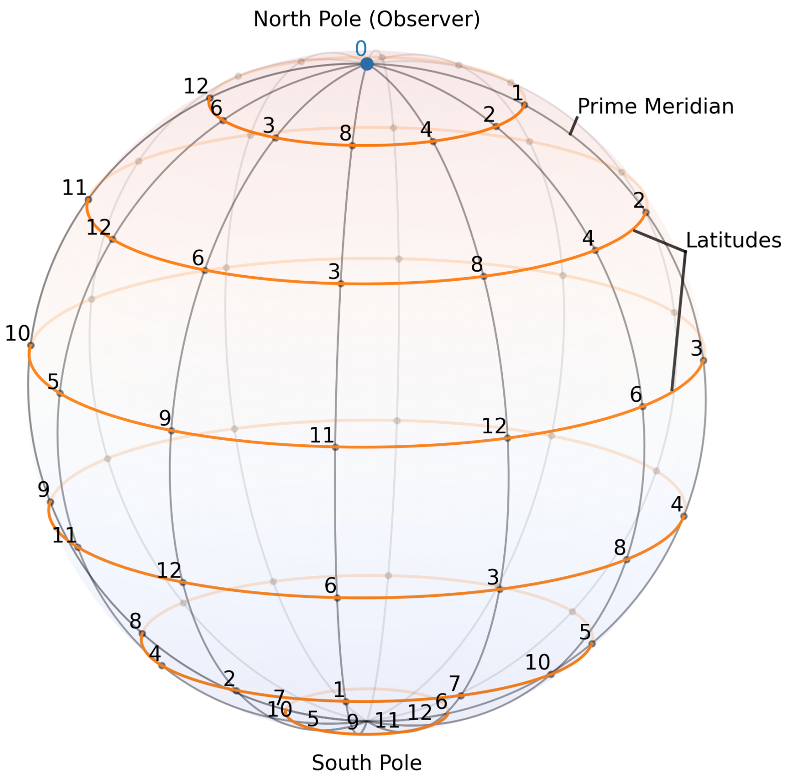

Figure 1. State diagram for framed finite field $\mathbb{F}_{13}(3; 0, 1, 2)$ as a 2D spheroid in $(1,3)$-D symmetry space $\mathcal{U}$ combining the additive symmetry along the meridians $M_n(a)$, as well as multiplicative symmetry along the latitudes $L_a(m)$ for multiplicative generator $g = 2$.

2.2. *Quadratic Extension and Emergence of Causality*

Within the physical interpretation of the FRC framework—which we refer to as Finite Ring Cosmology—the physical universe is modelled as an ensemble of finite arithmetic symmetry shells $\mathcal{U}_\mathsf{t}$ formed by a succession of finite algebraic rings $\mathbb{Z}_\mathsf{q}$ with $\mathsf{q} = 4\mathsf{t}+1$ and t being a time-like discrete radial chronon parameter, as illustrated in Figure 2. As has been demonstrated in [2], a genuine Lorentzian metric cannot be realized within a single space-like shell $\mathbb{F}_\mathsf{p}$. To separate time from space, one requires a coefficient c^2 belonging to the *opposite square class* of $\mathbb{F}_\mathsf{p}^\times$. If $\nu \in \mathbb{F}_\mathsf{p}^\times$ is a fixed nonsquare, then no element $c \in \mathbb{F}_\mathsf{p}$ satisfies $c^2 = \nu$. The minimal completion that restores algebraic closure is obtained by adjoining such a square root, forming the quadratic extension

$$\mathbb{F}_{\mathsf{p}^2} \;\cong\; \mathbb{F}_\mathsf{p}[X]/(X^2-\nu), \qquad c := X \bmod (X^2-\nu), \qquad c^2 = \nu. \tag{2}$$

The new field $\mathbb{F}_{\mathsf{p}^2}$ therefore acts as the *spacetime shell* of radius t, while $\mathbb{F}_\mathsf{p}$ remains its Euclidean subfield.

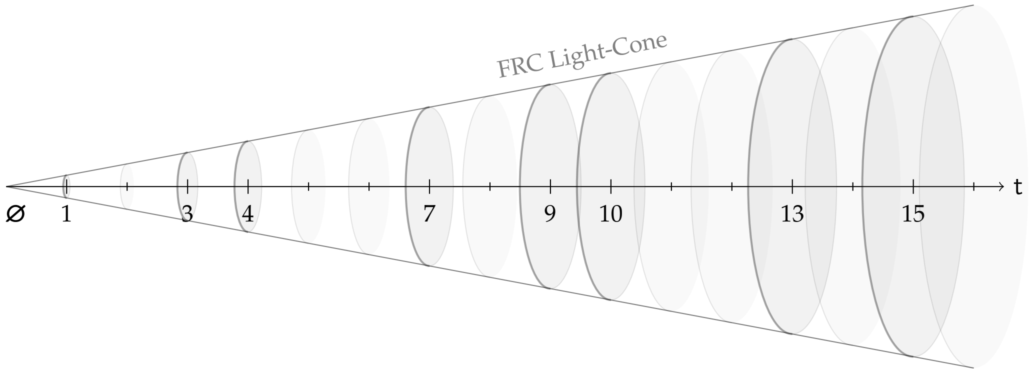

Figure 2. Schematic of the first 16 counts of the chronon parameter t and the corresponding arithmetic symmetry shells of the order $\mathsf{q} = 4\mathsf{t}+1$. The prime shells $\mathcal{S}_\mathsf{p}$ formed by the symmetry-complete fields $\mathbb{F}_\mathsf{p}$, where p is prime are emphasized.

Within $\mathbb{F}_{\mathsf{p}^2}$ we can define a Lorentzian quadratic form

$$Q_\nu(t,x,y,z) = -\,\nu\, t^2 + x^2 + y^2 + z^2, \tag{3}$$

whose coefficients now occupy two distinct square classes: the spatial coefficients 1 are quadratic residues, whereas the time coefficient $-\nu$ belongs to the complementary nonsquare class. Equation (3) thus realises the algebraic analogue of the Minkowski metric with signature $(1,3)$. The corresponding symmetry group

$$O(Q_\nu, \mathbb{F}_{\mathsf{p}^2}) = \{\, A \in \mathrm{GL}_4(\mathbb{F}_{\mathsf{p}^2}) \mid Q_\nu(Ax) = Q_\nu(x) \,\}$$

is a finite orthogonal group of split type (Witt index 1), playing the role of the Lorentz group in the finite-field setting.

2.3. *Algebraic Interpretation*

The passage from $\mathbb{F}_\mathsf{p}$ to $\mathbb{F}_{\mathsf{p}^2}$ constitutes the minimal *expansion step* of the Finite Ring Continuum. Inside a single shell, the powering operation P_ε can be many-to-one, compressing information by merging residue classes. The adjoining of c according to (2) restores these missing classes and expands the accessible symmetry space. Consequently, the quadratic extension not only enables the Lorentzian split but also represents an informational *innovation*—a new layer of algebraic degrees of freedom corresponding to the emergence of causal structure. This step underlies the transition from Euclidean kinematics within $\mathbb{F}_\mathsf{p}$ to Lorentzian dynamics within $\mathbb{F}_{\mathsf{p}^2}$, which will serve as the foundation for the discrete Dirac and Schrödinger equations developed in the following sections.

3. Discrete Derivatives and Framed Wavefunctions

3.1. Finite-Difference Operators

In FRC framework, all dynamical quantities are defined over discrete algebraic manifolds—the symmetry shells U_t introduced in [1]. Let $\mathbb{F}_{\mathsf{p}}$ be a symmetry-complete Euclidean shell and $\mathbb{F}_{\mathsf{p}^2}$ its Lorentzian extension defined by (2). Each element $x \in \mathbb{F}_{\mathsf{p}}$ can be regarded as a vertex of the orbital complex $\mathcal{S}_{\mathsf{p}}$, connected to its neighbours by additive shifts along the meridians and multiplicative shifts along the latitudes of $\mathcal{S}_{\mathsf{p}}$. This structure enables a purely algebraic definition of discrete derivative operators without invoking any limiting process.

For a scalar field $f : \mathbb{F}_{\mathsf{p}} \to \mathbb{F}_{\mathsf{p}^k}$, we define the *forward finite-difference operator* along the μ-th coordinate direction as

$$\partial_\mu f(x) := f(x+e_\mu) - f(x), \qquad e_\mu \in \mathbb{F}_{\mathsf{p}}, \tag{4}$$

where e_μ is a fixed generator of translation in the corresponding direction. Higher-order differences are obtained by repeated application of ∂_μ, and the discrete Laplacian is defined by

$$\Delta f(x) := \sum_{\mu=1}^{d} \big(f(x+e_\mu) - 2f(x) + f(x-e_\mu)\big), \tag{5}$$

where d is the spatial dimensionality of the shell. All derivatives are computed modulo p and therefore belong to the same finite algebraic domain as the original field. This construction is frame-covariant: a change of affine reference frame $(0,1) \mapsto (a,b)$ acts on the arguments by the bijection $\varphi(x) = a + bx$, preserving the difference relations $\partial_\mu f(\varphi(x)) = b\,\partial_\mu f(x)$.

3.2. Framed Complex and Real Domains

Wavefunctions in FRC are valued in the *framed complex field*

$${}^{\mathsf{f}}\mathbb{C} = {}^{\mathsf{f}}\mathbb{R}[\,i_t\,] = \{\,a + b\,i_t \mid a,b \in {}^{\mathsf{f}}\mathbb{R},\ i_t^2 = -1\,\} \subset \mathbb{F}_{\mathsf{p}},$$

which corresponds to the framed analogue of the ordinary complex plane, as introduced in [1]. The real subfield ${}^{\mathsf{f}}\mathbb{R}$ is formed by the closure of the framed rationals under modular convergence, while i_t denotes the structural imaginary unit associated with the symmetry-complete field ($p \equiv 1 \pmod 4$ guarantees $i_t \in \mathbb{F}_{\mathsf{p}}$). In the Lorentzian extension $\mathbb{F}_{\mathsf{p}^2}$ the additional element c defined by (2) coexists with i_t and serves as the algebraic link between the spatial and temporal square classes.

A *framed wavefunction* is a map

$$\psi : \mathbb{F}_{\mathsf{p}} \longrightarrow {}^{\mathsf{f}}\mathbb{C} \quad \text{or} \quad \psi : \mathbb{F}_{\mathsf{p}^2} \longrightarrow \mathbb{F}_{\mathsf{p}^2},$$

depending on whether the evolution is considered within a Euclidean or Lorentzian shell. The set of all such functions forms a finite-dimensional vector space over ${}^{\mathsf{f}}\mathbb{C}$ (or over $\mathbb{F}_{\mathsf{p}^2}$ in the Lorentzian case), endowed with the standard inner product

$$\langle \psi_1, \psi_2 \rangle := \sum_{x \in \mathbb{F}_{\mathsf{p}}} \psi_1(x)^* \, \psi_2(x) \in {}^{\mathsf{f}}\mathbb{C}, \tag{6}$$

where the conjugation ψ^* acts as $a + bi_t \mapsto a - bi_t$. Normalization of ψ is therefore well-defined and purely algebraic:

$$\langle \psi, \psi \rangle = 1 \quad \Longleftrightarrow \quad \sum_{x \in \mathbb{F}_{\mathsf{p}}} |\psi(x)|^2 = 1.$$

This allows unitary (norm-preserving) time evolution to be formulated entirely within the finite field, providing a consistent algebraic foundation for quantum dynamics.

3.3. Covariance Across Shells

The derivative operators (4) and (5) extend naturally from the Euclidean shell $\mathbb{F}_{\mathsf{p}}$ to the Lorentzian shell $\mathbb{F}_{\mathsf{p}^2}$ by linearity over the extended scalar field. If $f : \mathbb{F}_{\mathsf{p}^2} \to \mathbb{F}_{\mathsf{p}^2}$, then

$$\partial_\mu f(x) := f(x + c\, e_\mu) - f(x)$$

defines a covariant discrete derivative in the presence of the causal coefficient c. In the limit where the observer's horizon covers the entire field, the difference operators behave as algebraic analogues of differential operators on a continuous manifold, but without appealing to actual infinity. These discrete derivatives, together with the framed wavefunctions defined above, constitute the kinematic foundation for the Schrödinger and Dirac equations developed in the subsequent sections.

4. Schrödinger Equation in Euclidean Shell $\mathbb{F}_{\mathsf{p}}$

Within a single symmetry-complete Euclidean shell $\mathbb{F}_{\mathsf{p}}$, the space-time structure remains isotropic and reversible. Causality has not yet emerged, and all arithmetic symmetries $\langle T_a, S_m, P_\varepsilon \rangle$ act bijectively. In this regime, the appropriate dynamical law is the discrete analogue of the Schrödinger equation defined over the framed complex field ${}^{\mathsf{f}}\mathbb{C} \subset \mathbb{F}_{\mathsf{p}}$ introduced in Section 3.

Let $\psi : \mathbb{F}_{\mathsf{p}} \times \mathbb{F}_{\mathsf{p}} \to {}^{\mathsf{f}}\mathbb{C}$ denote a framed wavefunction with spatial coordinate $x \in \mathbb{F}_{\mathsf{p}}$ and discrete time index $t \in \mathbb{F}_{\mathsf{p}}$. Using the finite-difference operators (4) and (5), the Schrödinger equation in a potential $V(x)$ is written as

$$i_t\, \partial_t \psi(x,t) = -\frac{1}{2m}\, \Delta\psi(x,t) + V(x)\, \psi(x,t), \tag{7}$$

where $i_t \in \mathbb{F}_{\mathsf{p}}$ is the structural imaginary unit satisfying $i_t^2 = -1$ and $m \in {}^{\mathsf{f}}\mathbb{R}$ is the framed mass parameter.[1] All quantities are evaluated modulo p, and hence the evolution operator is a finite linear transformation on the vector space of wavefunctions $\psi(x,t)$.

Equation (7) preserves the inner product (6) exactly:

$$\langle \psi(t+\delta t), \psi(t+\delta t) \rangle = \langle \psi(t), \psi(t) \rangle,$$

so that time evolution is algebraically unitary. This property follows directly from the antisymmetry of the discrete Laplacian and the conjugation rule $i_t^* = -i_t$ in ${}^{\mathsf{f}}\mathbb{C}$.

4.1. Scale-Periodicity and Zoom Invariance

As has been established in [1], the class of framed rationals ${}^{\mathsf{f}}\mathbb{Q}$ possesses an intrinsic scale-periodicity under the *zoom transformation*

$$Z : [x, n] \longmapsto [x, n+1],$$

which rescales the local observation grid by a factor of the primitive root $g \in \mathbb{F}_{\mathsf{p}}^\times$. After $p-1$ iterations, the grid repeats exactly:

$$Z^{\,p-1}[x,n] = [x,n], \tag{8}$$

as a consequence of Fermat's little theorem $g^{p-1} = 1$. The same periodicity applies to the discrete Laplacian and therefore to the entire dynamical system (7). Hence the wavefunction obeys a *finite renormalization cycle*

$$\psi_{n+p-1}(x,t) = \psi_n(x,t),$$

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[1] The connection between the framed mass m and the FRC logarithmic complexity measure will be discussed in Section 7.

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demonstrating that the Schrödinger dynamics in $\mathbb{F}_p$ is scale-periodic and self-similar across zoom levels. From the physical perspective, this periodicity represents the *consolidation phase* of the FRC evolution: information is not lost but compressed into repeating symmetry cycles inside a fixed shell.

4.2. Example: Prime $p = 13$

To illustrate the construction, consider the symmetry-complete field $\mathbb{F}_{13}$ of radius $t = 3$, whose structural unit satisfies $i_t = 5$ since $5^2 \equiv -1 \pmod{13}$. Let the spatial domain consist of the 13 points $x \in \{0, 1, \ldots, 12\}$ with cyclic boundary conditions. Choose the potential $V(x) = 0$ and the initial state

$$\psi(x,0) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $m = 1$ and a discrete time step $\delta t = 1$, Equation (7) reduces to

$$\psi(x,t+1) = \psi(x,t) - i_t \frac{1}{2}\big(\psi(x+1,t) - 2\psi(x,t) + \psi(x-1,t)\big). \tag{9}$$

Iterating (9) over $t \in \mathbb{F}_{13}$ produces an exactly periodic sequence of states with period $p - 1 = 12$, confirming the scale-periodicity predicted by (8). The discrete probability distribution $|\psi(x,t)|^2$ is conserved for all t, providing a finite-field realization of unitary quantum evolution.

4.3. Interpretation

The discrete Schrödinger equation (7) encapsulates the Euclidean, reversible dynamics internal to a single FRC shell. All observables evolve unitarily within the bounded informational capacity of $\mathbb{F}_p$, and the periodic recurrence under zoom operations ensures closed, time-symmetric behaviour. The subsequent transition to the Lorentzian extension $\mathbb{F}_{p^2}$ introduces a new coefficient $c^2 = \nu$ and thereby enables the emergence of directional causality and the Dirac equation discussed in Section 5.

5. Dirac Equation in Quadratic Extension $\mathbb{F}_{p^2}$

5.1. Lorentzian Setting and Clifford Algebra

The quadratic extension $\mathbb{F}_{p^2}$ introduced in (2) provides the minimal algebraic domain in which a Lorentzian structure can exist. Inside this extension, the time-like coefficient $c \in \mathbb{F}_{p^2}$ satisfies $c^2 = \nu$, where $\nu \in \mathbb{F}_p^\times$ is a fixed nonsquare. The Lorentzian quadratic form

$$Q_\nu(t,x,y,z) = -\nu\, t^2 + x^2 + y^2 + z^2, \tag{10}$$

distinguishes the time coordinate from the spatial ones through the square-class separation of their coefficients and defines a finite-field analogue of the Minkowski metric with signature $(1,3)$.

To formulate dynamics that are covariant under the symmetry group $O(Q_\nu, \mathbb{F}_{p^2})$, we introduce the Clifford algebra $\mathrm{Cl}(1,3;\mathbb{F}_{p^2})$ generated by elements γ^μ $(\mu = 0,1,2,3)$ satisfying

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\mu\nu} I, \qquad \eta = \mathrm{diag}(-\nu,\ 1,\ 1,\ 1). \tag{11}$$

Since $\mathbb{F}_{p^2}$ is algebraically closed under multiplication, a representation of this algebra by 4×4 matrices with entries in $\mathbb{F}_{p^2}$ always exists. A convenient choice mimics the Weyl representation,

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \qquad k = 1,2,3,$$

where the σ^k are the standard Pauli matrices with entries in $\mathbb{F}_{p^2}$ and I is the 2×2 identity. These matrices satisfy (11) with all products taken modulo p.

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5.2. *Finite-Field Dirac Operator*

Let $\psi : \mathbb{F}^4_{\mathsf{p}^2} \to \mathbb{F}^4_{\mathsf{p}^2}$ be a four-component spinor field. The *finite-field Dirac equation* is defined as

$$(i_t\, \gamma^\mu \partial_\mu - m)\,\psi = 0, \tag{12}$$

where i_t is the structural imaginary unit inherited from the symmetry-complete shell $\mathbb{F}_{\mathsf{p}}$, $m \in \mho\mathbb{R}$ is the framed mass parameter, and the discrete derivative operators ∂_μ are those defined in (4), now extended linearly to the field $\mathbb{F}_{\mathsf{p}^2}$:

$$\partial_\mu f(x) := f(x + c\, e_\mu) - f(x), \qquad e_\mu \in \mathbb{F}_{\mathsf{p}}.$$

The equation acts on ψ componentwise and is entirely finite: all quantities belong to the algebraic closure $\mathbb{F}_{\mathsf{p}^2}$, and no limiting or differential calculus is required.

Applying the operator $(i_t\, \gamma^\mu \partial_\mu + m)$ from the left to (12) yields the finite-field Klein-Gordon relation,

$$\big(-\eta^{\mu\nu}\partial_\mu \partial_\nu + m^2\big)\,\psi = 0, \tag{13}$$

demonstrating that (12) is a first-order factorization of the Lorentz-invariant wave equation over $\mathbb{F}_{\mathsf{p}^2}$.

5.3. *Lorentz Covariance*

Let $\Lambda(u) \in O(Q_\nu, \mathbb{F}_{\mathsf{p}^2})$ denote the finite-field Lorentz boost parameterized by $u = g^{\Delta m} \in \mathbb{F}^\times_{\mathsf{p}}$, as defined in [2]. The boost acts on coordinates and spinors according to

$$x'^\mu = \Lambda^\mu_{\ \nu}(u)\, x^\nu, \qquad \psi'(x') = S(\Lambda)\, \psi(x),$$

where the spinor transformation matrix $S(\Lambda)$ satisfies

$$S^{-1}(\Lambda)\, \gamma^\mu\, S(\Lambda) = \Lambda^\mu_{\ \nu}\, \gamma^\nu.$$

Substituting these relations into (12) shows that the Dirac equation is invariant under the finite orthogonal group $O(Q_\nu, \mathbb{F}_{\mathsf{p}^2})$, establishing the exact analogue of Lorentz covariance in the FRC framework.

The associated conserved current is

$$j^\mu = \bar{\psi}\, \gamma^\mu \psi, \qquad \bar{\psi} = \psi^* \gamma^0, \tag{14}$$

which satisfies the discrete continuity equation $\partial_\mu j^\mu = 0$. The algebraic conjugation ψ^* acts as complex conjugation in $\mathbb{F}_{\mathsf{p}^2}$: $(a + b i_t)^* = a - b i_t$ and $c^* = c$.

5.4. *Example: Prime* $p = 13$ *and Nonsquare* $\nu = 2$

Consider again the illustrative case $p = 13$. The nonsquares in $\mathbb{F}_{13}$ are $\{2, 5, 6, 7, 8, 11\}$; choosing $\nu = 2$ defines the quadratic extension

$$\mathbb{F}_{13^2} \cong \mathbb{F}_{13}[X]/(X^2 - 2), \qquad c = X \bmod (X^2 - 2).$$

The Lorentzian quadratic form $Q_2(t, x, y, z) = -2t^2 + x^2 + y^2 + z^2$ and the corresponding finite Lorentz boost

$$\Lambda(u) = \begin{pmatrix} \gamma & a & 0 & 0 \\ 2a & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \frac{u + u^{-1}}{2},\ a = \frac{u - u^{-1}}{2c},$$

preserve Q_2 exactly: $Q_2(\Lambda(u)x) = Q_2(x)$. Explicit enumeration of the null set $Q_2 = 0$ confirms the existence of nontrivial light-like solutions in $\mathbb{F}_{13^2}$, providing a finite analogue of the light cone.

Substituting these data into (12) allows one to compute discrete spinor solutions $\psi(t, x)$ and verify that the bilinear current (14) remains conserved at each iteration, demonstrating algebraic unitarity and Lorentz covariance at finite p.

5.5. Interpretation

The Dirac Equation (12) describes the *innovation phase* of FRC evolution. Whereas the Schrödinger dynamics within $\mathbb{F}_p$ compress information into reversible cycles, the extension to $\mathbb{F}_{p^2}$ introduces the causal coefficient c and thereby expands the accessible algebraic degrees of freedom. Particle-antiparticle duality and causal propagation emerge as manifestations of this expanded square-class structure. In this sense, the Dirac formalism represents the first genuinely relativistic dynamics of the finite universe, corresponding to the transition from Euclidean consolidation to Lorentzian innovation in the algebraic exploration of the symmetry space U.

6. Consolidation-Innovation Cycle in FRC Dynamics

6.1. Compression and Expansion Across Shells

The results established in Sections 4 and 5 reveal two complementary regimes of evolution within the Finite Ring Continuum (FRC). The Schrödinger Equation (7) operates entirely inside a single Euclidean shell $\mathbb{F}_p$, where all arithmetic symmetries are bijective and reversible. The corresponding dynamics is time-symmetric, scale-periodic, and informationally closed: each zoom operation returns the system to its initial configuration after a finite number of steps (cf. (8)).

By contrast, the Dirac Equation (12) resides in the quadratic extension $\mathbb{F}_{p^2}$, where the coefficient $c^2 = \nu$ belonging to a nonsquare class introduces an explicit algebraic asymmetry between time and space. This asymmetry breaks the perfect cyclicity of the Euclidean regime and enables the emergence of causal order. In FRC terminology, the passage

$$\mathbb{F}_p \longrightarrow \mathbb{F}_{p^2}$$

represents the transition from the *consolidation* (or compression) phase to the *innovation* (or expansion) phase of the finite universe.

6.2. Algebraic Mechanism of the Cycle

Within a symmetry-complete shell $\mathbb{F}_p$, the powering operation $P_\varepsilon : x \mapsto x^\varepsilon$ can become non-bijective whenever $\gcd(\varepsilon, p-1) > 1$. This many-to-one mapping merges distinct residues into power-residue classes, producing an apparent loss of information. However, from the global viewpoint of the extended field $\mathbb{F}_{p^2}$, these merged classes are fully resolved by the new generator c that connects the two square classes of $\mathbb{F}_p$. Thus, the act of powering triggers a compression within a shell, while the subsequent quadratic extension restores and expands the symmetry structure:

$$P_\varepsilon : \mathbb{F}_p^\times \longrightarrow \left(\mathbb{F}_p^\times\right)^\varepsilon \quad \text{(compression)}, \qquad \mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2} \quad \text{(expansion)}. \tag{15}$$

Equation (15) formalizes the *consolidation-innovation cycle*: each shell first refines its internal symmetries through local information compression, then innovates by extending its algebraic horizon to include previously inaccessible elements. This mechanism provides an algebraic analogue of the cosmological arrow of time: the universe evolves by iteratively exploring and enlarging its own symmetry space.

6.3. Physical and Informational Interpretation

From the informational perspective, the Euclidean Schrödinger regime encodes reversible micro-dynamics inside a closed information manifold. All processes are unitary and conserve the total informational content of the shell. The transition to the Lorentzian Dirac regime corresponds to the *creation of new informational degrees of freedom* associated with the extension $\mathbb{F}_{p^2}$. To an observer confined within $\mathbb{F}_p$, this extension appears as an irreversible increase of entropy—a forward temporal evolution.

Yet from the global, algebraic standpoint of the full FRC structure, the overall information is conserved: compression and expansion are dual phases of a single relational process.

Geometrically, the consolidation phase manifests as periodic motion on the discrete spheroid $\mathcal{S}_{\mathsf{p}} \subset U$, while the innovation phase corresponds to the emergence of a new spheroid S_{p^2} of higher radius within the same symmetry space U. This expansion of the accessible orbit space represents the *growth of the universe* in purely algebraic terms.

6.4. Cycle Summary

The complete consolidation-innovation cycle can be summarized as follows:

Phase	**Algebraic Domain**	**Physical Character**
Refinement (Compression)	$\mathbb{F}_{\mathsf{p}}$	Euclidean, reversible, scale-periodic
Innovation (Expansion)	$\mathbb{F}_{\mathsf{p}^2}$	Lorentzian, causal, information-expanding

This alternation of consolidation and innovation provides a natural explanation for the emergence of both quantum coherence and causal structure in the Finite Ring Continuum. The Schrödinger and Dirac equations thus represent not two unrelated formalisms, but successive stages in the universe's algebraic self-development:

$$\text{Schrödinger dynamics within } \mathbb{F}_{\mathsf{p}} \xrightarrow{\text{extension}} \text{Dirac dynamics within } \mathbb{F}_{\mathsf{p}^2}.$$

The consolidation-innovation principle therefore unifies the reversible and irreversible aspects of physics within a single, finite, and informationally complete framework.

7. Discussion and Outlook

7.1. Synthesis of Results

The developments presented in this note extend the algebraic framework of the Finite Ring Continuum (FRC) from kinematics to dynamics. Within the Euclidean shell $\mathbb{F}_{\mathsf{p}}$, the discrete Schrödinger Equation (7) establishes a reversible, scale-periodic evolution of framed wavefunctions over the finite complex field ${}^{f}\mathbb{C}$. Within the Lorentzian extension $\mathbb{F}_{\mathsf{p}^2}$, the Dirac Equation (12) realizes the first relativistically covariant dynamics, uniting the finite-field Clifford algebra (11) with the causal structure encoded by the quadratic form (10). Together these two regimes demonstrate that both quantum and relativistic formalisms can be reconstructed in a purely finite and self-consistent arithmetic setting.

The central conceptual result is the identification of the *consolidation-innovation cycle*, summarized in Section 6. Each Euclidean shell $\mathbb{F}_{\mathsf{p}}$ supports reversible, information-preserving dynamics, while the quadratic extension to $\mathbb{F}_{\mathsf{p}^2}$ introduces causal asymmetry and informational expansion. This alternation of compression and expansion provides a natural algebraic origin for the coexistence of time-symmetric quantum processes and the directional flow of macroscopic causality.

7.2. Relation to Mass and Complexity

In the broader FRC program, the mass parameter m appearing in (7) and (12) is not a free constant but a frame-invariant scalar associated with the logarithmic measure of algebraic complexity [1]. From this perspective, the inertial role of m in the Dirac operator $(i_t\gamma^\mu\partial_\mu - m)\psi = 0$ corresponds to the resistance of the system to informational change: higher algebraic complexity implies greater inertia against reconfiguration under the symmetry group G. Establishing the quantitative equivalence between this log-complexity mass and conventional inertial or gravitational mass will be the next step in the program. Such a derivation would connect the finite algebraic formulation of quantum dynamics with measurable physical observables and close the conceptual gap between arithmetic information and physical matter.

7.3. Future Directions

Several extensions of the present framework suggest themselves:

Mass, entropy, and energy. The forthcoming extension of the FRC programme will formalize the role of the mass parameter m as a logarithmic measure of algebraic complexity and establish its relation to entropy and energy as quasi-invariant observables of the consolidation-innovation cycle.

Gauge couplings and interactions. Introducing local frame transformations $\psi(x) \mapsto U(x)\psi(x)$ with $U(x) \in {}^{\mathsf{f}}\mathbb{C}^{\times}$ will allow the definition of finite-field analogues of gauge potentials and covariant derivatives. This could lead to discrete versions of electromagnetic or Yang-Mills interactions within the FRC structure.

Composite shells and hierarchical dynamics. Extending the formalism from prime to composite moduli via the Chinese Remainder Theorem, as outlined in [1], will enable the study of coupled prime subshells $S_q = \prod_i S_{p_i}$ and the emergence of multi–scale phenomena.

Numerical enumeration and simulation. Finite-field Schrödinger and Dirac dynamics are exactly computable for small primes. Explicit enumeration of wavefunction evolution for $p = 13, 17, 29, \ldots$ can be used to verify conservation laws, Lorentz invariance, and periodicity, providing a constructive test of the formalism.

Continuum correspondence. The asymptotic limit of large p (always finite) connects the discrete quadratic form Q_v with the continuous Minkowski metric, and the finite-difference Dirac Equation (12) approaches its standard analytic counterpart. Exploring the rate of this convergence may offer new insight into how continuous physics arises as an effective description of finite algebraic reality.

7.4. Conceptual Outlook

The unification achieved here demonstrates that the essential structures of quantum mechanics and special relativity are not contingent on the continuum hypothesis. Both arise naturally from the arithmetic symmetries of finite fields once framed and extended according to the principles of relational finitude. The Schrödinger and Dirac equations thus represent two complementary faces of a single finite-informational ontology: the former describing reversible evolution within a bounded algebraic frame, and the latter describing its irreversible expansion into a causally ordered domain. From this viewpoint, the evolution of the universe is the process by which algebraic relations progressively unfold their own symmetry space—a self-consistent arithmetic exploration of existence.

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