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Article

Recurrence Relations and Characterization of Moments for the Extended Power Lindley Distribution Under Progressive First Failure Censoring

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Abstract: This study explores the development of recurrence relations (RRs) for single moments (SMs) and product moments (PMs) based on progressive first-failure censored data (PFFCD) using the extended power Lindley distribution (EPLD). These RRs enable efficient computation of key statistical measures, such as means and variances, across varying sample sizes, reducing computational effort. The proposed approach extends existing methodologies and adapts to progressive type-II censoring schemes (PTIICS), broadening its applicability. This advancement enhances analytical frameworks and computational tools for statistical inference in models involving censored data.

Keywords: characterization; extended power Lindley distribution; progressive first-failure censored

MSC: 62E10; 62N99

1. Introduction

In industrial experiments and reliability testing, it is important to save costs and money when observing product downtime. Censoring is the most appropriate technique to achieve this goal by using lifetime experiments where we observe some lifetimes or failure times and not all lifetimes of the units under test. There are different methods of censoring. Progressively censored sampling has gained considerable attention in statistical research due to its flexibility and applicability across various fields. Numerous studies have explored this approach, including investigations by Balakrishnan et al. [1], Balakrishnan and Sandhu [2], and Davis and Feldstein [3]. Singh et al. [4] proposed both classical and Bayesian inference methods for the extended exponential distribution under PTIICS with binomial removals. Aggarwala and Balakrishnan [5] derived RRs for SMs and PMs under PTIICS. Recent advancements include the work of Athar et al. [6], who examined moments under PTIICS for the Lindley distribution, and Mohie El-Din et al. [7], who developed E-Bayesian estimation techniques for the Gompertz distribution with practical applications. Kotb et al. [8] extended these methods to the Kumaraswamy distribution under PFFCD, while Abu-Moussa et al. [9] estimated reliability functions for the extended Rayleigh distribution using the same framework. Further contributions by Sharawy [10] included RRs of PFFCD and characterization of right-truncated exponential distributions. Additionally, Alsadat et al. [11] conducted a comprehensive study of RRs and characterizations based on PFFCD.

The PFFCD scheme operates by dividing the total n groups, each containing b items, into segments based on observed failures. At the first $X_{1:u:n,b}^{(k_1,\dots,k_u)}$, k_1 groups, along with the group containing the failed item, are removed. This process continues for subsequent failures $X_{2:u:n,b}^{(k_1,\dots,k_u)}$, $X_{3:u:n,b}^{(k_1,\dots,k_u)}$, and so on, until the u^{th} failure $X_{u:u:n,b}^{(k_1,\dots,k_u)}$, after which the remaining k_u groups are removed. The resulting order statistics $X_{1:u:n,b}^{(k_1,\dots,k_u)} < X_{2:u:n,b}^{(k_1,\dots,k_u)} < \dots < X_{u:u:n,b}^{(k_1,\dots,k_u)}$ are termed PFFCD order statistics under the PTIICS scheme, where $n - u = \sum_{i=1}^u k_i$.

Assuming the failure times of $b \times n$ items follow a continuous distribution characterized by the cumulative distribution function (CDF) $F(x)$, survival function (SF) $\bar{F}(x) = 1 - F(x)$, and probability density function (PDF) $f(x)$, and the joint PDF of the order statistics $X_{1:n,b}^{(k_1,\dots,k_u)}, X_{2:n,b}^{(k_1,\dots,k_u)}, \dots, X_{u:n,b}^{(k_1,\dots,k_u)}$ is expressed as follows:

$$f_{X_{1:n,b},\dots,X_{u:n,b}}(x_1, \dots, x_u) = Ab^u \prod_{i=1}^u f(x_i)[\bar{F}(x_i)]^{R_i}, \quad 0 < x_1 < \dots < x_u < \infty, \tag{1}$$

where,

$$x_i = x_{i:u:n,b}, A = n \prod_{i=1}^{u-1} \left(n - i - \sum_{j=1}^i k_j \right), i = 1, \dots, u - 1, \text{ and } R_i = b k_i + b - 1.$$

The EPLD is significant due to its flexibility in modeling various types of real-world data, particularly in reliability analysis and survival studies. Its ability to encompass multiple well-known distributions as special cases enhances its applicability across diverse statistical and probabilistic fields, see Alkarni [12]. The PDF and the CDF of EPLD are given, respectively, as follows.

$$f(x, \theta, \alpha, \beta) = \frac{\alpha \theta^2}{\theta + \beta} [1 + \beta x^\alpha] x^{\alpha-1} e^{-\theta x^\alpha}, \quad \theta, \alpha, \beta > 0, \quad x \geq 0, \tag{2}$$

$$F(x, \theta, \alpha, \beta) = 1 - \left[1 + \frac{\theta \beta}{\theta + \beta} x^\alpha \right] e^{-\theta x^\alpha}. \tag{3}$$

It is worth noting that the relationship between PDF and SF can be derived using Equations (2) and (3) and is expressed as follows:

$$[\theta + \beta + \theta \beta x^\alpha] f(x) = \alpha \theta^2 [1 + \beta x^\alpha] x^{\alpha-1} \bar{F}(x). \tag{4}$$

In Table 1 we present some distributions derived from the EPLD as special cases with different values of the parameters α, β and θ .

Table 1. Special cases of the EPLD.

Case	Parameters	Distribution
1	$\beta = 1$	Power Lindley
2	$\alpha = 1$ and $\beta = 1$	Lindley
3	$\beta = 0$	Weibull
4	$\alpha = 2$ and $\beta = 0$	Rayleigh
5	$\alpha = 1$ and $\beta = 0$	Exponential
6	$\beta = -\theta$	Modified Gamma
7	$\alpha = 2$ and $\beta = 1$	Modified Rayleigh
8	$\theta = 1$	Extended Lindley

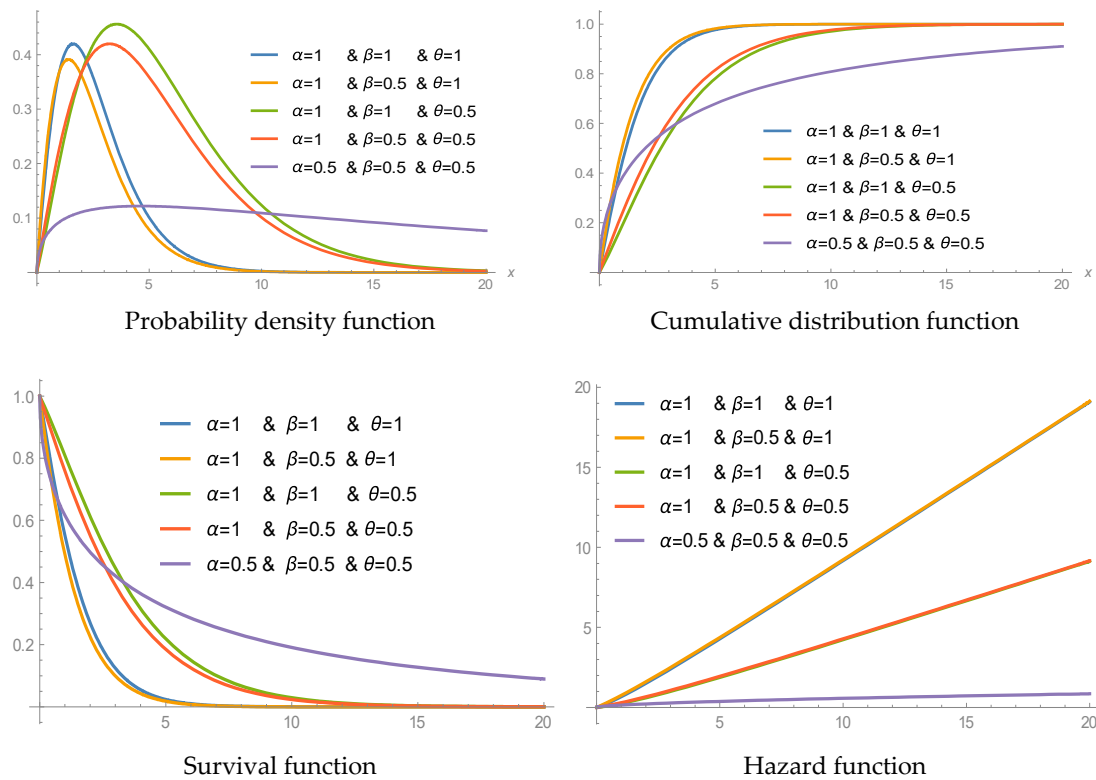


Figure 1. graphs of the PDF, CDF, SF and Hazard functions of EPLD.

For any continuous distribution, the i^{th} SMs of PFFCD can be represented, as outlined in Equation (1), as follows:

$$\mu_{t:u:n,b}^{(R_1,\dots,R_u)(i)} = E \left[X_{t:u:n,b}^{(R_1,\dots,R_u)} \right]^i = A \iint \dots \int_{0 < x_1 < \dots < x_u < \infty} x_t^i b^u \times f(x_1) [\bar{F}(x_1)]^{R_1} f(x_2) [\bar{F}(x_2)]^{R_2} \dots f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_u, \quad (5)$$

while the i^{th} and j^{th} product moment for $X_{t:u:n,b}$ and $X_{y:u:n,b}$ based on the PFFCD is defined as follows,

$$\mu_{t,y:u:n,b}^{(R_1,\dots,R_u)(ij)} = E \left[X_{t,y:u:n,b}^{(R_1,\dots,R_u)} X_{y:u:n,b}^{(R_1,\dots,R_u)} \right] = A \iint \dots \int_{0 < x_1 < \dots < x_u < \infty} x_t^i x_y^j b^u \times f(x_1) [\bar{F}(x_1)]^{R_1} f(x_2) [\bar{F}(x_2)]^{R_2} \dots f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_u. \quad (6)$$

2. Recurrence Relations for SMs and PMs

This section introduces RRs for SMs and PMs within the framework of PFFCD. These RRs simplify statistical computations by providing an iterative approach to determine expected values, variances, and higher-order moments as failures occur, significantly reducing computational effort. For product moments, the RRs facilitate the analysis of joint distributions and covariance structures under progressive censoring, making them particularly useful in areas such as reliability studies and clinical research. Employing these RRs offers valuable insights into system behavior under PFFCD, with practical applications in probabilistic modeling and risk evaluation [13]

The following theorem presents the RRs for SMs based on PFFCD.

Theorem 1. If $X_{1:n:b} \leq X_{2:n:b} \leq \dots \leq X_{n:n:b}$ be the order statistics for a random sample of size n drawn from EPLD, for $2 \leq r \leq u-1, u \leq n$ and $i \geq 0$, then

$$\begin{aligned} \mu_{r:n,b}^{(R_1, \dots, R_u)(i+2\alpha)} &= \frac{(i+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_r+1)} \mu_{r:n,b}^{(R_1, \dots, R_u)(i)} + \frac{(i+2\alpha)}{\alpha\theta(R_r+1)} \mu_{r:n,b}^{(R_1, \dots, R_u)(i+\alpha)} \\ &+ \frac{(n-k_1-\dots-k_{r-1}-r+1)}{(R_r+1)} \left[\mu_{r-1:n,b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1}+R_r-1), R_{r+1}, \dots, R_u)(i+2\alpha)} \right. \\ &+ \left. \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r-1:n,b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1}+R_r-1), R_{r+1}, \dots, R_u)(i+\alpha)} \right] - \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r:n,b}^{(R_1, R_2, \dots, R_u)(i+\alpha)} \\ &- \frac{(n-k_1-\dots-k_r-r)}{(R_r+1)} \left[\mu_{r:n,b}^{(R_1, R_2, \dots, R_{r-1}, (R_r+R_{r+1}-1), R_{r+2}, \dots, R_u)(i+2\alpha)} \right. \\ &- \left. \frac{(n-k_1-\dots-k_r-r)}{(R_r+1)} \left[\mu_{r:n,b}^{(R_1, R_2, \dots, R_{r-1}, (R_r+R_{r+1}-1), R_{r+2}, \dots, R_u)(i+\alpha)} \right. \right. \\ &\quad \left. \left. + \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r-1:n,b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1}+R_r-1), R_{r+1}, \dots, R_u)(i+\alpha)} \right] \right]. \quad (7) \end{aligned}$$

Proof of Theorem 1. Using Equations (1.4) and (1.5), we derive the following:

$$\begin{aligned} (\theta+\beta) \mu_{r:n,b}^{(R_1, \dots, R_u)(i)} + \theta\beta \mu_{r:n,b}^{(R_1, \dots, R_u)(i+\alpha)} &= A \iint \dots \int_{0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_u < \infty} b^u \dots \\ &\times K_1(x_{r-1}, x_{r+1}) f(x_1) [\bar{F}(x_1)]^{R_1} \dots f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots \\ &\times f(x_u) [1-F(x_u)]^{R_u} dx_1 dx_2 \dots dx_{r-1} dx_{r+1} \dots dx_u, \quad (8) \end{aligned}$$

where

$$K_1(x_{r-1}, x_{r+1}) = \alpha\theta^2 \int_{x_{r-1}}^{x_{r+1}} x_r^i [1+\beta x_r^\alpha] x_r^{\alpha-1} [\bar{F}(x_r)]^{R_r+1} dx_r. \quad (9)$$

By applying integration by parts, we arrive at the following result:

$$\begin{aligned} K_1(x_{r-1}, x_{r+1}) &= \alpha\theta^2 \left[\frac{x_{r+1}^{i+\alpha} [\bar{F}(x_{r+1})]^{R_r+1} - x_{r-1}^{i+\alpha} [\bar{F}(x_{r-1})]^{R_r+1}}{i+\alpha} \right. \\ &+ \frac{\beta x_{r+1}^{i+2\alpha} [\bar{F}(x_{r+1})]^{R_r+1} - \beta x_{r-1}^{i+2\alpha} [\bar{F}(x_{r-1})]^{R_r+1}}{i+2\alpha} \\ &+ \left. \left(\frac{R_r+1}{i+\alpha} \right) \int_{x_{r-1}}^{x_{r+1}} x_r^{i+\alpha} f(x_r) [\bar{F}(x_r)]^{R_r} dx_r \right. \\ &\quad \left. + \frac{\beta(R_r+1)}{i+2\alpha} \int_{x_{r-1}}^{x_{r+1}} x_r^{i+2\alpha} f(x_r) [\bar{F}(x_r)]^{R_r} dx_r \right]. \quad (10) \end{aligned}$$

Now substituting for the resultant expression of $K_1(x_{r-1}, x_{r+1})$ from Equation (10) in Equation (8), yields Equation (7), thus concluding the proof.

The following two theorems present RRs for PMs derived from PFFCD.

Theorem 2. If $X_{1:n:b} \leq X_{2:n:b} \leq \dots \leq X_{n:n:b}$ be the order statistics for a random sample of size n drawn from EPLD, for $1 \leq r < s \leq u-1, u \leq n$ then and $i, j \geq 0$,

$$\begin{aligned} \mu_{r,s:n,b}^{(R_1, \dots, R_u)(i+2\alpha, j)} &= \frac{(i+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_r+1)} \mu_{r,s:n,b}^{(R_1, \dots, R_u)(i, j)} \\ &+ \frac{(i+2\alpha)}{\alpha\theta(R_r+1)} \mu_{r,s:n,b}^{(R_1, \dots, R_u)(i+\alpha, j)} - \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r,s:n,b}^{(R_1, R_2, \dots, R_u)(i+\alpha, j)} \end{aligned}$$

$$\begin{aligned}
& + \frac{(n - k_1 - \dots - k_{r-1} - r + 1)}{(R_r + 1)} \left[\mu_{r-1, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1} + R_r - 1), R_{r+1}, \dots, R_u)^{(i+2\alpha, j)}} \right. \\
& + \frac{i + 2\alpha}{\beta(i + \alpha)} \mu_{r-1, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1} + R_r - 1), R_{r+1}, \dots, R_u)^{(i+\alpha, j)}} \left. \right] \\
& - \frac{(n - k_1 - \dots - k_r - r)}{(R_r + 1)} \left[\mu_{r, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{r-1}, (R_r + R_{r+1} - 1), R_{r+2}, \dots, R_u)^{(i+2\alpha, j)}} \right. \\
& \left. + \frac{i + 2\alpha}{\beta(i + \alpha)} \mu_{r, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{r-1}, (R_r + R_{r+1} - 1), R_{r+2}, \dots, R_u)^{(i+\alpha, j)}} \right] \quad (11)
\end{aligned}$$

Proof of Theorem 2. Using Equations (4) and (6), we derive the following:

$$\begin{aligned}
& (\theta + \beta) \mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i, j)}} + \theta \beta \mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i+\alpha, j)}} = \\
& A \iint \dots \int_{0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_u < \infty} x_s^j b^u K_1(x_{r-1}, x_{r+1}) f(x_1) [\bar{F}(x_1)]^{R_1} \dots \\
& \times f(x_{r-1}) [1 - F(x_{r-1})]^{R_{r-1}} f(x_{r+1}) [1 - F(x_{r+1})]^{R_{r+1}} \dots f(x_u) \\
& \times [1 - F(x_u)]^{R_u} dx_1 dx_2 \dots dx_{r-1} dx_{r+1} \dots dx_u, \quad (12)
\end{aligned}$$

substituting the resultant expression of $K_1(x_{r-1}, x_{r+1})$ from Equation (10) in Equation (12), yields Equation (11), thus concluding the proof.

Theorem 3. If $X_{1:n:n,b} \leq X_{2:n:n,b} \leq \dots \leq X_{n:n:n,b}$ be the order statistics for a random sample of size n drawn from EPLD, for $1 \leq r < s \leq u - 1, u \leq n$ and $i, j \geq 0$, then

$$\begin{aligned}
\mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i, j+2\alpha)}} & = \frac{(j + 2\alpha)(\theta + \beta)}{\beta\alpha\theta^2(R_s + 1)} \mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i, j)}} \\
& + \frac{(j + 2\alpha)}{\alpha\theta(R_s + 1)} \mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i, j+\alpha)}} - \frac{j + 2\alpha}{\beta(j + \alpha)} \mu_{r, s: u: n, b}^{(R_1, R_2, \dots, R_u)^{(i, j+\alpha)}} \\
& + \frac{(n - k_1 - \dots - k_{s-1} - s + 1)}{(R_s + 1)} \left[\mu_{r, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{s-2}, (R_{s-1} + R_s - 1), R_{s+1}, \dots, R_u)^{(i, j+2\alpha)}} \right. \\
& + \frac{j + 2\alpha}{\beta(j + \alpha)} \mu_{r, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{s-2}, (R_{s-1} + R_s - 1), R_{s+1}, \dots, R_u)^{(i, j+\alpha)}} \left. \right] \\
& - \frac{(n - k_1 - \dots - k_s - s)}{(R_s + 1)} \left[\mu_{r, s: u-1: n, b}^{(R_1, R_2, \dots, R_{s-1}, (R_s + R_{s+1} - 1), R_{s+2}, \dots, R_u)^{(i, j+2\alpha)}} \right. \\
& + \frac{j + 2\alpha}{\beta(j + \alpha)} \mu_{r, s: u-1: n, b}^{(R_1, R_2, \dots, R_{s-1}, (R_s + R_{s+1} - 1), R_{s+2}, \dots, R_u)^{(i, j+\alpha)}} \left. \right] \quad (13)
\end{aligned}$$

Proof of Theorem 3. Using Equations (4) and (6), we derive the following:

$$\begin{aligned}
& (\theta + \beta) \mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i, j)}} + \theta \beta \mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i, j+\alpha)}} = \\
& A \iint \dots \int_{0 < x_1 < \dots < x_{s-1} < x_{s+1} < \dots < x_u < \infty} x_s^i b^u K_2(x_{s-1}, x_{s+1}) f(x_1) [\bar{F}(x_1)]^{R_1} \dots f(x_{s-1}) \times \\
& [\bar{F}(x_{s-1})]^{R_{s-1}} f(x_{s+1}) [\bar{F}(x_{s+1})]^{R_{s+1}} \dots f(x_m) [\bar{F}(x_m)]^{R_m} dx_1 dx_2 \dots dx_{s-1} dx_{s+1} \dots dx_u, \quad (14)
\end{aligned}$$

where

$$K_2(x_{s-1}, x_{s+1}) = \alpha\theta^2 \int_{x_{s-1}}^{x_{s+1}} x_s^j (1 + \beta x_s^\alpha) x_s^{\alpha-1} [\bar{F}(x_s)]^{R_s+1} dx_s. \quad (15)$$

Now, integrating by parts gives

$$\begin{aligned}
K_2(x_{s-1}, x_{s+1}) &= \alpha \theta^2 \left[\frac{x_{s+1}^{j+\alpha} [\bar{F}(x_{s+1})]^{R_s+1} - x_{s-1}^{j+\alpha} [\bar{F}(x_{s-1})]^{R_s+1}}{j + \alpha} \right. \\
&+ \frac{\beta x_{s+1}^{j+2\alpha} [\bar{F}(x_{s+1})]^{R_s+1} - \beta x_{s-1}^{j+2\alpha} [\bar{F}(x_{s-1})]^{R_s+1}}{j + 2\alpha} \\
&+ \left(\frac{R_s + 1}{j + \alpha} \right) \int_{x_{s-1}}^{x_{s+1}} x_s^{j+\alpha} f(x_s) [1 - F(x_s)]^{R_s} dx_s \\
&\left. + \frac{\beta(R_s + 1)}{j + 2\alpha} \int_{x_{s-1}}^{x_{s+1}} x_s^{j+2\alpha} f(x_s) [\bar{F}(x_s)]^{R_s} dx_s \right]. \quad (16)
\end{aligned}$$

Now substituting for the resultant expression of $K_2(x_{s-1}, x_{s+1})$ from Equation (16) in Equation (14), yields Equation (13), thus concluding the proof.

3. The Characterizations

This section presents the characterizations of the EPLD by utilizing the relationship between its PDF and CDF, as well as through RRs for SMs and PMs under PFFCD.

3.1. Characterization via differential equation for the extended power Lindley distribution

The following theorem provides a characterization of the EPLD based on the connection between its PDF and SF.

Theorem 4. Let X be a random variable with PDF $f(\cdot)$ and SF $\bar{F}(x)$. Then X has EPLD iff

$$[\theta + \beta + \theta\beta x^\alpha]f(x) = \alpha\theta^2[1 + \beta x^\alpha]x^{\alpha-1}[\bar{F}(x)] \quad x \geq 0. \quad (17)$$

Proof of Theorem 4.

Necessity: From Equations (2) and (3) we can easily obtain Equation (17).

Sufficiency: Let X be a random variable with pdf $dF(x)$ and SF $\bar{F}(x)$. Assume that Equation (17) holds. Therefore, we have:

$$\begin{aligned}
\frac{-d\bar{F}(x)}{\bar{F}(x)} &= \frac{\alpha\theta[\theta + \theta\beta x^\alpha]x^{\alpha-1}}{\theta + \beta + \theta\beta x^\alpha} dx \\
&= \frac{\alpha\theta[\theta + \beta + \theta\beta x^\alpha - \beta]x^{\alpha-1}}{\theta + \beta + \theta\beta x^\alpha} dx \\
&= \alpha\theta x^{\alpha-1} dx - \frac{\alpha\beta\theta x^{\alpha-1}}{\theta + \beta + \theta\beta x^\alpha} dx.
\end{aligned}$$

On integrating, we get

$$-\ln |\bar{F}(x)| = \theta x^\alpha - \ln |\theta + \beta + \theta\beta x^\alpha| + C,$$

where C is an arbitrary constant.

Since $\bar{F}(0) = 1$, substituting $x = 0$ into this equation yields $C = \ln |\theta + \beta|$.

As a result,

$$\ln \left| \frac{[\theta + \beta + \theta\beta x^\alpha]}{[\theta + \beta]\bar{F}(x)} \right| = \theta x^\alpha,$$

or,

$$\frac{[\theta + \beta + \theta\beta x^\alpha]}{[\theta + \beta]\bar{F}(x)} = e^{\theta x^\alpha}.$$

Hence,

$$\bar{F}(x) = \left[1 + \frac{\theta\beta}{\theta + \beta}x^\alpha\right] e^{-\theta x^\alpha}.$$

That is the SF of EPLD, thus concluding the proof.

3.2. Characterization via RRs for SMs

The upcoming theorem provides a characterization of the EPLD by utilizing RRs for SMs derived from PFFCD. These relations offer a systematic approach for analyzing statistical properties and simplify computations under censored data schemes. This characterization enhances the understanding of the EPLD's behavior, making it valuable for applications in reliability analysis and survival studies, as outlined in the subsequent theorems.

Theorem 5. Let $X_{1:n:n;b} \leq X_{2:n:n;b} \leq \dots \leq X_{n:n:n;b}$ be the order statistics for a random sample of size n . Then X has EPLD iff, for $2 \leq r \leq u-1, u \leq n$ and $i \geq 0$,

$$\begin{aligned} \mu_{r:u:n,b}^{(R_1, \dots, R_u)(i+2\alpha)} &= \frac{(i+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_r+1)} \mu_{r:u:n,b}^{(R_1, \dots, R_u)(i)} \\ &+ \frac{(i+2\alpha)}{\alpha\theta(R_r+1)} \mu_{r:u:n,b}^{(R_1, \dots, R_u)(i+\alpha)} - \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r:u:n,b}^{(R_1, R_2, \dots, R_u)(i+\alpha)} \\ &+ \frac{(n-k_1-\dots-k_{r-1}-r+1)}{(R_r+1)} \left[\mu_{r-1:u-1:n,b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1}+R_r-1), R_{r+1}, \dots, R_u)(i+2\alpha)} \right. \\ &\left. + \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r-1:u-1:n,b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1}+R_r-1), R_{r+1}, \dots, R_u)(i+\alpha)} \right] \\ &- \frac{(n-k_1-\dots-k_r-r)}{(R_r+1)} \left[\mu_{r:u-1:n,b}^{(R_1, R_2, \dots, R_{r-1}, (R_r+R_{r+1}-1), R_{r+2}, \dots, R_u)(i+2\alpha)} \right. \\ &\left. + \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r:u-1:n,b}^{(R_1, R_2, \dots, R_{r-1}, (R_r+R_{r+1}-1), R_{r+2}, \dots, R_u)(i+\alpha)} \right]. \quad (18) \end{aligned}$$

Proof of Theorem 5.

Necessity: Theorem 1 proved the necessary part of this theorem

Sufficiency: Let X be a random variable with PDF $f(\cdot)$ and SF $\bar{F}(\cdot)$. Assuming that equation (18) is valid, we then obtain:

$$\begin{aligned} \mu_{r:u:n,b}^{(R_1, \dots, R_u)(i+2\alpha)} &= \frac{(i+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_r+1)} \mu_{r:u:n,b}^{(R_1, \dots, R_u)(i)} \\ &+ \frac{(i+2\alpha)}{\alpha\theta(R_r+1)} \mu_{r:u:n,b}^{(R_1, \dots, R_u)(i+\alpha)} - \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r:u:n,b}^{(R_1, R_2, \dots, R_u)(i+\alpha)} \\ &+ \frac{(n-k_1-\dots-k_{r-1}-r+1)}{(R_r+1)} \left[\mu_{r-1:u-1:n,b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1}+R_r-1), R_{r+1}, \dots, R_u)(i+2\alpha)} \right. \\ &\left. + \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r-1:u-1:n,b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1}+R_r-1), R_{r+1}, \dots, R_u)(i+\alpha)} \right] \\ &- \frac{(n-k_1-\dots-k_r-r)}{(R_r+1)} \left[\mu_{r:u-1:n,b}^{(R_1, R_2, \dots, R_{r-1}, (R_r+R_{r+1}-1), R_{r+2}, \dots, R_u)(i+2\alpha)} \right. \\ &\left. + \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r:u-1:n,b}^{(R_1, R_2, \dots, R_{r-1}, (R_r+R_{r+1}-1), R_{r+2}, \dots, R_u)(i+\alpha)} \right]. \quad (19) \end{aligned}$$

where,

$$\begin{aligned} \mu_{r:u:n,b}^{(R_1,\dots,R_u)(i+\alpha)} &= A \iint \dots \int_{0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_u < \infty} b^u K_3(x_{r-1}, x_{r+1}) f(x_1) [\bar{F}(x_1)]^{R_1} \dots \\ &\times f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots f(x_u) [\bar{F}(x_u)]^{R_u} \\ &\times dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_u, \quad (20) \end{aligned}$$

where

$$K_3(x_{r-1}, x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^{i+\alpha} f(x_r) [\bar{F}(x_r)]^{R_r} dx_r. \quad (21)$$

Now, integrating by parts gives

$$\begin{aligned} K_3(x_{r-1}, x_{r+1}) &= \frac{-1}{R_r + 1} x_{r+1}^{i+\alpha} [\bar{F}(x_{r+1})]^{1+R_r} + \frac{1}{1 + R_r} x_{r-1}^{i+\alpha} [\bar{F}(x_{r-1})]^{1+R_r} \\ &+ \frac{i + \alpha}{R_r + 1} \int_{x_{r-1}}^{x_{r+1}} x_r^{i+\alpha-1} [\bar{F}(x_r)]^{1+R_r} dx_r. \quad (22) \end{aligned}$$

Now by substituting in Equation (20), we get

$$\begin{aligned} \mu_{r:u:n,b}^{(R_1,\dots,R_u)(i+\alpha)} &= A \frac{i + \alpha}{R_r + 1} \iint \dots \int_{0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_u < \infty} b^u \\ &\times f(x_1) [\bar{F}(x_1)]^{R_1} \dots \int_{x_{r-1}}^{x_{r+1}} x_r^{i+\alpha-1} [\bar{F}(x_r)]^{1+R_r} dx_r f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} \\ &f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_u \\ &+ \frac{A}{R_r + 1} \iint \dots \int_{0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_u < \infty} x_{r+1}^{i+\alpha} b^u f(x_1) \\ &\times [\bar{F}(x_1)]^{R_1} \dots f(x_{r-1}) [\bar{F}(x_{r-1})]^{1+R_r+R_{r-1}} f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots \\ &\times f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_u \\ &- \frac{A}{R_r + 1} \iint \dots \int_{0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_u < \infty} x_{r-1}^{i+\alpha} b^u f(x_1) \\ &\times [\bar{F}(x_1)]^{R_1} \dots f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} f(x_{r+1}) [\bar{F}(x_{r+1})]^{1+R_r+R_{r+1}} \dots \\ &\times f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_u \\ &= A \frac{i + \alpha}{R_r + 1} \iint \dots \int_{0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_u < \infty} b^u f(x_{k+1}) [\bar{F}(x_1)]^{R_1} \dots \\ &\times \int_{x_{r-1}}^{x_{r+1}} x_r^{i+\alpha-1} [\bar{F}(x_r)]^{1+R_r} dx_r f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} \\ &f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_u \\ &+ \frac{(n - R_{k+1} - \dots - R_r - r)}{R_r + 1} \mu_{r:u-1:n,b}^{(R_1,\dots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\dots,R_u)(i+\alpha)} \\ &- \frac{(n - R_{k+1} - \dots - R_{r-1} - r + 1)}{R_r + 1} \mu_{r-1:u-1:n,b}^{(R_1,\dots,R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\dots,R_u)(i+\alpha)}, \quad (23) \end{aligned}$$

and

$$\begin{aligned} \mu_{r:u:n,b}^{(R_1,R_2,\dots,R_u)(i+2\alpha)} &= A \frac{i + 2\alpha}{R_r + 1} \iint \dots \int_{0 < x_1 < \dots < x_{r-1} < x_{r+1} < \dots < x_u < \infty} b^u f(x_1) [\bar{F}(x_1)]^{R_1} \dots \times \\ &\int_{x_{r-1}}^{x_{r+1}} x_r^{i+2\alpha-1} [\bar{F}(x_r)]^{1+R_r} dx_r f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} \\ &f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots f(x_m) [\bar{F}(x_m)]^{R_m} dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_u \end{aligned}$$

$$+ \frac{(n - R_{k+1} - \dots - R_r - r)}{R_r + 1} \mu_{r:u-1:n,b}^{(R_1, \dots, R_{r-1}, (R_r + R_{r+1} + 1), R_{r+2}, \dots, R_u)^{(i+2\alpha)}} \\ - \frac{(n - R_{k+1} - \dots - R_{r-1} - r + 1)}{R_r + 1} \mu_{r-1:u-1:n,b}^{(R_1, \dots, R_{r-2}, (R_{r-1} + R_r + 1), R_{r+1}, \dots, R_u)^{(i+2\alpha)}}. \quad (24)$$

Now by substituting for $\mu_{r:u:n,b}^{(R_1, R_2, \dots, R_u)^{(i+\alpha)}$ and $\mu_{r:u:n,b}^{(R_1, R_2, \dots, R_u)^{(i+2\alpha)}$ by substituting Equations (23) and (24) into Equation (19), we obtain:

$$(\theta + \beta) \mu_{r:u:n,b}^{(R_1, R_2, \dots, R_u)^{(i)}} + \theta \beta \mu_{r:u:n,b}^{(R_1, R_2, \dots, R_u)^{(i+\alpha)}} = \\ A\alpha\theta^2 b^u \iint \dots \int_{0 < x_1 < \dots < x_u < \infty} x_r^i (x_r^{\alpha-1} + \beta x_r^{2\alpha-1}) [\bar{F}(x_r)]^{R_r+1} f(x_1) [\bar{F}(x_{k+1})]^{R_1} \dots \\ \times f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_u.$$

We get

$$Ab^u \iint \dots \int_{0 < x_1 < \dots < x_m < \infty} x_r^i (\theta + \beta + \theta \beta x_r^\alpha) f(x_r) [\bar{F}(x_r)]^{R_r} \times \\ f(x_1) [\bar{F}(x_1)]^{R_1} \dots f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_u \\ = A\alpha\theta^2 b^u \iint \dots \int_{0 < x_1 < \dots < x_m < \infty} x_r^i (x_r^{\alpha-1} + \beta x_r^{2\alpha-1}) [\bar{F}(x_r)]^{R_r+1} \\ \times f(x_1) [\bar{F}(x_1)]^{R_1} \dots f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots \\ \times f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_u. \quad (25)$$

We get

$$Ab^u \iint \dots \int_{0 < x_1 < \dots < x_u < \infty} [(\theta + \beta + \theta \beta x_r^\alpha) f(x_r) - \alpha\theta^2 (x_r^{\alpha-1} + \beta x_r^{2\alpha-1}) [\bar{F}(x_r)]] \\ \times x_r^i [\bar{F}(x_r)]^{R_r} f(x_1) [\bar{F}(x_1)]^{R_1} \dots f(x_{r-1}) [\bar{F}(x_{r-1})]^{R_{r-1}} \\ \times f(x_{r+1}) [\bar{F}(x_{r+1})]^{R_{r+1}} \dots f(x_u) [\bar{F}(x_u)]^{R_u} dx_1 \dots dx_u = 0.$$

By applying the Muntz-Szasz theorem (see Hwang and Lin [14]), we obtain:

$$(\theta + \beta + \theta \beta x_r^\alpha) f(x_r) = \alpha\theta^2 (x_r^{\alpha-1} + \beta x_r^{2\alpha-1}) [\bar{F}(x_r)].$$

Using theorem 4, we get the SF of $u \leq n$ and $i \geq 0$, EPLD.

$$\bar{F}(x) = \left[1 + \frac{\theta\beta}{\theta + \beta} x^\alpha \right] e^{-\theta x^\alpha}.$$

This completes the proof.

3.3. Characterization via RRs for PMs

In this subsection we characterize the EPLD using RRs for PMs based on PFFCD

Theorem 6. Let $X_{1:u:n,b} \leq X_{2:u:n,b} \leq \dots \leq X_{n:u:n,b}$ be the order statistics for a random sample of size n . Then X has EPLD iff, for $1 \leq r < s \leq u-1, u \leq n$ and $i, j \geq 0$,

$$\mu_{r,s:u:n,b}^{(R_1, \dots, R_u)^{(i+2\alpha, j)}} = \frac{(i+2\alpha)(\theta + \beta)}{\beta\alpha\theta^2(R_r + 1)} \mu_{r,s:u:n,b}^{(R_1, \dots, R_u)^{(i, j)}} \\ + \frac{(i+2\alpha)}{\alpha\theta(R_r + 1)} \mu_{r,s:u:n,b}^{(R_1, \dots, R_u)^{(i+\alpha, j)}} - \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r,s:u:n,b}^{(R_1, R_2, \dots, R_u)^{(i+\alpha, j)}}$$

$$\begin{aligned}
& + \frac{(n - k_1 - \dots - k_{r-1} - r + 1)}{(R_r + 1)} \left[\mu_{r-1, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1} + R_r - 1), R_{r+1}, \dots, R_u)^{(i+2\alpha, j)}} \right. \\
& + \frac{i + 2\alpha}{\beta(i + \alpha)} \mu_{r-1, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{r-2}, (R_{r-1} + R_r - 1), R_{r+1}, \dots, R_u)^{(i+\alpha, j)}} \left. \right] \\
& - \frac{(n - k_1 - \dots - k_r - r)}{(R_r + 1)} \left[\mu_{r, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{r-1}, (R_r + R_{r+1} - 1), R_{r+2}, \dots, R_u)^{(i+2\alpha, j)}} \right. \\
& \quad \left. + \frac{i + 2\alpha}{\beta(i + \alpha)} \mu_{r, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{r-1}, (R_r + R_{r+1} - 1), R_{r+2}, \dots, R_u)^{(i+\alpha, j)}} \right]. \quad (26)
\end{aligned}$$

Proof of Theorem 6.

Necessity: The necessary condition of this theorem was established in theorem 2.

Sufficiency: Following the proof in theorem 5, we derive the SF of the EPLD, given by:

$$\bar{F}(x) = \left[1 + \frac{\theta\beta}{\theta + \beta} x^\alpha \right] e^{-\theta x^\alpha}.$$

This concludes the proof.

Theorem 7. Let $X_{1:n:n,b} \leq X_{2:n:n,b} \leq \dots \leq X_{n:n:n,b}$ be the order statistics for a random sample of size n . Then X has EPLD iff, for $1 \leq r < s \leq u - 1, u \leq n$ and $i, j \geq 0$,

$$\begin{aligned}
\mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i, j+2\alpha)}} &= \frac{(j + 2\alpha)(\theta + \beta)}{\beta\alpha\theta^2(R_s + 1)} \mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i, j)}} \\
& + \frac{(j + 2\alpha)}{\alpha\theta(R_s + 1)} \mu_{r, s: u: n, b}^{(R_1, \dots, R_u)^{(i, j+\alpha)}} - \frac{j + 2\alpha}{\beta(j + \alpha)} \mu_{r, s: u: n, b}^{(R_1, R_2, \dots, R_u)^{(i, j+\alpha)}} \\
& + \frac{(n - k_1 - \dots - k_{s-1} - s + 1)}{(R_s + 1)} \left[\mu_{r, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{s-2}, (R_{s-1} + R_s - 1), R_{s+1}, \dots, R_u)^{(i, j+2\alpha)}} \right. \\
& + \frac{j + 2\alpha}{\beta(j + \alpha)} \mu_{r, s-1: u-1: n, b}^{(R_1, R_2, \dots, R_{s-2}, (R_{s-1} + R_s - 1), R_{s+1}, \dots, R_u)^{(i, j+\alpha)}} \left. \right] \\
& - \frac{(n - k_1 - \dots - k_s - s)}{(R_s + 1)} \left[\mu_{r, s: u-1: n, b}^{(R_1, R_2, \dots, R_{s-1}, (R_s + R_{s+1} - 1), R_{s+2}, \dots, R_u)^{(i, j+2\alpha)}} \right. \\
& + \frac{j + 2\alpha}{\beta(j + \alpha)} \mu_{r, s: u-1: n, b}^{(R_1, R_2, \dots, R_{s-1}, (R_s + R_{s+1} - 1), R_{s+2}, \dots, R_u)^{(i, j+\alpha)}} \left. \right]. \quad (27)
\end{aligned}$$

Necessity: theorem 7 proved the necessary part of this theorem the necessary condition of this theorem was established in theorem 3.

Sufficiency: Following the proof in theorem 5, we derive the SF of the EPLD, given by:

$$\bar{F}(x) = \left[1 + \frac{\theta\beta}{\theta + \beta} x^\alpha \right] e^{-\theta x^\alpha}.$$

This concludes the proof.

Author contributions: Ali M. Sharawy: Methodology, validation, resources, writing original draft preparation, writing review and editing, analysis. All authors have read and approved the final version of the manuscript for publication.

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