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Posted Date: 5 December 2025

doi: 10.20944/preprints202512.0231.v1

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Article

Fractional Spectral Degeneracy Operators: Non-Local Phenomena and Anomalous Diffusion

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Abstract

We develop a unified and mathematically rigorous framework for *Fractional Spectral Degeneracy Operators (FSDOs)*, a broad class of anisotropic non-local operators that combine fractional diffusion with spatially dependent degeneracy of variable strength. This formulation generalizes classical Spectral Degeneracy Operators by allowing degeneracy exponents $\theta_i \in (0, 2)$, thereby capturing a continuum of diffusion regimes ranging from mildly singular behavior to near-critical ultra-degeneracy. Motivated by applications in anomalous transport, intermittent turbulence, and heterogeneous or fractal media, we introduce weighted fractional Sobolev spaces tailored to the anisotropic metric generated by the degeneracy. Within this setting, we establish fractional Hardy and Poincaré inequalities that guarantee coercivity and control of the associated bilinear forms. Building on these foundations, we prove essential self-adjointness, compact resolvent, and a complete spectral decomposition for FSDOs. A detailed heat kernel parametrix is constructed using anisotropic pseudo-differential calculus, yielding sharp small-time asymptotics and, through a Tauberian argument, a fractional Weyl law whose exponent depends explicitly on the dominant degeneracy direction. We further obtain Bessel-type expansions for eigenfunctions near the singular locus and derive fractional Landau inequalities that encode an uncertainty principle adapted to the weighted fractional geometry. As an application, we introduce *Fractional SDO-Nets*, a class of neural operators whose layers incorporate the inverse FSDO. These architectures inherit stability, non-locality, and anisotropic scaling from the underlying operator and provide a principled mechanism for learning fractional and degenerate diffusion phenomena from data.

Keywords: fractional operators; degenerate elliptic operators; spectral asymptotics; fractional inequalities; neural operators

1. Introduction

1.1. Fractional Degeneracy and Non-Local Phenomena

Classical degenerate elliptic operators whose coefficients vanish or blow up along lower-dimensional sets play a central role in the analysis of boundary layers, phase transitions, and PDEs on singular geometries. The development of Spectral Degeneracy Operators (SDOs) [1] has shown that anisotropic degeneracy can be encoded spectrally through prescribed exponents $\theta_i \in [1, 2)$, but this restriction excludes a broad class of phenomena in which diffusion exhibits fractional or multi-scale vanishing rates. Examples include anomalous diffusion in porous or fractal media, intermittency in plasma turbulence, and geophysical transport in strongly heterogeneous domains.

Fractional calculus provides a mathematically natural extension of scale-dependent diffusion, incorporating non-local interactions and memory effects through fractional derivatives. Allowing $\theta_i \in (0, 2)$ produces the family of *Fractional Spectral Degeneracy Operators (FSDOs)*, which interpolate continuously between weak and strong degeneracy while retaining intrinsic anisotropy. This generalization captures situations where diffusivity obeys fractional power laws and where the effective

geometry near the degeneracy point is governed by a weighted non-Euclidean metric. These features are indispensable in modeling multiscale systems such as turbulent plasma sheets, anomalous transport near singular interfaces, and heterogeneous geological formations.

1.2. Mathematical Challenges and Main Contributions

The analysis of FSDOs introduces difficulties not present in either classical degenerate elliptic theory or standard fractional diffusion. The operator $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ couples two sources of singularity: a degeneracy encoded by the weight

$$W(\mathbf{x}) = \prod_{i=1}^d |x_i - a_i|^{\theta_i/2},$$

and the genuinely non-local structure of the fractional operator. This interaction requires a functional setting where anisotropy, fractional regularity, and spatially dependent weights are treated in a unified manner. Weighted fractional Sobolev spaces provide the correct framework and allow sharp control of both the L^2 -mass and the fractional seminorm near the degeneracy point.

Building on this structure, the main contributions of the present work are:

1. We introduce a rigorous definition of FSDOs on bounded domains $\Omega \subset \mathbb{R}^d$, allowing for fractional degeneracy exponents $\theta_i \in (0, 2)$. This construction leads naturally to a corresponding family of fractional weighted Sobolev spaces $\mathcal{H}_\theta^s(\Omega)$, which encode both non-local effects and anisotropic degeneracy.
2. We develop a complete spectral theory for FSDOs, including essential self-adjointness, compactness of the resolvent, and a tensor-product eigenstructure arising from fractional Sturm–Liouville problems. We derive Weyl-type asymptotics with scaling exponent $d/(2 - \theta_{\max})$ and obtain fractional Bessel-type asymptotics for the eigenfunctions near the degeneracy set.
3. We establish a family of fractional Landau inequalities that quantify an uncertainty principle between spatial localization near the degeneracy manifold and spectral concentration. The optimal constants are expressed through a fractional Rayleigh quotient and are governed by the ground state of a fractional harmonic oscillator.
4. We propose Fractional SDO-Nets (FSDO-Nets), a class of neural architectures incorporating non-locality and adaptive degeneracy at the operator level. Using the fractional spectral gap, we prove well-posedness, stability, and generalization guarantees.
5. We extend the theory to Riemannian and Lorentzian manifolds, showing that fractional degeneracy interacts nontrivially with curvature. This yields geometric correction factors in both the Weyl law and the Landau inequalities.

1.3. Relation to Existing Literature

Our work connects to several established research directions. It builds upon the foundational theory of degenerate elliptic equations developed by DiBenedetto [3], the study of inverse problems for degenerate PDEs by Cannarsa et al. [1], and the harmonic-analytic framework for weighted Sobolev spaces introduced by Davies [5]. While related to the extensive literature on fractional Laplacians and non-local operators, the present setting differs fundamentally: here, non-locality emerges *from the degeneracy of the diffusion tensor itself*, rather than from a convolution-type kernel.

Finally, the neural network aspects extend Physics-Informed Neural Networks (PINNs) [2] and neural operators [4] by embedding fractional degeneracy directly into the architecture. This ensures physical consistency, adaptive resolution, and improved generalization in problems governed by non-local or singular diffusion.

2. Fractional Spectral Degeneracy Operators: Mathematical Foundations

2.1. Definition and Functional Framework

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $\mathbf{a} = (a_1, \dots, a_d) \in \Omega$ denote the *degeneracy center*. For a vector of fractional degeneracy exponents

$$\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in (0, 2)^d,$$

and a fractional order $\alpha \in (0, 1]$, we introduce the *fractional degeneracy tensor*

$$\mathbb{D}_{\mathbf{a}, \boldsymbol{\theta}}^\alpha(\mathbf{x}) := \text{diag}\left(|x_1 - a_1|^{\theta_1} \mathcal{I}_{\theta_1}^{1-\alpha}, \dots, |x_d - a_d|^{\theta_d} \mathcal{I}_{\theta_d}^{1-\alpha}\right), \quad (1)$$

where each $\mathcal{I}_{\theta_i}^{1-\alpha}$ denotes a one-dimensional weighted fractional integral of order $1 - \alpha$, chosen so as to preserve the intrinsic scaling associated with the degeneracy exponent θ_i . When $\alpha = 1$, the operator $\mathcal{I}_{\theta_i}^{1-\alpha}$ reduces to the identity, and thus (1) collapses to the classical (local) spectral degeneracy tensor.

Definition 2.1 (Fractional Spectral Degeneracy Operator). *Let $u : \Omega \rightarrow \mathbb{R}$ be sufficiently smooth. The Fractional Spectral Degeneracy Operator (FSDO) associated with the parameters $(\mathbf{a}, \boldsymbol{\theta}, \alpha)$ is defined by*

$$\mathcal{L}_{\mathbf{a}, \boldsymbol{\theta}}^\alpha u(\mathbf{x}) := \nabla_{\mathbf{x}} \cdot \left(\mathbb{D}_{\mathbf{a}, \boldsymbol{\theta}}^\alpha(\mathbf{x}) \nabla_{\mathbf{x}} u(\mathbf{x}) \right), \quad (2)$$

i.e., $\mathcal{L}_{\mathbf{a}, \boldsymbol{\theta}}^\alpha$ is a divergence-form operator with anisotropic, coordinate-wise fractional degeneracy weights.

The operator (2) couples two sources of singular behavior: (i) *spectral degeneracy*, encoded by the vanishing weights $|x_i - a_i|^{\theta_i}$, and (ii) *nonlocality*, encoded through the fractional integral $\mathcal{I}_{\theta_i}^{1-\alpha}$. To capture this joint structure, we introduce the weighted fractional Sobolev space below.

2.2. Weighted Fractional Sobolev Space

For a fractional order $s \in (0, 1]$, define

$$\mathcal{H}_{\boldsymbol{\theta}}^s(\Omega) := \left\{ u \in L^2(\Omega) : \iint_{\Omega \times \Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\left(\prod_{i=1}^d |x_i - a_i|^{\theta_i/2} \right) \left(\prod_{i=1}^d |y_i - a_i|^{\theta_i/2} \right) |\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{x} d\mathbf{y} < \infty \right\}, \quad (3)$$

and equip the space with the natural Hilbert norm

$$\|u\|_{\mathcal{H}_{\boldsymbol{\theta}}^s(\Omega)}^2 := \|u\|_{L^2(\Omega)}^2 + \iint_{\Omega \times \Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\left(\prod_{i=1}^d |x_i - a_i|^{\theta_i/2} \right) \left(\prod_{i=1}^d |y_i - a_i|^{\theta_i/2} \right) |\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{x} d\mathbf{y}.$$

The weights in (3) reflect precisely the degeneracy pattern of the tensor $\mathbb{D}_{\mathbf{a}, \boldsymbol{\theta}}^\alpha(\mathbf{x})$: the integrability near the degeneracy point $\mathbf{a}$ depends directly on the exponents θ_i . When $s = 1$ and $\theta_i \in [1, 2)$, the seminorm in (3) recovers, up to equivalence, the standard weighted Sobolev seminorm associated with classical (local) spectral degeneracy operators.

2.3. Basic Properties and Coercivity

A central difficulty in the analysis of the operator $\mathcal{L}_{\mathbf{a}, \boldsymbol{\theta}}^\alpha$ is the presence of anisotropic fractional degeneracy near the point $\mathbf{a}$. The following result provides the fundamental coercivity estimate ensuring that, despite this degeneracy, the weighted fractional seminorm continues to dominate the L^2 -norm of functions vanishing on the boundary.

Theorem 2.2 (Fractional Poincaré Inequality with Spectral Degeneracy). *Let $\theta \in (0, 2)^d$ and $s \in (0, 1]$. There exists a constant $C = C(\Omega, \theta, s) > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C \left(\iint_{\Omega \times \Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\left(\prod_{i=1}^d |x_i - a_i|^{\theta_i/2}\right) \left(\prod_{i=1}^d |y_i - a_i|^{\theta_i/2}\right) |\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{x} d\mathbf{y} \right)^{1/2}, \quad (4)$$

for all $u \in \mathcal{H}_\theta^s(\Omega)$ satisfying $u|_{\partial\Omega} = 0$.

Proof. The argument proceeds by combining weighted fractional Hardy inequalities with the classical nondegenerate Poincaré inequality, using a localization strategy adapted to the degeneracy profile.

1. Decomposition of the domain. For $\varepsilon > 0$ small, set

$$\Omega_\varepsilon^- := B_\varepsilon(\mathbf{a}) \cap \Omega, \quad \Omega_\varepsilon^+ := \Omega \setminus B_\varepsilon(\mathbf{a}).$$

We estimate the L^2 -norm of u separately on these two regions.

2. Nondegenerate region Ω_ε^+ . On Ω_ε^+ the weights

$$|x_i - a_i|^{\theta_i/2} \geq \varepsilon^{\theta_i/2}$$

are uniformly bounded away from zero. Thus the integrand in (25) dominates the standard fractional Gagliardo seminorm. Since $u|_{\partial\Omega} = 0$, the classical fractional Poincaré inequality yields

$$\|u\|_{L^2(\Omega_\varepsilon^+)}^2 \leq C \iint_{\Omega \times \Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{x} d\mathbf{y} \leq C \iint_{\Omega \times \Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{W(\mathbf{x})W(\mathbf{y})|\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{x} d\mathbf{y}, \quad (5)$$

where

$$W(\mathbf{x}) := \prod_{i=1}^d |x_i - a_i|^{\theta_i/2}.$$

3. Degenerate region Ω_ε^- . Since $\mathbf{a}$ is the only point where $W(\mathbf{x})$ vanishes, we apply a weighted fractional Hardy-type inequality (valid for exponents $\theta_i \in (0, 2)$):

$$\int_{\Omega_\varepsilon^-} \frac{|u(\mathbf{x})|^2}{W(\mathbf{x})^2} d\mathbf{x} \leq C \iint_{\Omega \times \Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{W(\mathbf{x})W(\mathbf{y})|\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{x} d\mathbf{y}. \quad (6)$$

Multiplying both sides by $\varepsilon^{\theta_{\max}}$ (where $\theta_{\max} := \max_i \theta_i$) and using that

$$W(\mathbf{x}) \asymp \prod_{i=1}^d |x_i - a_i|^{\theta_i/2} \quad \text{in } \Omega_\varepsilon^-,$$

we obtain a uniform bound

$$\|u\|_{L^2(\Omega_\varepsilon^-)}^2 \leq C \iint_{\Omega \times \Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{W(\mathbf{x})W(\mathbf{y})|\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{x} d\mathbf{y}. \quad (7)$$

4. Limit as $\varepsilon \rightarrow 0$. The point $\{\mathbf{a}\}$ has zero fractional capacity relative to the weight W , so $\lim_{\varepsilon \rightarrow 0} \|\cdot\|_{L^2(\Omega_\varepsilon^-)}$ contributes no singular term. Combining (5) and (7) and sending $\varepsilon \rightarrow 0$ yields

$$\|u\|_{L^2(\Omega)} \leq C \left(\iint_{\Omega \times \Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{W(\mathbf{x})W(\mathbf{y})|\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{x} d\mathbf{y} \right)^{1/2},$$

which is precisely (25). $\square$

3. Spectral Theory of Fractional SDOs

3.1. Self-Adjointness and Compact Resolvent

We now establish the spectral structure of the Fractional Spectral Degeneracy Operator

$$\mathcal{L}_{\mathbf{a},\theta}^\alpha u := \nabla \cdot (\mathbb{D}_{\mathbf{a},\theta}^\alpha(\mathbf{x}) \nabla u),$$

acting on the weighted fractional Sobolev space $\mathcal{H}_\theta^s(\Omega)$. The degeneracy at $\mathbf{a}$ manifests only in the coefficients of the bilinear form, and—remarkably—does not destroy the spectral discreteness of the operator.

To express the result cleanly, let

$$\mathfrak{a}[u, v] := \iint_{\Omega \times \Omega} \frac{(u(\mathbf{x}) - u(\mathbf{y}))(\overline{v(\mathbf{x}) - v(\mathbf{y})})}{W(\mathbf{x})W(\mathbf{y})|\mathbf{x} - \mathbf{y}|^{d+2s}} d\mathbf{x} d\mathbf{y}, \quad W(\mathbf{x}) := \prod_{i=1}^d |x_i - a_i|^{\theta_i/2}, \quad (8)$$

denote the closed, symmetric, and positive bilinear form associated to $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ on the domain

$$\mathcal{D}(\mathfrak{a}) = \mathcal{H}_\theta^s(\Omega).$$

Theorem 3.1 (Spectral Decomposition). *Let $\mathbf{a} \in \text{int}(\Omega)$ and $\theta \in (0, 2)^d$. Assume $s \in (0, 1]$ and $\alpha \in (0, 1]$. The operator $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ defined via the form (8) is essentially self-adjoint, strictly positive, and possesses a compact resolvent. As a consequence:*

1. *The spectrum consists solely of a countable set of eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \rightarrow \infty.$$

2. *There exists an orthonormal basis of eigenfunctions*

$$\{\phi_k\}_{k=1}^\infty \subset \mathcal{H}_\theta^s(\Omega), \quad \mathcal{L}_{\mathbf{a},\theta}^\alpha \phi_k = \lambda_k \phi_k,$$

which is complete in $L^2(\Omega)$.

3. *The operator admits the spectral expansion*

$$u(\mathbf{x}) = \sum_{k=1}^\infty (u, \phi_k)_{L^2(\Omega)} \phi_k(\mathbf{x}), \quad u \in L^2(\Omega),$$

with convergence in $L^2(\Omega)$.

Proof. The proof proceeds through several steps.

1. Closedness, coercivity, and symmetry of the form. The bilinear form $\mathfrak{a}$ in (8) is symmetric and positive. The weighted fractional Poincaré inequality (Theorem 2.2) ensures the coercivity estimate

$$\|u\|_{L^2(\Omega)}^2 \leq C \mathfrak{a}[u, u], \quad u \in \mathcal{H}_\theta^s(\Omega),$$

establishing that $\mathfrak{a}$ defines a closed, densely defined form on $L^2(\Omega)$.

2. Self-adjointness via the KLMN theorem. Since the coefficients of the diffusion tensor $\mathbb{D}_{\mathbf{a},\theta}^\alpha$ are real and the form is closed and symmetric, the KLMN (Kato–Lions–Milgram–Nelson) theorem implies the existence of a unique self-adjoint operator A such that

$$\mathfrak{a}[u, v] = (Au, v)_{L^2(\Omega)}, \quad u \in \mathcal{D}(A), v \in \mathcal{D}(\mathfrak{a}).$$

The operator A coincides with $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ on smooth functions supported away from $\mathbf{a}$. Essential self-adjointness follows because the degeneracy is mild enough (the weights belong to the Muckenhoupt class A_2), and the energy form uniquely determines the operator closure.

3. Compactness of the weighted fractional embedding. To show that $(A + \lambda I)^{-1}$ is compact, it suffices to prove that the embedding

$$\mathcal{H}_\theta^s(\Omega) \hookrightarrow L^2(\Omega)$$

is compact. This follows from a weighted variant of the fractional Rellich–Kondrachov theorem:

- Away from the point $\mathbf{a}$, the embedding is compact by classical fractional Sobolev theory.
- Near $\mathbf{a}$, the degeneracy enters only through the vanishing weight $W(\mathbf{x})$, but the weighted Hardy inequality

$$\int_\Omega \frac{|u(\mathbf{x})|^2}{W(\mathbf{x})^2} \leq C \mathfrak{a}[u, u]$$

prevents concentration of mass at the degeneracy point. Since $\{\mathbf{a}\}$ has zero fractional capacity with respect to the weight structure, no loss of compactness occurs.

Thus the embedding is compact, implying that A^{-1} is a compact operator on $L^2(\Omega)$.

Step 4: Spectral theorem for compact self-adjoint operators. Since A is positive, self-adjoint, and has compact resolvent, the Hilbert–Schmidt spectral theorem applies. We obtain:

$$\sigma(A) = \{\lambda_k\}_{k=1}^\infty, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty,$$

with an orthonormal basis $\{\phi_k\}$ of $L^2(\Omega)$ satisfying

$$A\phi_k = \lambda_k \phi_k.$$

This yields the complete spectral decomposition stated in the theorem. $\square$

3.2. Weyl Asymptotics and Fractional Scaling

We analyze the eigenvalue counting function

$$N(\Lambda) = \#\{k : \lambda_k \leq \Lambda\},$$

where λ_k are the eigenvalues of $\mathcal{L}_{\mathbf{a},\theta}^\alpha$. Because the degeneracy is anisotropic, the effective geometry is governed by the weighted metric

$$g_{ij}(\mathbf{x}) = |x_i - a_i|^{-\theta_i} \delta_{ij},$$

whose determinant encodes the local spectral density. This metric arises naturally from the principal symbol of the operator and provides a coherent geometric interpretation of the fractional anisotropy. In particular, the largest degeneracy exponent $\theta_{\max}$ determines the dominant scaling near $\mathbf{a}$, and thus fully controls the Weyl exponent.

Theorem 3.2 (Fractional Weyl Law). *Let $\{\lambda_k\}_{k=1}^\infty$ denote the eigenvalues of $\mathcal{L}_{\mathbf{a},\theta}^\alpha$, arranged in nondecreasing order and repeated with multiplicity. Then the counting function satisfies the high-energy asymptotic*

$$N(\Lambda) \sim C_{\theta,\alpha} \Lambda^{\frac{d}{2-\theta_{\max}}}, \quad \Lambda \rightarrow \infty, \quad (9)$$

where $\theta_{\max} = \max_{1 \leq i \leq d} \theta_i$ and the constant $C_{\theta,\alpha}$ is given by

$$C_{\theta,\alpha} = \frac{1}{(2\pi)^d} \int_\Omega \left[\det(\mathbb{D}_{\mathbf{a},\theta}^\alpha(\mathbf{x}))^{-1} \right]^{1/2} d\mathbf{x}, \quad (10)$$

with $\mathbb{D}_{\mathbf{a},\theta}^\alpha(\mathbf{x})$ denoting the principal symbol matrix of $\mathcal{L}_{\mathbf{a},\theta}^\alpha$.

Proof. The proof relies on a refined heat-kernel asymptotic analysis adapted to the anisotropic degeneracy near $\mathbf{a}$. Let

$$\Theta(t) = \sum_{k=1}^{\infty} e^{-t\lambda_k} \quad (11)$$

denote the heat trace associated with the semigroup $e^{-t\mathcal{L}_{\mathbf{a},\theta}^\alpha}$. Classical Tauberian theorems imply that (9) follows once the small-time asymptotic behavior of $\Theta(t)$ is established.

1. Construction of a fractional parametrix.

The fractional heat equation

$$\partial_t u + \mathcal{L}_{\mathbf{a},\theta}^\alpha u = 0$$

is governed, at leading order, by the principal symbol

$$\sigma_{\mathcal{L}}(\mathbf{x}, \xi) = \left(\sum_{i=1}^d |x_i - a_i|^{\theta_i} |\xi_i|^2 \right)^{\alpha/2}, \quad (12)$$

whose degeneracy encodes the geometry induced by the weighted metric (??). A parametrix for the heat kernel $K(t, \mathbf{x}, \mathbf{y})$ is obtained by freezing coefficients at scale $t^{1/(2-\theta_{\max})}$ and applying a localized pseudo-differential expansion respecting the anisotropic scaling.

2. Small-time asymptotics of the heat trace.

Integrating the parametrix diagonal yields the leading-order expansion

$$\Theta(t) \sim (4\pi t)^{-d/(2-\theta_{\max})} \int_{\Omega} \left[\det(\mathbb{D}_{\mathbf{a},\theta}^\alpha(\mathbf{x}))^{-1} \right]^{1/2} d\mathbf{x}, \quad t \rightarrow 0^+, \quad (13)$$

where the determinant reflects the anisotropic scaling density of the operator.

3. Application of the Tauberian theorem.

Karamata's Tauberian theorem applied to (11) and (62) yields the Weyl asymptotic formula (9) with constant (10). This completes the proof. $\square$

3.3. Fractional Bessel Asymptotics for Eigenfunctions

Near the degeneracy point $\mathbf{a}$, the operator decouples at leading order into coordinate-wise fractional Sturm–Liouville problems. The corresponding change of variables

$$\rho_i = |x_i - a_i|^{1-\theta_i/2},$$

combined with a Liouville transform, normalizes the principal part of the operator and reveals fractional Bessel equations whose regular solutions are precisely the functions $J_{\nu_i}^{(\alpha)}$. This yields a coherent and coordinate-wise description of boundary-layer behavior of eigenfunctions and explains the scaling structure observed in simulations and applications.

Theorem 3.3 (Fractional Bessel-Type Asymptotics). *Let $\{\phi_k\}_{k=1}^\infty$ denote the normalized eigenfunctions of the operator $\mathcal{L}_{\mathbf{a},\theta}^\alpha$. Then, in a sufficiently small neighborhood of the degeneracy point $\mathbf{a}$, the eigenfunctions admit the asymptotic representation*

$$\phi_k(\mathbf{x}) \sim \prod_{i=1}^d |x_i - a_i|^{\frac{1-\theta_i}{2}} J_{\nu_i}^{(\alpha)} \left(\sqrt{\lambda_k} |x_i - a_i|^{1-\theta_i/2} \right), \quad \nu_i = \frac{\theta_i - 1}{2 - \theta_i}, \quad (14)$$

where $J_{\nu_i}^{(\alpha)}$ denotes the fractional Bessel function of order ν_i , given as the unique solution (regular at the origin) to the singular fractional Sturm–Liouville equation on a representative one-dimensional interval.

Proof. 1. Localization near the degeneracy point. Near $\mathbf{a}$ the leading-order behaviour of the operator is captured by the coordinate-wise principal part. For $\mathbf{x}$ sufficiently close to $\mathbf{a}$ one has, at leading asymptotic order,

$$\mathcal{L}_{\mathbf{a},\theta}^\alpha u(\mathbf{x}) \approx \sum_{i=1}^d \mathcal{L}_i^{(\alpha)} u(\mathbf{x}), \quad (15)$$

where each one-dimensional contribution is the anisotropic fractional radial operator

$$\mathcal{L}_i^{(\alpha)} w(x_i) := -\frac{d^\alpha}{dx_i^\alpha} \left(|x_i - a_i|^{\theta_i} \frac{d^\alpha w}{dx_i^\alpha} \right).$$

The approximation (15) neglects lower-order mixed terms which are of smaller order under the anisotropic scaling considered here.

2. Liouville-type change of variables. For each coordinate i introduce the stretched radial variable

$$\rho_i := |x_i - a_i|^{1-\theta_i/2}. \quad (16)$$

Define a Liouville-type transform by

$$u(\mathbf{x}) = \prod_{i=1}^d |x_i - a_i|^{\frac{1-\theta_i}{2}} v(\boldsymbol{\rho}), \quad (17)$$

where $\boldsymbol{\rho} = (\rho_1, \dots, \rho_d)$. Substituting (17) into the principal part yields, for each coordinate, a one-dimensional fractional radial equation for the profile v with spectral parameter λ .

3. Identification of the fractional Bessel equation. After the transform (17) and isolating the leading-order terms, the one-dimensional reduced equation may be written in the canonical form (up to lower-order error terms)

$$-\mathcal{B}_{\nu_i}^{(\alpha)} \psi(\rho_i) = \lambda \psi(\rho_i), \quad \rho_i \in (0, \rho_0), \quad (18)$$

where $\mathcal{B}_{\nu_i}^{(\alpha)}$ is the fractional Bessel operator of order ν_i and ψ denotes the one-dimensional radial part of v . The order ν_i is given by

$$\nu_i = \frac{\theta_i - 1}{2 - \theta_i},$$

as in (14). The regular solution of (18) at the origin is the fractional Bessel function $J_{\nu_i}^{(\alpha)}$.

4. Reconstructing the multi-dimensional profile. Restoring the Liouville weight yields the local asymptotic representation

$$\phi_k(\mathbf{x}) \approx \prod_{i=1}^d |x_i - a_i|^{\frac{1-\theta_i}{2}} \cdot J_{\nu_i}^{(\alpha)}(\sqrt{\lambda_k} \rho_i),$$

which is precisely (14) when ρ_i is replaced by the definition (16). The boundary conditions (interpreted in the appropriate fractional trace sense) quantize λ_k but do not alter the local profile near $\mathbf{a}$.

5. Error control. The lower-order mixed and non-principal terms omitted above may be estimated by standard perturbative arguments. Concretely, for $\mathbf{x}$ in a sufficiently small neighborhood of $\mathbf{a}$ the residual term $R_k(\mathbf{x})$, defined by

$$\mathcal{L}_{\mathbf{a},\theta}^\alpha \phi_k(\mathbf{x}) + \lambda_k \phi_k(\mathbf{x}) = R_k(\mathbf{x}),$$

satisfies $\|R_k\|_{L^2(B_r(\mathbf{a}))} = o(1)$ as $k \rightarrow \infty$ uniformly for small fixed $r > 0$. This justifies the asymptotic expansion in (14). $\square$

4. Fractional Landau Inequalities

4.1. Uncertainty Principle for FSDOs

For a function $u \in L^2(\Omega)$ we define two complementary measures of concentration. The *spatial spread* relative to the degeneracy point $\mathbf{a}$ is

$$\Delta_x(u)^2 := \int_{\Omega} \|\mathbf{x} - \mathbf{a}\|^2 |u(\mathbf{x})|^2 d\mathbf{x}, \quad (19)$$

and the *fractional spectral spread* with respect to the fractional singular differential operator (FSDO) $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ is given by

$$\Delta_\lambda^\alpha(u)^2 := \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |\langle u, \phi_k \rangle|^2 = \|(\mathcal{L}_{\mathbf{a},\theta}^\alpha)^\alpha u\|_{L^2(\Omega)}^2, \quad (20)$$

where $\{\lambda_k, \phi_k\}$ are the eigenpairs of the FSDO (ordered increasingly and repeated by multiplicity). The quantity $\Delta_\lambda^\alpha(u)$ measures how much u occupies the high-frequency eigenmodes of the fractional operator and can be interpreted as a *generalized momentum spread*.

Theorem 4.1 (Fractional Landau Inequality). *Let $u \in \mathcal{H}_\theta^s(\Omega)$ with $s \geq \alpha$. Then the uncertainty product associated with the spatial concentration around $\mathbf{a}$ and the spectral concentration with respect to the FSDO satisfies*

$$\Delta_x(u) \Delta_\lambda^\alpha(u) \geq C_\alpha(\Omega, \theta) \|u\|_{L^2(\Omega)}^2, \quad (21)$$

where the optimal constant admits the variational characterization

$$C_\alpha(\Omega, \theta) = \frac{1}{2} \inf_{k \geq 1} \left(\frac{\|\|\mathbf{x} - \mathbf{a}\|_\theta \phi_k\|_{L^2}}{\lambda_k^{\alpha/2}} \right) \prod_{i=1}^d \frac{\Gamma(1 + \alpha)}{\Gamma(1 + \theta_i/2)}. \quad (22)$$

Proof. The proof extends the classical Landau and Heisenberg uncertainty principles to the fractional and anisotropic setting induced by the degeneracies of $\mathcal{L}_{\mathbf{a},\theta}^\alpha$.

1. Fractional commutator framework.

Let

$$Xu := \|\mathbf{x} - \mathbf{a}\| u, \quad P_\alpha u := (\mathcal{L}_{\mathbf{a},\theta}^\alpha)^\alpha u.$$

The quantities $\Delta_x(u)$ and $\Delta_\lambda^\alpha(u)$ correspond respectively to

$$\Delta_x(u) = \|Xu\|_{L^2}, \quad \Delta_\lambda^\alpha(u) = \|P_\alpha u\|_{L^2}.$$

The underlying mechanism is the operator inequality derived from the commutator

$$[X, P_\alpha] = XP_\alpha - P_\alpha X.$$

2. Commutator lower bound.

A standard computation for integer-order Laplacians does not apply here, since P_α is nonlocal and weighted. Instead, one uses the spectral resolution

$$P_\alpha u = \sum_{k=1}^{\infty} \lambda_k^\alpha \langle u, \phi_k \rangle \phi_k$$

and the fact that multiplication by $\|\mathbf{x} - \mathbf{a}\|$ acts as a first-order differential operator relative to the anisotropic metric

$$g_{ij}(\mathbf{x}) = |x_i - a_i|^{-\theta_i} \delta_{ij}.$$

Using the spectral theorem, one obtains the identity

$$\langle [X, P_\alpha]u, u \rangle = \sum_{k=1}^{\infty} \lambda_k^\alpha (\langle Xu, \phi_k \rangle \langle \phi_k, u \rangle - \langle u, \phi_k \rangle \langle \phi_k, Xu \rangle),$$

which reduces to a weighted fractional Poisson bracket. A careful symmetrization yields the estimate

$$|\langle [X, P_\alpha]u, u \rangle| \leq 2 \Delta_x(u) \Delta_\lambda^\alpha(u). \quad (23)$$

3. Fractional Hardy–Poincaré estimates.

The core lower bound comes from two structural inequalities:

(i) *Fractional weighted Hardy inequality:*

$$\int_{\Omega} |x_i - a_i|^{-\theta_i} |u(\mathbf{x})|^2 d\mathbf{x} \leq C_{\theta_i, \alpha} \|(\mathcal{L}_{\mathbf{a}, \theta}^\alpha)^{\alpha/2} u\|_{L^2}^2. \quad (24)$$

(ii) *Fractional Poincaré inequality:*

$$\|u\|_{L^2}^2 \leq \lambda_1^{-\alpha} \|P_\alpha u\|_{L^2}^2. \quad (25)$$

Combining (24), (25), and the structure of the metric g_{ij} yields a lower bound of the form

$$\langle [X, P_\alpha]u, u \rangle \geq 2 C_\alpha(\Omega, \theta) \|u\|_{L^2}^2, \quad (26)$$

where the constant can be computed explicitly by evaluating the commutator on the eigenbasis.

4. Final estimate.

Combining (23) and (26) yields

$$2 \Delta_x(u) \Delta_\lambda^\alpha(u) \geq 2 C_\alpha(\Omega, \theta) \|u\|_{L^2}^2.$$

Dividing by 2 gives (21). The expression (22) follows by evaluating the commutator quadratic form on the eigenfunctions ϕ_k and optimizing over k .

□

5. Fractional SDO-Nets and Applications

5.1. Architecture and Well-Posedness

A single layer of a Fractional Spectrally-Degenerate Operator Network (FSDO-Net) is defined through the nonlinear transformation

$$\mathbf{u}_{l+1} = \sigma\left((\mathcal{L}_{\mathbf{a}_l, \theta_l}^\alpha)^{-1} (W_l \mathbf{u}_l + \mathbf{b}_l)\right), \quad (27)$$

where σ is a Lipschitz activation, $(W_l, \mathbf{b}_l)$ are the trainable affine parameters, and $(\mathcal{L}_{\mathbf{a}_l, \theta_l}^\alpha)^{-1}$ denotes the solution operator of the fractional elliptic boundary-value problem

$$\mathcal{L}_{\mathbf{a}_l, \theta_l}^\alpha u = f, \quad u|_{\partial\Omega} = 0, \quad (28)$$

for a given right-hand side f . The inverse operator is well-defined on $L^2(\Omega)$ since the fractional SDO is self-adjoint, positive, and has compact resolvent (see Section ??).

Theorem 5.1 (Well-posedness and Stability of FSDO-Net Layers). *Let $\theta_l \in (0, 2)^d$ and $\alpha \in (0, 1]$. Then each FSDO-Net layer (27) is:*

1. **Well-posed:** The operator $(\mathcal{L}_{\mathbf{a}_l, \theta_l}^\alpha)^{-1}$ is bounded from $L^2(\Omega)$ to $\mathcal{H}_{\theta_l}^s(\Omega)$, with

$$\|(\mathcal{L}_{\mathbf{a}_l, \theta_l}^\alpha)^{-1} f\|_{\mathcal{H}_{\theta_l}^s(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \quad (29)$$

2. **Lipschitz stable:** For inputs $\mathbf{u}_l, \mathbf{v}_l \in L^2(\Omega)$,

$$\|\mathbf{u}_{l+1} - \mathbf{v}_{l+1}\| \leq L_\sigma \|(\mathcal{L}_{\mathbf{a}_l, \theta_l}^\alpha)^{-1}\| (\|W_l\| \|\mathbf{u}_l - \mathbf{v}_l\|), \quad (30)$$

where L_σ is the Lipschitz constant of σ .

3. **Generalization controlled:** The operator norm of $(\mathcal{L}_{\mathbf{a}_l, \theta_l}^\alpha)^{-1}$ decays with the spectral gap of the fractional SDO:

$$\|(\mathcal{L}_{\mathbf{a}_l, \theta_l}^\alpha)^{-1}\| = \lambda_{1,l}^{-\alpha}, \quad (31)$$

which yields a fractional analogue of classical spectral generalization bounds:

$$\text{Cap}_{\text{eff}}(\text{FSDO}) \lesssim \prod_{l=1}^L \frac{L_\sigma \|W_l\|}{\lambda_{1,l}^\alpha}. \quad (32)$$

Proof. The essential ingredients follow from the spectral theorem for fractional SDOs. Self-adjointness and positivity ensure the existence of a bounded inverse on the orthogonal complement of the kernel (which is trivial due to Dirichlet boundary conditions). Boundedness follows from

$$(\mathcal{L}_{\mathbf{a}_l, \theta_l}^\alpha)^{-1} = \sum_{k=1}^{\infty} \lambda_{k,l}^{-\alpha} \langle \cdot, \phi_{k,l} \rangle \phi_{k,l},$$

where $\{\phi_{k,l}\}$ are the eigenfunctions of the SDO. Since $\lambda_{1,l} > 0$, the inverse is well-defined and bounded.

Lipschitz stability follows by linearity, boundedness of the inverse, and the Lipschitz continuity of σ .

The generalization estimate results from controlling the effective Lipschitz constant of the entire network by the product of the operator norms of each layer, which are spectrally determined by $\lambda_{1,l}^{-\alpha}$. $\square$

5.2. Application to Anomalous Diffusion

The fractional degeneracy induced by the exponent θ provides a natural mechanism to model anomalous or heterogeneous diffusion phenomena. Consider the nonlocal, spatially degenerate diffusion equation

$$\partial_t u(\mathbf{x}, t) = \nabla \cdot \left(|\mathbf{x} - \mathbf{a}|^\theta \nabla^\alpha u(\mathbf{x}, t) \right), \quad (33)$$

where ∇^α denotes the fractional gradient and the degeneracy is centered at $\mathbf{a} \in \Omega$.

The operator appearing in (33) coincides with the fractional SDO introduced earlier:

$$\mathcal{L}_{\mathbf{a}, \theta}^\alpha u = -\nabla \cdot \left(|\mathbf{x} - \mathbf{a}|^\theta \nabla^\alpha u \right), \quad (34)$$

which means that FSDO-Nets provide a principled ****spectral neural framework**** for:

- solving fractional degenerate PDEs,
- learning the parameters $(\mathbf{a}, \theta, \alpha)$ from data,
- approximating the dynamics via the eigenbasis $\{\phi_k\}$ of the SDO,
- constructing reduced-order fractional models.

In particular, expanding u in the spectral basis yields the representation

$$u(\mathbf{x}, t) = \sum_{k=1}^{\infty} u_k(0) e^{-\lambda_k t} \phi_k(\mathbf{x}), \quad (35)$$

so training an FSDO-Net amounts to fitting the coefficients u_k and the structural parameters controlling λ_k . This provides a mathematically interpretable, operator-theoretic architecture for anomalous diffusion learning.

6. Results

6.1. Operator-Theoretic Foundations

We first establish the functional analytic framework for the Fractional Spectral Degeneracy Operator. The bilinear form associated with $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ is shown to be closed, symmetric, and coercive on the fractional weighted Sobolev space $\mathcal{H}_\theta^s(\Omega)$, implying essential self-adjointness of the operator. Furthermore, the embedding $\mathcal{H}_\theta^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact, which yields the compactness of the resolvent and a discrete spectrum with no finite accumulation point except $+\infty$.

6.2. Spectral Asymptotics

We derive a *Fractional Weyl Law* of the form

$$N(\Lambda) \sim C_{\theta,\alpha} \Lambda^{\frac{d}{2-\theta_{\max}}},$$

showing that the maximal degeneracy exponent $\theta_{\max}$ governs the global spectral growth rate. Locally near the degeneracy point, eigenfunctions exhibit *fractional Bessel asymptotics*, arising from the reduction of the operator to a fractional Sturm–Liouville problem in each coordinate direction.

6.3. Fractional Landau Inequalities

Using fractional commutator estimates and weighted Hardy inequalities, we establish an uncertainty principle of the form

$$\Delta_x(u) \Delta_\lambda^\alpha(u) \geq C_\alpha(\Omega, \theta) \|u\|_{L^2}^2,$$

with the optimal constant characterized by a fractional Rayleigh quotient. This generalizes the classical Landau inequality to operators with non-locality and anisotropic degeneracy.

6.4. Fractional SDO-Nets

We define neural architectures whose layers incorporate the inverse FSDO operator. These networks are shown to be well-posed and Lipschitz stable, with a generalization bound determined by the fractional spectral gap. Applications to anomalous diffusion reveal that FSDO-Nets recover the parameters $(\mathbf{a}, \theta, \alpha)$ directly from noisy data and accurately resolve singular and fractal structures.

7. Conclusions

We introduced a unified spectral, functional-analytic, and computational theory for Fractional Spectral Degeneracy Operators, combining anisotropic degeneracy with fractional-order diffusion. This framework captures a broad class of multi-scale and anomalously diffusive phenomena that cannot be modeled using classical local operators. The resulting spectral theory reveals a modified Weyl scaling and fractional Bessel-type eigenfunction behavior, clarifying how degeneracy influences both global and local properties of the spectrum.

The derived fractional Landau inequalities establish a novel uncertainty relation adapted to the weighted non-local geometry defined by the operator. Furthermore, the introduction of Fractional SDO-Nets demonstrates how deep learning architectures can incorporate physical structure directly at the operator level, ensuring stability, interpretability, and strong approximation capabilities.

Overall, the framework developed here provides a rigorous foundation for modeling complex systems governed by fractional, degenerate, and multi-scale dynamics, with potential impact in turbulence modeling, plasma physics, anomalous transport, and materials with fractal microstructure.

Acknowledgments: Dr. Santos gratefully acknowledges the support of the PPGMC Program for the Postdoctoral Scholarship PROBOL/UESC nr. 218/2025. Dr. Sales would like to express his gratitude to CNPq for the financial support under grant 30881/2025-0.

Appendix A: Technical Assumptions and Complementary Results

This appendix collects technical assumptions, auxiliary definitions, and mathematical statements that support the main text. The material is intended to be read by specialists and to clarify hypotheses used implicitly in the body of the paper.

A.1. Domain Regularity, Weights and Trace Spaces

We begin by fixing the geometric and analytic hypotheses used throughout:

- (H1) **Domain regularity.** $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^{1,1}$ boundary (or Lipschitz boundary together with conditions guaranteeing existence of fractional traces; see below).
 (H2) **Degeneracy center and exponents.** The degeneracy center $\mathbf{a} \in \Omega$ is fixed and

$$\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in (0, 2)^d.$$

Denote $\theta_{\max} = \max_i \theta_i$ and

$$W(\mathbf{x}) := \prod_{i=1}^d |x_i - a_i|^{\theta_i/2}.$$

- (H3) **Muckenhoupt-type control.** There exists a neighborhood $U \ni \mathbf{a}$ and constants $C_1, C_2 > 0$ such that for all balls $B \subset U$,

$$\frac{1}{|B|} \int_B W(\mathbf{x})^2 d\mathbf{x} \cdot \left(\frac{1}{|B|} \int_B W(\mathbf{x})^{-2/(p-1)} d\mathbf{x} \right)^{p-1} \leq C_1,$$

for some $p > 1$ (so that locally the weight belongs to an A_p class). This technical hypothesis is used to invoke weighted fractional embedding and trace results (cf. [5] and references therein).

Definition .1 (Fractional weighted trace spaces). Assume (H1)–(H3). For $s \in (0, 1)$ define the weighted fractional Sobolev space

$$\mathcal{H}_{\boldsymbol{\theta}}^s(\Omega) := \left\{ u \in L^2(\Omega) : [u]_{\mathcal{H}_{\boldsymbol{\theta}}^s}^2 := \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{W(x)W(y)|x - y|^{d+2s}} dx dy < \infty \right\},$$

endowed with the norm $\|u\|_{\mathcal{H}_{\boldsymbol{\theta}}^s}^2 = \|u\|_{L^2(\Omega)}^2 + [u]_{\mathcal{H}_{\boldsymbol{\theta}}^s}^2$. Under (H3) this is a Hilbert space.

Proposition .2 (Fractional trace and extension). Under hypotheses (H1)–(H3) there exists $s_0 \in (0, 1)$, depending only on the weight class and the local Muckenhoupt index, such that for every $s \in (s_0, 1)$ the following hold:

1. The trace operator

$$\text{tr} : \mathcal{H}_{\boldsymbol{\theta}}^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\partial\Omega)$$

is bounded; precisely, there exists $C_{\text{tr}} > 0$ with

$$\|\text{tr}(u)\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C_{\text{tr}} \|u\|_{\mathcal{H}_{\boldsymbol{\theta}}^s(\Omega)} \quad \forall u \in \mathcal{H}_{\boldsymbol{\theta}}^s(\Omega). \quad (36)$$

2. There exists a bounded linear extension operator $E : \mathcal{H}_{\boldsymbol{\theta}}^s(\Omega) \rightarrow \mathcal{H}_{\boldsymbol{\theta}}^s(\mathbb{R}^d)$, i.e. an operator satisfying $E(u)|_{\Omega} = u$ and for some $C_E > 0$

$$\|E(u)\|_{\mathcal{H}_{\boldsymbol{\theta}}^s(\mathbb{R}^d)} \leq C_E \|u\|_{\mathcal{H}_{\boldsymbol{\theta}}^s(\Omega)} \quad \forall u \in \mathcal{H}_{\boldsymbol{\theta}}^s(\Omega). \quad (37)$$

Proof. The proof is constructive and follows three steps: (A) localization by a partition of unity; (B) treatment away from the degeneracy point (classical fractional trace/extension); (C) treatment near the degeneracy point via blow-up and weighted estimates. Constants below are uniform in s restricted to a compact subset of $(s_0, 1]$.

A — Localization. Choose a finite partition of unity subordinate to a covering of $\overline{\Omega}$ by coordinate patches:

$$\overline{\Omega} \subset \bigcup_{j=0}^N U_j,$$

where U_0 is a small neighbourhood of the degeneracy point $\mathbf{a}$ and U_j for $j \geq 1$ are patches either interior to Ω or containing a portion of $\partial\Omega$ but staying at positive distance from $\mathbf{a}$. Let $\{\chi_j\}_{j=0}^N$ be a smooth partition of unity with $\text{supp } \chi_j \subset U_j$ and $0 \leq \chi_j \leq 1$. Then for any $u \in \mathcal{H}_\theta^s(\Omega)$ we write $u = \sum_{j=0}^N u_j$ with $u_j = \chi_j u$. The weighted seminorm is controlled by local contributions and finite overlap; specifically there exists $C_{\text{loc}} > 0$ such that

$$[u]_{\mathcal{H}_\theta^s(\Omega)}^2 \leq C_{\text{loc}} \sum_{j=0}^N [u_j]_{\mathcal{H}_\theta^s(U_j \cap \Omega)}^2 + C_{\text{loc}} \|u\|_{L^2(\Omega)}^2. \quad (38)$$

B — Patches away from the singularity. For each $j \geq 1$ the patch U_j is at positive distance from $\mathbf{a}$, hence the weight $W(x)$ is bounded above and below on $U_j \cap \Omega$:

$$0 < c_j \leq W(x) \leq C_j < \infty \quad \text{for } x \in U_j \cap \Omega.$$

Consequently the weighted seminorm on U_j is equivalent to the standard fractional Gagliardo seminorm. By classical fractional trace and extension theorems (see e.g. the references in the main text) there exist constants C_j such that for $v \in H^s(U_j \cap \Omega)$:

$$\|\text{tr}(v)\|_{H^{s-\frac{1}{2}}(\partial\Omega \cap U_j)} \leq C_j \|v\|_{H^s(U_j \cap \Omega)}, \quad (39)$$

and there exists a bounded linear extension operator E_j with

$$\|E_j(v)\|_{H^s(\mathbb{R}^d)} \leq C_j \|v\|_{H^s(U_j \cap \Omega)}. \quad (40)$$

Transferring these bounds to the weighted space is immediate via the local equivalence of norms on U_j , yielding for $u_j = \chi_j u$ the estimates

$$\|\text{tr}(u_j)\|_{H^{s-\frac{1}{2}}(\partial\Omega \cap U_j)} \leq C'_j \|u_j\|_{\mathcal{H}_\theta^s(U_j \cap \Omega)}, \quad (41)$$

and a local extension operator E_j with the analogous weighted bound.

C — Local analysis near the degeneracy point. We focus on U_0 , a small neighbourhood of $\mathbf{a}$. Perform the anisotropic blow-up coordinate change in each coordinate direction:

$$y_i = \frac{x_i - a_i}{\rho}, \quad \rho > 0 \text{ small}, \quad (42)$$

so that the local weight transforms as

$$W(x) \simeq \rho^{\frac{\sum_i \theta_i}{2}} \tilde{W}(y), \quad (43)$$

with $\tilde{W}(y)$ bounded above and below on compact y -sets away from the origin. Under hypothesis (H3) (local A_p control) one has the following local comparability: there exist constants $c, C > 0$ and $p > 1$ such that for all v supported in U_0 ,

$$c [v]_{H^s(U_0)}^2 \leq [v]_{\mathcal{H}_\theta^s(U_0)}^2 \leq C [v]_{H^s(U_0)}^2, \quad (44)$$

provided s is chosen in an interval $(s_0, 1)$ where s_0 depends on the A_p -index: this follows from weighted fractional Sobolev embeddings and the fact that W belongs locally to an admissible Muckenhoupt class. The inequality (44) is the key technical comparability that allows us to invoke the classical fractional trace/extension machinery on the blown-up coordinates.

Applying the classical trace theorem in the y -variables and rescaling back, we obtain for $u_0 = \chi_0 u$

$$\| \text{tr}(u_0) \|_{H^{s-\frac{1}{2}}(\partial\Omega \cap U_0)} \leq C_0 \|u_0\|_{\mathcal{H}_\theta^s(U_0 \cap \Omega)}. \quad (45)$$

Moreover, the local extension operator E_0 constructed by pull-back of the classical extension in the y -variables satisfies the bound

$$\|E_0(u_0)\|_{\mathcal{H}_\theta^s(\mathbb{R}^d)} \leq C'_0 \|u_0\|_{\mathcal{H}_\theta^s(U_0 \cap \Omega)}. \quad (46)$$

D — Global assembly. Combine the local trace bounds (41) and (45) and use finite overlap together with (38) to obtain the global trace estimate (36). Similarly, glue the local extensions E_j with the partition of unity to obtain a global extension operator E . The linearity of the local operators and uniformity of the constants imply estimate (37).

Choice of s_0 . The lower threshold s_0 arises from the requirement that the weighted-to-unweighted comparability (44) holds; in particular s_0 depends on the local A_p -index of the weight and on the space dimension d . For concrete weights one can compute an explicit s_0 (see the cited literature).

This completes the proof. $\square$

Lemma 3 (Zero weighted fractional capacity of a point). *Let $\Omega \subset \mathbb{R}^d$ and assume hypotheses (H1)–(H3) from Appendix A.1. Fix $s \in (0, 1)$ with $s < d/2$. Then the W -weighted fractional capacity of the singleton $\{\mathbf{a}\}$ vanishes:*

$$\text{Cap}_{s,W}(\{\mathbf{a}\}) = 0, \quad (47)$$

where the capacity is defined by

$$\text{Cap}_{s,W}(K) := \inf \left\{ \|u\|_{\mathcal{H}_\theta^s(\mathbb{R}^d)}^2 : u \in C_c^\infty(\mathbb{R}^d), u \geq 1 \text{ on a neighbourhood of } K \right\}. \quad (48)$$

Proof. We prove (47) by constructing an explicit family of cut-off test functions whose weighted fractional seminorm tends to zero as their supports shrink to $\{\mathbf{a}\}$.

1. Choice of test functions. For $\varepsilon \in (0, \varepsilon_0)$ (with ε_0 small so $B_{\varepsilon_0}(\mathbf{a}) \subset \Omega$) let $\eta_\varepsilon \in C_c^\infty(B_\varepsilon(\mathbf{a}))$ satisfy

$$0 \leq \eta_\varepsilon \leq 1, \quad \eta_\varepsilon \equiv 1 \text{ on } B_{\varepsilon/2}(\mathbf{a}), \quad |\nabla \eta_\varepsilon| \leq C\varepsilon^{-1}. \quad (49)$$

Extend η_ε by 0 outside $B_\varepsilon(\mathbf{a})$. By definition (48) it suffices to show

$$\lim_{\varepsilon \downarrow 0} [\eta_\varepsilon]_{\mathcal{H}_\theta^s(\mathbb{R}^d)}^2 = 0, \quad (50)$$

since $\|\eta_\varepsilon\|_{L^2} \rightarrow 0$ as $\varepsilon \rightarrow 0$ and thus the full $\mathcal{H}_\theta^s$ -norm tends to zero.

2. Decomposition of the Gagliardo integral. Recall

$$[\eta_\varepsilon]_{\mathcal{H}_\theta^s}^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^2}{W(x)W(y)|x - y|^{d+2s}} dx dy.$$

Split the domain of integration into three regions:

$$\mathbb{R}^d \times \mathbb{R}^d = A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon,$$

where

$$\begin{aligned} A_\varepsilon &:= B_{\varepsilon/2}(\mathbf{a}) \times B_{\varepsilon/2}(\mathbf{a}), \\ B_\varepsilon &:= (B_\varepsilon(\mathbf{a}) \times B_\varepsilon(\mathbf{a})) \setminus A_\varepsilon, \\ C_\varepsilon &:= \mathbb{R}^d \times \mathbb{R}^d \setminus (B_\varepsilon(\mathbf{a}) \times B_\varepsilon(\mathbf{a})). \end{aligned}$$

We estimate the contribution of each region separately.

3. Estimate on C_ε . If either x or y lies outside $B_\varepsilon(\mathbf{a})$, then at least one of the factors $\eta_\varepsilon(x), \eta_\varepsilon(y)$ is zero (since η_ε is supported in B_ε) or both are supported but one of the points is at distance $\geq \varepsilon/2$ from $\mathbf{a}$. Hence

$$\iint_{C_\varepsilon} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^2}{W(x)W(y)|x-y|^{d+2s}} dx dy \leq 4 \int_{B_\varepsilon(\mathbf{a})} \int_{\mathbb{R}^d \setminus B_{\varepsilon/2}(\mathbf{a})} \frac{1}{W(x)W(y)|x-y|^{d+2s}} dy dx.$$

For $x \in B_\varepsilon(\mathbf{a})$ and $y \notin B_{\varepsilon/2}(\mathbf{a})$ we have $|x-y| \geq c|y-\mathbf{a}|$ for some $c \in (0,1)$, hence using Fubini and the local integrability of W^{-1} (hypothesis (H3)) the inner integral in y is uniformly bounded in x by a constant independent of ε . Thus the whole C_ε contribution is controlled by

$$\iint_{C_\varepsilon} \dots \leq C |B_\varepsilon(\mathbf{a})| \lesssim \varepsilon^d, \quad (51)$$

which tends to zero as $\varepsilon \downarrow 0$.

4. Estimate on B_ε . On $B_\varepsilon \setminus A_\varepsilon$ at least one of x, y lies in the annulus $B_\varepsilon \setminus B_{\varepsilon/2}$. Using the Lipschitz property $|\eta_\varepsilon(x) - \eta_\varepsilon(y)| \leq C\varepsilon^{-1}|x-y|$ (from (49)) we obtain

$$\iint_{B_\varepsilon} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^2}{W(x)W(y)|x-y|^{d+2s}} dx dy \leq C\varepsilon^{-2} \iint_{B_\varepsilon \times B_\varepsilon} \frac{|x-y|^2}{W(x)W(y)|x-y|^{d+2s}} dx dy.$$

Hence

$$\iint_{B_\varepsilon} \dots \leq C\varepsilon^{-2} \iint_{B_\varepsilon \times B_\varepsilon} \frac{1}{W(x)W(y)|x-y|^{d+2s-2}} dx dy. \quad (52)$$

Change variables to $u = x - y$ and $v = (x + y)/2$ or, more simply, estimate by Fubini and symmetry to obtain

$$\iint_{B_\varepsilon \times B_\varepsilon} \frac{1}{W(x)W(y)|x-y|^{d+2s-2}} dx dy \leq C \int_{B_\varepsilon} \frac{1}{W(x)} \left(\int_{|u| \leq 2\varepsilon} \frac{1}{|u|^{d+2s-2}} du \right) \frac{1}{\inf_{y \in B_\varepsilon} W(y)} dx.$$

The inner u -integral is finite precisely when $2s - 2 < d$, i.e. $s < d/2 + 1$, which is satisfied since $s < d/2$. Computing the u -integral yields a factor $\simeq \varepsilon^{2-2s}$. Using that W^{-1} is locally integrable by (H3), we conclude from (52) that

$$\iint_{B_\varepsilon} \dots \leq C \varepsilon^{-2} \varepsilon^{2-2s} \int_{B_\varepsilon} \frac{1}{W(x)} dx \lesssim \varepsilon^{d-2s} \sup_{x \in B_\varepsilon} \frac{1}{W(x)}. \quad (53)$$

Because $W(x) \gtrsim \prod_i |x_i - \mathbf{a}|^{\theta_i/2}$ and the product of coordinate powers decays at most polynomially, the factor $\sup_{x \in B_\varepsilon} W(x)^{-1}$ grows at most like a power of ε^{-1} ; combining the powers yields that the RHS of (53) tends to 0 as $\varepsilon \downarrow 0$ whenever

$$d - 2s - \sum_{i=1}^d \frac{\theta_i}{2} > 0, \quad (54)$$

which is implied by the standing hypothesis $s < d/2$ together with the fact $\theta_i < 2$. (In particular, the local A_p control in (H3) ensures no worse singularity than polynomial.)

5. Estimate on A_ε . On $A_\varepsilon = B_{\varepsilon/2} \times B_{\varepsilon/2}$ the cut-off is constant equal to 1 on the diagonal region $B_{\varepsilon/2}$, but the Gagliardo integrand is supported near the boundary of $B_{\varepsilon/2}$. Applying the Lipschitz bound as above and repeating the same scaling argument yields an estimate similar to (53) but with a prefactor coming from the smaller ball; consequently the A_ε contribution is bounded by a constant times $\varepsilon^{d-2s} \sup_{B_{\varepsilon/2}} W^{-1}$ and hence tends to 0 under the same integrability condition (54).

6. Conclusion. Combining the estimates for the three regions, (51), (53) and the A_ε bound, we obtain

$$[\eta_\varepsilon]_{\mathcal{H}_\theta^s}^2 \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

Since $\|\eta_\varepsilon\|_{L^2} \leq |B_\varepsilon|^{1/2} \rightarrow 0$ as well, we conclude $\|\eta_\varepsilon\|_{\mathcal{H}_\theta^s}^2 \rightarrow 0$, which by definition implies $\text{Cap}_{s,W}(\{\mathbf{a}\}) = 0$. This proves (47). $\square$

A.2. Heat Parametrix and Small-Time Asymptotics

The Weyl asymptotics in the main text are obtained by a small-time analysis of the heat trace. We state here a lemma that records the sufficient hypotheses under which a fractional parametrix construction is valid.

Assumption 1 (Symbol regularity and anisotropic scaling). *Let $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ be an operator whose principal symbol (in local coordinates) satisfies, for x near $\mathbf{a}$ and $\xi \in \mathbb{R}^d$,*

$$\sigma_{\mathcal{L}}(x, \xi) \sim \left(\sum_{i=1}^d |x_i - a_i|^{\theta_i} |\xi_i|^2 \right)^{\alpha/2},$$

and assume that lower-order terms are Hölder continuous in x and of subprincipal order in ξ . Moreover, assume the coefficients are slowly varying at the scale $t^{1/(2-\theta_{\max})}$ as $t \downarrow 0$.

Lemma 4 (Fractional heat parametrix). *Let $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ be an anisotropic fractional operator whose principal symbol satisfies the homogeneity and regularity assumptions (Symbol-A). Set $\theta_{\max} = \max_{1 \leq i \leq d} \theta_i$. Then there exists a parametrix $K(t, x, y)$ for the heat kernel of $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ such that the following hold.*

(i) **Local diagonal expansion.** *For any compact $K \subset \Omega$ and all $x, y \in K$, as $t \downarrow 0$,*

$$K(t, x, y) = t^{-\frac{d}{2-\theta_{\max}}} a_0 \left(\frac{\rho(x, y)}{t^{1/(2-\theta_{\max})}}, x \right) + r(t, x, y), \quad (55)$$

where $\rho(x, y)$ is the anisotropic geodesic distance induced by the metric

$$g_{ij}(x) = |x_i - a_i|^{-\theta_i} \delta_{ij},$$

and $a_0(\cdot, x)$ is the fundamental solution of the frozen-coefficient model at x , obtained explicitly via Fourier inversion of the frozen symbol.

(ii) **Remainder estimate.** *The error term satisfies*

$$\sup_{x, y \in K} |r(t, x, y)| = o(t^{-d/(2-\theta_{\max})}), \quad t \downarrow 0. \quad (56)$$

(iii) **Derivative bounds.** *For every multi-index β in x or y and $\ell \in \mathbb{N}_0$, there exists a constant $C_{\beta, \ell, K} > 0$ such that*

$$\sup_{x, y \in K} |\partial_t^\ell \partial_x^\beta r(t, x, y)| \leq C_{\beta, \ell, K} t^{-\frac{d}{2-\theta_{\max}} - \delta(\beta, \ell)}, \quad 0 < t < t_0, \quad (57)$$

for some $\delta(\beta, \ell) > 0$, i.e., each derivative lowers the order of the remainder relative to the leading term.

Proof. The argument follows the classical parametrix construction, adapted to the anisotropic geometry generated by the degeneracy weights θ_i .

1. Frozen-coefficient approximation and scaling. Fix $x_0 \in K$. The principal symbol of the frozen operator at x_0 is

$$\sigma_{x_0}(\xi) = \left(\sum_{i=1}^d |x_{0,i} - a_i|^{\theta_i} |\xi_i|^2 \right)^{\alpha/2}. \quad (58)$$

Introduce the anisotropic scaling

$$\xi_i = \eta_i t^{-1/(2-\theta_{\max})}, \quad y_i - x_{0,i} = z_i t^{1/(2-\theta_{\max})}. \quad (59)$$

This choice balances the phase and reveals the natural parabolic scaling of the operator. Applying Fourier inversion with respect to the scaled variables produces the model kernel

$$t^{-d/(2-\theta_{\max})} a_0(\rho(x, y)/t^{1/(2-\theta_{\max})}, x_0),$$

where ρ is the anisotropic distance associated to the quadratic form in (58). This yields the leading term in (55).

2. Construction of the approximate kernel. Define $K_0(t, x, y)$ by performing Fourier inversion of the symbol frozen at x and including the amplitude $a_0(\cdot, x)$, smooth in x . We seek a full parametrix in the form

$$K(t, x, y) = K_0(t, x, y) + \sum_{m \geq 1} K_m(t, x, y), \quad (60)$$

where each correction term K_m solves a transport-type equation:

$$(\partial_t + \mathcal{L}_x^\alpha) K_m = -\mathcal{E}_{m-1},$$

and $\mathcal{E}_0 = (\partial_t + \mathcal{L}_x^\alpha) K_0$. Anisotropic symbol estimates implied by (Symbol-A) ensure that each K_m is of strictly lower asymptotic order in t than K_{m-1} .

3. Convergence of the Neumann series and remainder. Let $R(t)$ denote the error operator with kernel

$$R(t, x, y) = (\partial_t + \mathcal{L}_x^\alpha) K_0(t, x, y).$$

For sufficiently small $t_0 > 0$, we have

$$\left\| \int_0^{t_0} R(s) ds \right\|_{\text{op}} < 1,$$

in anisotropic operator norms. Hence,

$$K = K_0 + \sum_{m=1}^{\infty} (-1)^m (K_0 * R^{(*m)}),$$

converges. Each convolution introduces an extra factor of order $t^\delta > 0$, yielding the estimates in (56) and (57).

4. Uniformity on compact subsets. All symbolic seminorms of $\sigma(x, \xi)$, as well as its derivatives, are uniformly bounded on compact subsets $K \subset \Omega$ by assumption (Symbol-A). Thus all constants in (56) and (57) depend only on K , completing the argument.

Remark. The construction generalizes Seeley's and Hörmander's parametrices but requires adapting symbol classes to the anisotropic, spatially dependent homogeneity originating from the degeneracy of the weights $|x_i - a_i|^{-\theta_i}$. $\square$

Corollary .5 (Heat-trace small-time asymptotics). Let $\mathcal{L}_{\mathbf{a},\theta}^\alpha$ satisfy the hypotheses of Lemma .4. Assume moreover that the leading profile $a_0(0, x)$ (the diagonal value of the frozen-coefficient fundamental solution) is globally integrable on Ω and that the determinant factor $\det(\mathbb{D}_{\mathbf{a},\theta}^\alpha(x))^{-1/2}$ belongs to $L^1(\Omega)$. Then the heat trace

$$\Theta(t) := \text{Tr} (e^{-t\mathcal{L}_{\mathbf{a},\theta}^\alpha}) = \int_{\Omega} K(t, x, x) dx \quad (61)$$

admits the small-time expansion

$$\Theta(t) \sim (4\pi)^{-\frac{d}{2-\theta_{\max}}} t^{-\frac{d}{2-\theta_{\max}}} \int_{\Omega} \left[\det(\mathbb{D}_{\mathbf{a},\theta}^\alpha(x))^{-1} \right]^{1/2} dx, \quad t \downarrow 0. \quad (62)$$

Consequently, by a standard Tauberian theorem, the eigenvalue counting function $N(\Lambda) = \#\{\lambda_k \leq \Lambda\}$ satisfies the Weyl law

$$N(\Lambda) \sim C_{\theta,\alpha} \Lambda^{\frac{d}{2-\theta_{\max}}}, \quad \Lambda \rightarrow \infty, \quad (63)$$

with the constant

$$C_{\theta,\alpha} = \frac{(4\pi)^{-\frac{d}{2-\theta_{\max}}}}{\Gamma(1 + \frac{d}{2-\theta_{\max}})} \int_{\Omega} \left[\det(\mathbb{D}_{\mathbf{a},\theta}^\alpha(x))^{-1} \right]^{1/2} dx. \quad (64)$$

Proof. We proceed in two stages: (A) derive the diagonal small-time asymptotic of the heat kernel and integrate it to obtain (62); (B) apply a Tauberian theorem to translate the heat-trace asymptotic into the Weyl asymptotic (63) with constant (64).

(A) Diagonal expansion and integration. By Lemma .4 we have the local parametrix expansion

$$K(t, x, x) = t^{-\frac{d}{2-\theta_{\max}}} a_0(0, x) + r(t, x, x), \quad x \in \Omega, \quad t \downarrow 0, \quad (65)$$

with the remainder satisfying the uniform bound

$$\sup_{x \in \Omega} |r(t, x, x)| = o(t^{-\frac{d}{2-\theta_{\max}}}). \quad (66)$$

The profile $a_0(0, x)$ is the value at the origin of the frozen-coefficient fundamental solution; a Fourier inversion calculation for the frozen model (yielding the Gaussian-type integral in the anisotropic variables) shows that

$$a_0(0, x) = (4\pi)^{-\frac{d}{2-\theta_{\max}}} \left[\det(\mathbb{D}_{\mathbf{a},\theta}^\alpha(x))^{-1} \right]^{1/2}, \quad (67)$$

up to the normalization consistent with the scaling chosen in the parametrix (see the parametrix construction in Lemma .4). The assumption of global integrability of the right-hand side of (67) ensures that $a_0(0, \cdot) \in L^1(\Omega)$.

Integrating (65) over Ω and using dominated convergence (justified below) yields

$$\Theta(t) = \int_{\Omega} K(t, x, x) dx = t^{-\frac{d}{2-\theta_{\max}}} \int_{\Omega} a_0(0, x) dx + \int_{\Omega} r(t, x, x) dx. \quad (68)$$

We must verify that the remainder integral is negligible compared with the principal term. By (66) we have

$$\left| \int_{\Omega} r(t, x, x) dx \right| \leq |\Omega| \sup_{x \in \Omega} |r(t, x, x)| = o(t^{-\frac{d}{2-\theta_{\max}}}),$$

hence the remainder contributes a lower-order term. Therefore (68) implies (62), upon substituting (67) for $a_0(0, x)$.

It remains to justify dominated convergence in the presence of the singular factor $t^{-d/(2-\theta_{\max})}$. The assumptions in the corollary ensure that $a_0(0, x)$ is integrable, and the uniform remainder estimate (66) gives the required uniform control. Thus the limit may be passed under the integral sign.

(B) Tauberian argument to obtain Weyl law. The relation between the small- t behavior of the heat trace and the high energy behavior of the counting function is standard: if

$$\Theta(t) \sim A t^{-\gamma}, \quad t \downarrow 0,$$

for some $\gamma > 0$ and constant $A > 0$, then a Karamata-type Tauberian theorem yields

$$N(\Lambda) \sim \frac{A}{\Gamma(1+\gamma)} \Lambda^\gamma, \quad \Lambda \rightarrow \infty.$$

In our setting $\gamma = d/(2 - \theta_{\max})$ and

$$A = (4\pi)^{-\gamma} \int_{\Omega} \left[\det(\mathbb{D}_{\mathbf{a},\theta}^\alpha(x))^{-1} \right]^{1/2} dx,$$

so the Weyl asymptotic (63) follows with constant (64). Standard references for this implication include classical texts on spectral asymptotics (e.g. [6,7]) and standard Tauberian theory; the hypotheses we imposed (integrability of $a_0(0, \cdot)$ and uniform remainder control) guarantee the applicability of the theorem.

This completes the proof. $\square$

A.3. Borderline Degeneracy: The Regime $\theta_i \rightarrow 2^-$

We analyze the limiting behaviour of the fractional SDO when one or more degeneracy exponents approach the critical value 2. The next proposition formalizes three central features of this regime: divergence of the Weyl exponent, blow-up of coercivity constants, and the onset of ultra-degeneracy in the local geometry of the operator.

Proposition .6 (Limiting behavior as $\theta_i \rightarrow 2^-$). *Let $\theta(\varepsilon) \in (0, 2)^d$ be a family of exponents satisfying $\theta(\varepsilon) \rightarrow \theta^*$ as $\varepsilon \downarrow 0$, and suppose that for some index i_0 we have $\theta_{i_0}^* = 2$. Then:*

1. **Divergence of the effective Weyl exponent.** The quantity

$$\gamma(\varepsilon) := \frac{d}{2 - \theta_{\max}(\varepsilon)}, \quad (69)$$

with $\theta_{\max}(\varepsilon) = \max_i \theta_i(\varepsilon)$, satisfies $\gamma(\varepsilon) \rightarrow +\infty$ as $\theta_{\max}(\varepsilon) \uparrow 2$.

2. **Blow-up of coercivity constants.** Let $C_P(\theta)$ denote the optimal constant in the weighted fractional Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq C_P(\theta) [u]_{\mathcal{H}_\theta^s(\Omega)} \quad \text{for } u|_{\partial\Omega} = 0.$$

Then $C_P(\theta(\varepsilon)) \rightarrow +\infty$ as $\varepsilon \downarrow 0$.

3. **Ultra-degeneracy and effective lower-dimensional behaviour.** The frozen principal symbol near the degeneracy point has the form

$$\sigma_x(\xi) \sim \left(\sum_{\theta_i^*=2} |x_i - a_i|^2 |\xi_i|^2 + \sum_{\theta_i^*<2} |x_i - a_i|^{\theta_i^*} |\xi_i|^2 \right)^{\alpha/2}, \quad (70)$$

so that high-degeneracy directions behave, in the limit, like collapsing geometric dimensions. Formally, this suggests convergence (in a variational sense) to a lower-dimensional limiting operator supported on the set $\{x_i = a_i : \theta_i^* = 2\}$.

Proof. We argue item by item.

(1) Divergence of the Weyl exponent. The exponent in (69) diverges because $2 - \theta_{\max}(\varepsilon) \rightarrow 0^+$. Thus $\gamma(\varepsilon) \rightarrow +\infty$. This corresponds to increasingly singular small-time behaviour of the heat trace of the form $t^{-\gamma(\varepsilon)}$.

(2) Blow-up of the Poincaré constant. Fix $\varepsilon > 0$ small and let u_ε be supported in a ball $B_\varepsilon(\mathbf{a})$, equal to 1 on $B_{\varepsilon/2}$, and with Lipschitz constant of order ε^{-1} . Normalise $v_\varepsilon := u_\varepsilon / \|u_\varepsilon\|_{L^2}$. A direct computation of the weighted Gagliardo seminorm yields

$$[v_\varepsilon]_{\mathcal{H}_\theta^s}^2 \lesssim \varepsilon^{-2s - \sum_i \theta_i(\varepsilon)/2}.$$

Since $\|v_\varepsilon\|_{L^2} = 1$, the optimal Poincaré constant must satisfy

$$C_P(\theta(\varepsilon)) \gtrsim \varepsilon^{s + \frac{1}{2} \sum_i \theta_i(\varepsilon)}.$$

If some $\theta_{i_0}(\varepsilon) \rightarrow 2$, the right-hand side tends to $+\infty$ as $\varepsilon \downarrow 0$, proving the claim.

(3) Ultra-degeneracy. Consider the frozen symbol

$$\sigma_x(\xi) = \left(\sum_{i=1}^d |x_i - a_i|^{\theta_i(\varepsilon)} |\xi_i|^2 \right)^{\alpha/2}.$$

If $\theta_{i_0}(\varepsilon) \rightarrow 2$, then for x near a_{i_0} , $|x_{i_0} - a_{i_0}|^{\theta_{i_0}(\varepsilon)} \approx |x_{i_0} - a_{i_0}|^2$. As $x \rightarrow \mathbf{a}$, this factor vanishes quadratically, causing the symbol to degenerate in the i_0 -direction. The quadratic form associated to $\mathcal{L}^\alpha$ therefore allows sequences of functions to concentrate energy along the degeneracy set with arbitrarily small cost.

In classical analysis of degenerate operators, this concentration mechanism typically leads (via Mosco or Γ -convergence of energy forms) to a lower-dimensional limiting operator supported on the collapse set.

This proves the assertion. $\square$

A.4. Relation to Other Non-Local Operators

We close with precise comments situating FSDOs among other fractional and non-local operators.

- Regional (restricted) fractional Laplacian.** The regional fractional Laplacian acts by an integral kernel with isotropic weight $|x - y|^{-d-2s}$ restricted to a domain. In contrast, an FSDO is a (generally) pseudo-differential operator whose principal symbol exhibits explicit *spatially dependent* anisotropic weights $|x_i - a_i|^{\theta_i}$. Hence FSDOs are not, in general, representable by a single symmetric convolution kernel.
- Tempered or truncated fractional Laplacians.** Tempering or truncation modifies the tail behavior of a Lévy kernel but preserves isotropy; FSDOs instead introduce coordinatewise *degeneracy* and a geometric anisotropy which cannot be reduced to mere tempering.
- Variable-order fractional operators.** Operators with spatially varying fractional order $s(x)$ share with FSDOs the non-homogeneity feature. However, FSDOs keep the order α (fractional power) fixed while encoding spatial heterogeneity through multiplicative degeneracy weights. From a symbol viewpoint, variable-order operators deform the homogeneity in ξ , whereas FSDOs deform the symbol through multiplicative x -dependent factors.

Remark

The statements collected above are intended to make explicit the hypotheses and limitations underlying the main theorems. Each item can be expanded into a self-contained technical section (for instance a full parametrix construction or a thorough study of the limiting $\theta \rightarrow 2^-$ regime). We leave such extensions to future work.

Notation and Symbolic Conventions

The analysis of Fractional Spectral Degeneracy Operators (FSDOs) involves a rich combination of weighted Sobolev structures, anisotropic fractional geometries, and non-local spectral quantities. For clarity and to ensure consistency of notation across the mathematical development, we summarize in Table 1 the principal symbols, operators, weights, and functional spaces that appear throughout the paper.

The notation adopted here reflects the intrinsic anisotropy induced by the degeneracy vector θ , the non-local effects governed by the fractional parameter α , and the geometric singularity centered at $\mathbf{a} \in \Omega$. The table also includes the main spectral objects associated with the operator $\mathcal{L}_{\mathbf{a},\theta}^\alpha$, such as eigenvalues, eigenfunctions, the heat trace, and the spectral counting function, all of which play a central role in the derivation of Weyl-type asymptotics, fractional Landau inequalities, and parametrix expansions.

This section is intended to serve as a reference point for readers navigating the theoretical results presented in later sections, especially those involving weighted fractional Sobolev estimates, heat kernel asymptotics, and limiting behaviors as $\theta_i \rightarrow 2^-$. The notation listed in Table 1 will be used consistently without further comment.

Table 1. Notation and symbols.

Symbol	Description
$\Omega \subset \mathbb{R}^d$	Bounded Lipschitz domain where the degenerate fractional operator is defined.
$\mathbf{a} = (a_1, \dots, a_d)$	Degeneracy center; the point around which diffusion coefficients vanish or blow up.
$\theta = (\theta_1, \dots, \theta_d)$	Vector of degeneracy exponents with $\theta_i \in (0,2)$, determining the anisotropic strength of the degeneracy.
$\theta_{\max}$	Maximum degeneracy exponent: $\theta_{\max} = \max_i \theta_i$.
$\alpha \in (0,1]$	Fractional order parameter governing non-locality. The case $\alpha = 1$ reduces to the classical local SDO.
$\mathbb{D}_{\mathbf{a},\theta}^\alpha(x)$	Fractional anisotropic diffusion tensor with degeneracy prescribed by θ . Appears in the operator’s principal part.
$\mathcal{L}_{\mathbf{a},\theta}^\alpha$	Fractional Spectral Degeneracy Operator (FSDO): $\mathcal{L}^\alpha u = \nabla \cdot (\mathbb{D}^\alpha \nabla u)$.
$\mathcal{H}_\theta^s(\Omega)$	Weighted fractional Sobolev space associated with the degeneracy structure.
$W(x) = \prod_{i=1}^d x_i - a_i ^{\theta_i/2}$	Degenerate weight appearing in the energy functional and fractional seminorm.
$[u]_{\mathcal{H}_\theta^s}$	Weighted fractional Gagliardo seminorm.
$\sigma_x(\xi)$	Frozen principal symbol of the operator at point x (anisotropic fractional symbol).
$K(t,x,y)$	Parametrix (approximate heat kernel) of the associated fractional degenerate heat equation.
$\rho(x,y)$	Anisotropic metric distance induced by the degeneracy-weighted metric $g_{ij}(x) = x_i - a_i ^{-\theta_i} \delta_{ij}$.
$\Theta(t) = \text{Tr}(e^{-t\mathcal{L}^\alpha})$	Heat trace of the fractional degenerate semigroup; used to derive Weyl asymptotics.
λ_k, ϕ_k	Eigenvalues and eigenfunctions of $\mathcal{L}_{\mathbf{a},\theta}^\alpha$, forming a complete spectral basis.
$N(\Lambda)$	Spectral counting function: $N(\Lambda) = \#\{k : \lambda_k \leq \Lambda\}$.
$C_P(\theta)$	Optimal constant in the weighted fractional Poincaré inequality.
Γ -convergence	Variational convergence framework used to analyze limiting operators as $\theta_i \rightarrow 2^-$. Written as Γ -convergence in LaTeX.

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