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Strip-Analytic Abu-Ghuwaleh Transforms: Continuous Dilation-Convolution, Wiener–Mellin Inversion, and Stable Log-Scale Recovery

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Abstract

We develop the strip-analytic sequel to the master-integral-transform program with entire kernels by replacing the discrete Taylor-spectrum model with a continuous spectral model on the dilation side. The central object is a Hardy-strip orbit kernel whose boundary representation induces a continuous dilation-convolution operator acting on the Fourier transform of a weighted signal. In this setting, the Abu-Ghuwaleh transform admits two complementary inversion mechanisms: Mellin contour inversion and contour-free Wiener–Mellin inversion. We prove exact factorization formulas on named weighted function spaces, derive branchwise Mellin diagonalization formulas, obtain inversion theorems under nonvanishing assumptions on the continuous symbol, and show that logarithmic coordinates convert the transform into an additive convolution equation. This yields a practical FFT-based inversion framework together with a stability bound on frequency windows away from zeros of the multiplier. We also prove an explicit injectivity-and-stability proposition for a resolvent-type kernel family with Gamma-type symbol. The paper is designed as the natural continuous-spectrum successor to the entire-kernel and finite-Laurent stages of the program.

Keywords: strip-analytic integral transforms; continuous dilation-convolution; Mellin transform; Wiener-Mellin inversion; log-scale Fourier inversion; Hardy-strip kernels; exact inversion; harmonic analysis; stability

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1. Introduction

The master integral transform with entire kernels established a broad transform calculus in which an entire kernel generates a family of oscillatory integral transforms together with completeness, injectivity, and Mellin–Fourier inversion under density hypotheses on the nonzero Taylor indices [1]. A natural first sequel is the finite-principal-part Laurent theory, where the discrete spectral set becomes two-sided. The next conceptual step is different in character: one passes from a discrete spectral model to a continuous one.

The purpose of the present paper is to develop that continuous sequel. We introduce a strip-analytic version of the Abu-Ghuwaleh transform (AGT) in which the relevant kernel is described not by a Taylor or Laurent expansion, but by a boundary-frequency representation along the AGT orbit. Under Hardy-strip assumptions, the forward transform becomes a continuous dilation-convolution operator acting on the Fourier transform of a weighted signal. In other words, the discrete sums

$$\sum_{k\geq 1} a(k)F(k\theta)$$

of the entire and Laurent stages are replaced by continuous multiplicative averages of the form

$$\int_0^\infty \kappa(\xi)F(\xi\theta)\,\frac{d\xi}{\xi}.$$

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This is the correct continuous counterpart of the earlier theory.

Two complementary inversion mechanisms then emerge. The first is a Mellin-symbol calculus: the forward operator is diagonalized by the Mellin transform, and inversion is achieved by division by a continuous symbol. The second is a contour-free Wiener–Mellin theory: after passing to logarithmic coordinates, the dilation equation becomes an additive convolution equation, so inversion may be recovered by Wiener-type arguments and implemented by FFT-based numerical methods. The paper shows that these viewpoints are not rivals; they are different factorizations of the same operator.

The novelty relative to the 2025 MIT paper is therefore structural rather than ornamental. The MIT paper already handles the broad entire-kernel umbrella with BM-density-based completeness and injectivity, explicit Mellin–Fourier inversion, and structural transform laws for arbitrary entire kernels of finite order [1]. The present paper does not revisit that umbrella with new examples. Instead, it develops the first genuinely continuous-spectrum sequel to the program. Its main contribution is to identify the correct strip-analytic orbit model, prove the exact continuous dilation-convolution factorization, establish Wiener–Mellin inversion, and provide a practical stability statement for the log-scale inversion.

A second point deserves emphasis. The paper is intentionally written at the level of a first rigorous continuous-spectrum theorem package, not as the most general imaginable strip theory. The kernel class is chosen so that the boundary representation is explicit, the exchanges of integration order are justified by named hypotheses, and the inversion mechanisms are exact rather than merely formal. This deliberate restraint gives the paper its mathematical bite.

Main results

The main results may be summarized as follows.

(1) We define a Hardy-strip orbit class of kernels and prove that the associated AGT is a continuous dilation-convolution operator on the Fourier side.
(2) We derive branchwise Mellin diagonalization formulas and the corresponding Mellin contour inversions.
(3) We prove a contour-free Wiener–Mellin inversion theorem by passing to logarithmic coordinates and invoking the Wiener lemma on the additive side.
(4) We derive the log-scale Fourier multiplier formula and a practical stability estimate away from zeros of the multiplier.
(5) We prove an explicit injectivity-and-stability proposition for a resolvent-type strip kernel family with Gamma-type symbol.

Roadmap of the Paper

Section 2 fixes the weighted signal spaces and the basic AGT notation. Section 3 introduces the Hardy-strip orbit kernels and proves the continuous factorization theorem. Section 4 establishes branchwise Mellin diagonalization and contour inversion formulas. Section 5 develops the Wiener–Mellin inversion framework. Section 6 derives the log-scale Fourier multiplier representation and a stability bound away from zeros. Section 7 presents explicit kernel families and an application proposition with Gamma-type symbol. Section 8 records a compact numerical experiment blueprint. The final sections discuss the meaning of the continuous-spectrum sequel inside the larger post-MIT research path.

2. Preliminaries and Weighted Signal Spaces

Fix $p \in (-1, 1)$ and parameters $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$. For a measurable function $f : \mathbb{R} \to \mathbb{C}$, define the weighted signal

$$\Phi(x) := |x|^p f(x) e^{-i\frac{\pi p}{2}\operatorname{sgn} x}. \tag{2.1}$$

Its Fourier transform is

$$F(\omega) := \widehat{\Phi}(\omega) = \int_{\mathbb{R}} \Phi(x) e^{i\omega x}\, dx,$$

whenever the integral is classically defined.

We use the following named spaces.

Definition 2.1 (Signal spaces)**.** *Let*

$$X_p := \{f : \mathbb{R} \to \mathbb{C} : \ |x|^p f(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}.$$

For Mellin inversion we use the subclass $X_p^{\mathrm{M}} \subset X_p$ consisting of functions for which the weighted even and odd parts satisfy

$$\Phi_e, \Phi_o \in L^1(0, \infty; x^{-c}\, dx) \cap L^1(0, \infty; x^{-c'}\, dx)$$

for some $c < c'$, so that their Mellin transforms are holomorphic in a vertical strip. For logarithmic Fourier inversion we use the class $X_p^{\log} \subset X_p$ for which the profile

$$g_0(t) := F(e^t)$$

belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Remark 2.2. *The spaces X_p^{M} and $X_p^{\log}$ are working spaces introduced to pin down exactly where Mellin and log-Fourier diagonalizations are justified. They are not claimed to be optimal.*

Definition 2.3 (Base AGT)**.** *Let g be a kernel defined on a neighborhood of the AGT orbit. The Abu-Ghuwaleh transform associated with (α, β, p, g) is*

$$\mathcal{G}_{\alpha,\beta}^{(p)}[f](\theta) := \frac{1}{2\pi} \int_{\mathbb{R}} |x|^p f(x) e^{-i\frac{\pi p}{2}\operatorname{sgn} x} g\big(\alpha + \beta e^{i\theta x}\big)\, dx = \frac{1}{2\pi} \int_{\mathbb{R}} \Phi(x) g\big(\alpha + \beta e^{i\theta x}\big)\, dx. \tag{2.2}$$

In the strip-analytic theory we restrict to $\theta > 0$, which is the natural scale variable for Mellin analysis.

3. Hardy-Strip Orbit Kernels and Continuous Factorization

The continuous sequel is best formulated in terms of the orbit profile

$$H(\tau) := g\big(\alpha + \beta e^{i\tau}\big), \qquad \tau \in \mathbb{R}.$$

Rather than assuming a Taylor or Laurent expansion for g, we assume a boundary-frequency representation for H.

Definition 3.1 (Hardy-strip orbit kernel)**.** *Let $\sigma > 0$. A kernel g is called* strip-admissible along the AGT orbit *if there exists a measurable function*

$$\kappa : (0, \infty) \to \mathbb{C}$$

such that

$$\int_0^\infty |\kappa(\xi)| e^{\sigma \xi} \frac{d\xi}{\xi} < \infty \tag{3.1}$$

and

$$H(\tau) = g\big(\alpha + \beta e^{i\tau}\big) = \int_0^\infty \kappa(\xi) e^{i\xi\tau} \frac{d\xi}{\xi} \qquad (\tau \in \mathbb{R}). \tag{3.2}$$

The exponential moment condition implies that the integral in (3.2) converges absolutely and defines a holomorphic extension of H to the strip

$$\Sigma_\sigma := \{\tau \in \mathbb{C} : |\Im \tau| < \sigma\}.$$

Remark 3.2. *This formulation is deliberately conservative. The point is not to characterize every possible strip-analytic kernel, but to work in a class in which the continuous-spectrum representation is explicit and all exchanges of integral order are justified.*

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Theorem 3.3 (Continuous dilation-convolution factorization). *Let $f \in X_p$ and let g be strip-admissible with boundary density κ as in (3.2). Then for every $\theta > 0$,*

$$\mathcal{G}(\theta) = \frac{1}{2\pi} \int_0^\infty \kappa(\xi) F(\xi\theta) \frac{d\xi}{\xi}. \tag{3.3}$$

Equivalently,

$$\mathcal{G} = \mathcal{T}_\kappa F, \qquad (\mathcal{T}_\kappa F)(\theta) := \frac{1}{2\pi} \int_0^\infty \kappa(\xi) F(\xi\theta) \frac{d\xi}{\xi}. \tag{3.4}$$

The integral converges absolutely and locally uniformly in $\theta > 0$.

Proof. For fixed $\theta > 0$ and $x \in \mathbb{R}$, the representation (3.2) gives

$$|\Phi(x) H(\theta x)| \le |\Phi(x)| \int_0^\infty |\kappa(\xi)| \frac{d\xi}{\xi}.$$

Since $\Phi \in L^1(\mathbb{R})$ and (3.1) implies $\kappa \in L^1((0,\infty), d\xi/\xi)$, Tonelli's theorem permits interchange of the x- and ξ-integrals in (2.2):

$$\mathcal{G}(\theta) = \frac{1}{2\pi} \int_0^\infty \kappa(\xi) \left(\int_{\mathbb{R}} \Phi(x) e^{i\xi\theta x} \, dx \right) \frac{d\xi}{\xi}.$$

The inner integral is $F(\xi\theta)$, proving (3.3). Local uniform convergence follows from the bound

$$\frac{1}{2\pi} \int_0^\infty |\kappa(\xi)| |F(\xi\theta)| \frac{d\xi}{\xi} \le \frac{\|\Phi\|_{L^1}}{2\pi} \int_0^\infty |\kappa(\xi)| \frac{d\xi}{\xi}.$$

□

Remark 3.4. *This theorem is the continuous analogue of the discrete AGT formulas from the entire and Laurent stages. The discrete coefficient sequence is replaced by a boundary density on the multiplicative scale variable.*

4. Branchwise Mellin Diagonalization and Contour Inversion

As in the discrete theory, parity separation clarifies the inversion formulas. Write

$$\Phi = \Phi_e + \Phi_o, \qquad \Phi_e(x) = \frac{\Phi(x) + \Phi(-x)}{2}, \qquad \Phi_o(x) = \frac{\Phi(x) - \Phi(-x)}{2}.$$

Then

$$F(\omega) = 2 \int_0^\infty \Phi_e(x) \cos(\omega x) \, dx + 2i \int_0^\infty \Phi_o(x) \sin(\omega x) \, dx.$$

Define the continuous Mellin symbol

$$m(s) := \int_0^\infty \kappa(\xi) \xi^{-s} \frac{d\xi}{\xi}, \tag{4.1}$$

whenever the integral converges.

Theorem 4.1 (Mellin diagonalization). *Assume the hypotheses of Theorem 3.3. Let $f \in X_p^{\mathrm{M}}$, and suppose there exists a strip $a < \Re s < b$ such that the symbol (4.1) converges absolutely there. Then, for every s with $a < \Re s < b$,*

$$\mathcal{M}_\theta\{\mathcal{G}^c\}(s) = \frac{\Gamma(s)\cos(\pi s/2)}{\pi} m(s) \, \mathcal{M}\{\Phi_e\}(1-s), \tag{4.2}$$

$$\mathcal{M}_\theta\{\mathcal{G}^s\}(s) = \frac{\Gamma(s)\sin(\pi s/2)}{i\pi} m(s) \, \mathcal{M}\{\Phi_o\}(1-s). \tag{4.3}$$

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Proof. Fix s with $a < \Re s < b$. By the assumptions on f and on the symbol strip, all exchanges of integral order are justified by Fubini–Tonelli. Starting from (3.3) and using parity decomposition, one obtains the cosine and sine branch formulas after the change of variables $u = \xi\theta x$ together with the classical Mellin integrals

$$\int_0^\infty u^{s-1}\cos u\,du = \Gamma(s)\cos(\pi s/2), \qquad \int_0^\infty u^{s-1}\sin u\,du = \Gamma(s)\sin(\pi s/2).$$

□

Theorem 4.2 (Mellin contour inversion)**.** *Assume the hypotheses of Theorem 4.1. Suppose there exists $c \in (a,b)$ such that $m(1-u) \neq 0$ on the line $\Re u = c$ and such that the branch quotients belong to $L^1(\mathbb{R}_v)$ along that line. Then*

$$\Phi_e(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} |x|^{-u}\frac{\pi\,\mathcal{M}_\theta\{\mathcal{G}^c\}(1-u)}{\Gamma(1-u)\sin(\pi u/2)m(1-u)}\,du, \tag{4.4}$$

$$\Phi_o(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} |x|^{-u}\frac{i\pi\,\mathcal{M}_\theta\{\mathcal{G}^s\}(1-u)}{\Gamma(1-u)\cos(\pi u/2)m(1-u)}\,du. \tag{4.5}$$

Consequently,

$$f(x) = e^{i\frac{\pi p}{2}\operatorname{sgn} x}|x|^{-p}\big(\Phi_e(x) + \Phi_o(x)\big) \tag{4.6}$$

for almost every $x \in \mathbb{R}\setminus\{0\}$.

Proof. Solve (4.2) and (4.3) for the Mellin transforms of Φ_e and Φ_o, then apply Mellin inversion. The factor $e^{i\frac{\pi p}{2}\operatorname{sgn} x}|x|^{-p}$ comes from (2.1). □

5. Wiener–Mellin Inversion

The continuous factorization admits a contour-free inversion theory when rewritten in logarithmic coordinates.

Set

$$\theta = e^t, \qquad h(t) := \mathcal{G}(e^t), \qquad g_0(t) := F(e^t), \qquad q(u) := \kappa(e^{-u}). \tag{5.1}$$

Then (3.3) becomes the additive convolution equation

$$h(t) = \frac{1}{2\pi}\int_{\mathbb{R}} q(u)g_0(t-u)\,du. \tag{5.2}$$

Thus the multiplicative AGT equation becomes a standard convolution equation on the additive line.

Theorem 5.1 (Log-scale Fourier multiplier)**.** *Assume the hypotheses of Theorem 3.3 and let $f \in X_p^{\log}$. Then $h, g_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the identity (5.2) holds in $L^1 \cap L^2$, and Fourier transformation in t yields*

$$\widehat{h}(\eta) = A(\eta)\widehat{g_0}(\eta), \qquad A(\eta) := \frac{1}{2\pi}\int_0^\infty \kappa(\xi)\xi^{-i\eta}\frac{d\xi}{\xi} = \frac{1}{2\pi}m(i\eta). \tag{5.3}$$

If $1/A \in L^\infty(\mathbb{R})$, then

$$g_0(t) = \mathcal{F}_t^{-1}\left[\frac{\widehat{h}(\eta)}{A(\eta)}\right](t), \qquad F(e^t) = g_0(t). \tag{5.4}$$

Proof. Since (3.1) implies $q \in L^1(\mathbb{R})$ and $g_0 \in L^1 \cap L^2$, the convolution in (5.2) is well-defined in $L^1 \cap L^2$. Fourier transformation gives

$$\widehat{h}(\eta) = \frac{1}{2\pi}\widehat{q}(\eta)\widehat{g_0}(\eta).$$

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A change of variables $\xi = e^{-u}$ shows that

$$\widehat{q}(\eta) = \int_{\mathbb{R}} \kappa(e^{-u})e^{i\eta u}\, du = \int_0^\infty \kappa(\xi)\xi^{-i\eta}\, \frac{d\xi}{\xi} = m(i\eta),$$

which yields (5.3). Division by $A(\eta)$ and inverse Fourier transformation give (5.4). □

Theorem 5.2 (Wiener–Mellin inversion). *Assume the hypotheses of Theorem 5.1. Suppose in addition that $q \in L^1(\mathbb{R})$ and that its Fourier transform does not vanish:*

$$\widehat{q}(\eta) \neq 0 \qquad \text{for all } \eta \in \mathbb{R}.$$

Then there exists $\ell \in L^1(\mathbb{R})$ such that

$$\widehat{\ell}(\eta) = \frac{2\pi}{\widehat{q}(\eta)},$$

and the inverse of the continuous AGT is given by

$$F(\theta) = \int_0^\infty \nu(\xi)\, \mathcal{G}(\theta/\xi)\, \frac{d\xi}{\xi}, \qquad \nu(\xi) := \ell(\log \xi). \tag{5.5}$$

Consequently,

$$\Phi = \mathcal{F}^{-1}[F], \qquad f(x) = e^{i\frac{\pi p}{2}\operatorname{sgn} x}|x|^{-p}\Phi(x)$$

for almost every $x \neq 0$.

Proof. By Wiener's theorem on $L^1(\mathbb{R})$ there exists $\ell \in L^1(\mathbb{R})$ with Fourier transform equal to $2\pi/\widehat{q}$. Since (5.2) is convolution-type on the additive side, one has

$$g_0(t) = \int_{\mathbb{R}} \ell(u)h(t-u)\, du.$$

Returning to the multiplicative variable $\theta = e^t$ and writing $\xi = e^u$ gives exactly (5.5). □

Remark 5.3. *This theorem is the continuous analogue of the discrete Dirichlet–Wiener inversion from the earlier AGT stages. The arithmetic inverse is replaced by the Wiener inverse of the logarithmic kernel.*

6. Stability Away from Zeros of the Multiplier

The log-scale multiplier formulation makes the location of numerical ill-conditioning transparent.

Proposition 6.1 (Practical stability on frequency windows). *Assume the hypotheses of Theorem 5.1. Let $I \subset \mathbb{R}$ be measurable and suppose there exists $\eta_0 > 0$ such that*

$$|A(\eta)| \geq \eta_0 \qquad \text{for a.e. } \eta \in I.$$

For exact data $\widehat{h}$ and noisy data $\widehat{h}_\delta$, define the frequency-windowed reconstructions

$$g_{0,I} := \mathcal{F}_t^{-1}\left[\mathbf{1}_I \frac{\widehat{h}}{A}\right], \qquad g_{0,I}^{\delta} := \mathcal{F}_t^{-1}\left[\mathbf{1}_I \frac{\widehat{h}_\delta}{A}\right].$$

Then

$$\|g_{0,I}^{\delta} - g_{0,I}\|_{L^2(\mathbb{R})} \leq \eta_0^{-1}\, \|\mathbf{1}_I(\widehat{h}_\delta - \widehat{h})\|_{L^2(\mathbb{R})}. \tag{6.1}$$

In particular, the log-scale inversion is L^2-stable on any frequency window on which the multiplier is bounded away from zero.

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Proof. By Plancherel's theorem,

$$\|g_{0,I}^{\delta} - g_{0,I}\|_{L^2} = \left\| \mathbf{1}_I \frac{\widehat{h}_\delta - \widehat{h}}{A} \right\|_{L^2} \le \left\| \frac{\mathbf{1}_I}{A} \right\|_{L^\infty} \|\mathbf{1}_I(\widehat{h}_\delta - \widehat{h})\|_{L^2} \le \eta_0^{-1} \|\mathbf{1}_I(\widehat{h}_\delta - \widehat{h})\|_{L^2}.$$

□

Remark 6.2. *This proposition isolates the practical numerical message: stable recovery occurs on windows where the multiplier stays away from zero, while ill-conditioning is concentrated near its small values.*

7. Kernel Families and a Sharper Application

We record explicit model kernels illustrating the theory.

Example 7.1 (Resolvent-type strip kernel). *Let*

$$\kappa(\xi) = \xi e^{-\xi}, \qquad \xi > 0.$$

Then

$$H(\tau) = \int_0^\infty \xi e^{-\xi} e^{i\xi\tau} \frac{d\xi}{\xi} = \int_0^\infty e^{-(1-i\tau)\xi}\, d\xi = \frac{1}{1-i\tau}.$$

Hence the orbit kernel extends holomorphically to the strip $|\Im\tau| < 1$. The corresponding AGT is

$$\mathcal{G}(\theta) = \frac{1}{2\pi} \int_0^\infty e^{-\xi} F(\xi\theta)\, d\xi.$$

The Mellin symbol is explicitly

$$m(s) = \int_0^\infty \xi e^{-\xi} \xi^{-s} \frac{d\xi}{\xi} = \Gamma(1-s), \qquad \Re s < 1.$$

Since the Gamma function has no zeros, this model is globally nondegenerate at the symbol level.

Example 7.2 (Exponential cutoff family). *Let*

$$\kappa_{a,b}(\xi) = \xi^a e^{-b\xi}, \qquad a > 0,\ b > 0.$$

Then

$$H_{a,b}(\tau) = \int_0^\infty \xi^a e^{-b\xi} e^{i\xi\tau} \frac{d\xi}{\xi} = \Gamma(a)(b - i\tau)^{-a},$$

so the strip width is controlled by b. The corresponding continuous symbol is

$$m_{a,b}(s) = \int_0^\infty \xi^a e^{-b\xi} \xi^{-s} \frac{d\xi}{\xi} = b^{s-a}\Gamma(a-s), \qquad \Re s < a.$$

The log-scale multiplier is therefore

$$A_{a,b}(\eta) = \frac{1}{2\pi} b^{i\eta - a} \Gamma(a - i\eta).$$

Proposition 7.3 (Explicit injectivity and stability for the Gamma family). *Fix $a, b > 0$ and let $\kappa_{a,b}$ be as in Example 7.2. Then the associated strip-analytic AGT has the following properties.*

(a) *For every vertical line $\Re u = c$ with $1 - c < a$, the Mellin symbol $m_{a,b}(1-u)$ has no zeros on that line. Hence the Mellin contour inversion is injective on the corresponding class X_p^{M}.*

(b) *For every $R > 0$, the log-scale inversion on the window $I_R = [-R, R]$ is L^2-stable with constant*

$$C_{a,b,R} := \frac{2\pi b^a}{\min_{|\eta|\le R} |\Gamma(a - i\eta)|}. \tag{7.1}$$

That is,

$$\|g^{\delta}_{0,I_R} - g_{0,I_R}\|_{L^2(\mathbb{R})} \le C_{a,b,R}\, \|\mathbf{1}_{I_R}(\widehat{h}_\delta - \widehat{h})\|_{L^2(\mathbb{R})}. \tag{7.2}$$

Proof. Because the Gamma function has no zeros in the complex plane, the symbol

$$m_{a,b}(1-u) = b^{1-u-a}\Gamma(a-1+u)$$

is nonzero on every vertical line contained in the half-plane $\Re(1-u) < a$. This proves part (a). For part (b),

$$|A_{a,b}(\eta)| = \frac{b^{-a}}{2\pi}|\Gamma(a - i\eta)|,$$

so on the compact window I_R one has

$$|A_{a,b}(\eta)| \ge \frac{1}{2\pi b^a} \min_{|\eta|\le R} |\Gamma(a - i\eta)| > 0.$$

Applying Proposition 6.1 yields (7.2) with the constant (7.1). □

Example 7.4 (Worked symbolic model with closed-form forward transform)**.** *Take the weighted signal*

$$\Phi(x) = e^{-x^2} \qquad \text{so that} \qquad F(\omega) = \widehat{\Phi}(\omega) = \sqrt{\pi}\, e^{-\omega^2/4},$$

and the resolvent-type kernel from Example 7.1. Then

$$\mathcal{G}(\theta) = \frac{1}{2\pi}\int_0^\infty e^{-\xi}\sqrt{\pi}\, e^{-\xi^2\theta^2/4}\, d\xi. \tag{7.3}$$

Using the standard Gaussian integral

$$\int_0^\infty e^{-a\xi^2 - b\xi}\, d\xi = \frac{\sqrt{\pi}}{2\sqrt{a}} \exp\left(\frac{b^2}{4a}\right) \operatorname{erfc}\left(\frac{b}{2\sqrt{a}}\right), \qquad a, b > 0,$$

one obtains the explicit formula

$$\mathcal{G}(\theta) = \frac{1}{2\theta} e^{1/\theta^2} \operatorname{erfc}(1/\theta), \qquad \theta > 0. \tag{7.4}$$

Thus the forward SAGT can be written in closed form, while the inverse is controlled by the multiplier

$$A(\eta) = \frac{1}{2\pi}\Gamma(1 - i\eta).$$

This model is especially useful for numerical testing because both the exact forward map and the multiplier are explicit.

8. A Mini Numerical Experiment Blueprint

We record a compact numerical thought-experiment based on the worked model above.

Take the weighted signal $\Phi(x) = e^{-x^2}$ and the strip kernel with density $\kappa(\xi) = \xi e^{-\xi}$. Then the exact forward data are given by (7.4). Passing to logarithmic scale yields

$$h(t) = \mathcal{G}(e^t), \qquad g_0(t) = F(e^t) = \sqrt{\pi}\, e^{-e^{2t}/4}.$$

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A practical inversion proceeds as follows.

(1) Sample $h(t_j)$ on a uniform grid $t_j = t_{\min} + j\Delta t$.
(2) Compute an FFT approximation of $\widehat{h}(\eta)$.
(3) Divide by the exact multiplier

$$A(\eta) = \frac{1}{2\pi}\Gamma(1 - i\eta).$$

(4) Apply the inverse FFT to recover an approximation of $g_0(t_j)$.
(5) Reconstruct $F(e^{t_j})$ and then recover f by inverse Fourier inversion of F.

Because the Gamma function has no zeros, this model avoids multiplier singularities and isolates the pure discretization and truncation errors of the log-scale algorithm. It therefore provides a clean first benchmark for the continuous SAGT inversion theory.

9. Discussion and Position in the Research Program

The present paper should be read as the continuous-spectrum sequel to the discrete AGT stages. The entire-kernel theory is governed by discrete Taylor indices. The finite-Laurent theory produces a two-sided discrete spectrum. The strip-analytic theory developed here replaces those discrete spectra by a continuous multiplicative frequency model.

The merit of this step is not merely that it enlarges the class of kernels. It changes the inversion architecture. In the discrete papers, one may speak of Dirichlet symbols and arithmetic inverses. In the present paper, the natural objects are the continuous Mellin symbol and the Wiener inverse of the logarithmic kernel. This is the correct analytic counterpart of the earlier theory.

The paper is also deliberately conservative. It does not attempt to solve the full general strip-analytic problem without explicit orbit representation, nor does it treat operator-valued or multivariate strip kernels. Those directions are natural next steps, but the point here is to pin down the first rigorous continuous-spectrum theorem package before the theory is widened further.

10. Future Directions

The next steps in the program are now clear.

(1) Develop an operator-valued SAGT in which the continuous symbol takes values in $\mathcal{B}(\mathcal{H})$ and inversion is controlled by operator-valued Wiener and Mellin criteria.
(2) Study multivariate strip-analytic AGT, where the frequency variable becomes vector-valued and anisotropic Mellin geometry enters.
(3) Relax the explicit orbit-representation hypothesis and move toward a more intrinsic reproducing-kernel or de Branges-type formulation of the continuous image space.
(4) Develop a full noisy-data regularization theory for windowed log-scale inversion near multiplier zeros.

11. Conclusions

We have developed the strip-analytic successor to the master-integral-transform theory with entire kernels. The central result is that, under Hardy-strip orbit assumptions, the Abu-Ghuwaleh transform is a continuous dilation-convolution operator on the Fourier side. This single observation produces two complementary inversion theories: Mellin contour inversion and contour-free Wiener–Mellin inversion. In logarithmic coordinates the transform becomes an additive convolution equation, which leads naturally to Fourier-multiplier analysis, FFT-based recovery, and practical stability estimates away from multiplier zeros.

The principal message may be compressed into one line:

$$\boxed{\mathcal{G} = \mathcal{T}_\kappa(\widehat{\Phi}), \qquad \text{and inversion means inverting the continuous Mellin/log-Fourier symbol of } \mathcal{T}_\kappa.}$$

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This is the correct continuous-spectrum sequel to the discrete AGT stages and, in that sense, the natural next paper in the post-MIT research path.

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