Short Note

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Short Note

Non-Archimedean Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality

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Abstract

In 1992, Hudzik and Landes derived a breakthrough generalization of the triangle inequality for two nonzero elements in normed linear spaces, which was generalized to finitely m any nonzero elements independently in 2006 by Dragomir and in 2007 by Kato, Saito and Tamura. We derive a non-Archimedean version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura inequality.

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MSC: 12J25; 46S10

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1. Introduction

Let $\mathcal{X}$ be a normed linear space (NLS). From the definition of the norm, we have the triangle inequality

$$\|x+y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathcal{X}. \tag{1}$$

In 1992, Hudzik and Landes derived a breakthrough generalization of Inequality (1) which is valid for any two nonzero elements in a NLS [1].

Theorem 1. *[1] (**Hudzik-Landes Inequlaity**) Let $\mathcal{X}$ be a NLS. Then for all $x, y \in \mathcal{X} \setminus \{0\}$,*

$$\|x+y\| \leq \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, \|y\|\}. \tag{2}$$

We note that, in 2006, Maligranda independently derived Inequality (2) [2]. It is natural to ask for a generalization of Inequality (2) to more than two non-zero vectors. This is done independently by Dragomir in 2006 [3] and by Kato, Saito and Tamura in 2007 [4].

Theorem 2. *[3,4] (**Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality**) Let $\mathcal{X}$ be a NLS and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$, we have*

$$\left\|\sum_{j=1}^n x_j\right\| \leq \sum_{j=1}^n \|x_j\| - \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\leq k\leq n} \|x_k\|.$$

It is natural and important to ask what are non-Archimedean versions of Theorems 1 and 2? We answer the question by deriving non-Archimedean version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality (Theorem 4).

2. Non-Archimedean Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality

Let $\mathbb{K}$ be a field. Recall that a map $|\cdot| : \mathbb{K} \to [0, \infty)$ is said to be a non-Archimedean valuation if following conditions holds.

- If $\lambda \in \mathbb{K}$ is such that $|\lambda| = 0$, then $\lambda = 0$.
- $|\lambda\mu| = |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{K}$.
- (Ultra-triangle inequality) $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.

In this case, $\mathbb{K}$ is called as non-Archimedean valued field [5]. Let $\mathcal{X}$ be a vector space over a non-Archimedean valued field $\mathbb{K}$ with valuation $|\cdot|$. Recall that a map $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is said to be a non-Archimedean norm if following conditions holds.

- If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
- $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}$, for all $x \in \mathcal{X}$.
- (Ultra-norm inequality) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{X}$.

In this case, $\mathcal{X}$ is called as non-Archimedean linear space (NALS) [6]. We first derive non-Archimedean version of Inequality (2).

Theorem 3. *(**Non-Archimedean Hudzik-Landes Inequality**) Let $\mathcal{X}$ be a NALS. Then for all $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathcal{X}$ it holds*

$$\|x + y\| \leq \min\left\{\|x\| \max\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left|\frac{\|y\|}{\|x\|} - 1\right|\right\}, \|y\| \max\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left|\frac{\|x\|}{\|y\|} - 1\right|\right\}\right\}$$
$$\leq \max\{\|x\|, \|y\|\}.$$

Proof. Let $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathcal{X}$. Then

$$\begin{aligned}
\|x + y\| &= \left\|\|x\|\left(\frac{x}{\|x\|} + \frac{y}{\|y\|}\right) + \left(1 - \frac{\|x\|}{\|y\|}\right)y\right\| \\
&\leq \max\left\{\|x\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left\|\left(1 - \frac{\|x\|}{\|y\|}\right)y\right\|\right\} \\
&= \max\left\{\|x\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, |\|y\| - \|x\||\right\}
\end{aligned}$$

and

$$\begin{aligned}
\|x + y\| &= \left\|\left(1 - \frac{\|y\|}{\|x\|}\right)x + \|y\|\left(\frac{x}{\|x\|} + \frac{y}{\|y\|}\right)\right\| \\
&\leq \max\left\{\left\|\left(1 - \frac{\|y\|}{\|x\|}\right)x\right\|, \|y\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right\} \\
&= \max\left\{|\|x\| - \|y\||, \|y\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right\}.
\end{aligned}$$

Therefore

$$\|x + y\| \leq \|x\| \max\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left|\frac{\|y\|}{\|x\|} - 1\right|\right\} \tag{3}$$

and

$$\|x + y\| \leq \|y\| \max\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left|\frac{\|x\|}{\|y\|} - 1\right|\right\}. \tag{4}$$

Inequalities (3)and (4) give

$$\|x + y\| \leq \min\left\{\|x\| \max\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left|\frac{\|y\|}{\|x\|} - 1\right|\right\}, \|y\| \max\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left|\frac{\|x\|}{\|y\|} - 1\right|\right\}\right\}.$$

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□

Note the additional assumption $\|x\|, \|y\| \in \mathcal{X}$ in the previous theorem. The reason is that, since the norm is a real number, we generally do not have a guarantee that it belongs to the given non-Archimedean field. Now we derive non-Archimedean version of Theorem 2.

Theorem 4. *(**Non-Archimedean Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality**) Let $\mathcal{X}$ be a NALS and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathcal{X}$ it holds*

$$\left\|\sum_{j=1}^{n} x_j\right\| \leq \min_{1\leq k\leq n}\left\{\|x_k\| \max\left\{\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|, \max_{1\leq j\leq n}\left|\frac{\|x_j\|}{\|x_k\|}-1\right|\right\}\right\} \leq \max_{1\leq j\leq n}\|x_j\|.$$

Proof. Let $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathcal{X}$. Let $1 \leq k \leq n$ be fixed. Then

$$\begin{aligned}
\left\|\sum_{j=1}^{n} x_j\right\| &= \left\|\sum_{j=1}^{n}\frac{\|x_k\| x_j}{\|x_j\|} + \sum_{j=1}^{n}\left(1-\frac{\|x_k\|}{\|x_j\|}\right)x_j\right\| \\
&\leq \max\left\{\left\|\sum_{j=1}^{n}\frac{\|x_k\| x_j}{\|x_j\|}\right\|, \left\|\sum_{j=1}^{n}\left(1-\frac{\|x_k\|}{\|x_j\|}\right)x_j\right\|\right\} \\
&= \max\left\{\|x_k\|\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|, \left\|\sum_{j=1}^{n}\left(1-\frac{\|x_k\|}{\|x_j\|}\right)x_j\right\|\right\} \\
&\leq \max\left\{\|x_k\|\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|, \max_{1\leq j\leq n}\left\|\left(1-\frac{\|x_k\|}{\|x_j\|}\right)x_j\right\|\right\} \\
&= \max\left\{\|x_k\|\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|, \max_{1\leq j\leq n}\left|\|x_j\|-\|x_k\|\right|\right\} \\
&= \|x_k\| \max\left\{\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|, \max_{1\leq j\leq n}\left|\frac{\|x_j\|}{\|x_k\|}-1\right|\right\}.
\end{aligned}$$

By varying k and taking minimum in the right side of previous inequality gives

$$\left\|\sum_{j=1}^{n} x_j\right\| \leq \min_{1\leq k\leq n}\left\{\|x_k\| \max\left\{\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|, \max_{1\leq j\leq n}\left|\frac{\|x_j\|}{\|x_k\|}-1\right|\right\}\right\}.$$

□

Now we derive continuous version of Theorem 4.

Theorem 5. *Let $\mathcal{X}$ be a NALS and (Ω, μ) be a non-Archimedean measure space. Let $\phi : \Omega \to \mathcal{X} \setminus \{0\}$ be a measurable function such that $\|\phi(\alpha)\| \in \mathcal{X}$ for every $\alpha \in \Omega$. Then*

$$\left\|\int_{\Omega}\phi(\alpha)\,d\mu(\alpha)\right\| \leq \inf_{\beta\in\Omega}\left\{\|\phi(\beta)\| \sup\left\{\left\|\int_{\Omega}\frac{\phi(\alpha)}{\|\phi(\alpha)\|}\,d\mu(\alpha)\right\|, \sup_{\alpha\in\Omega}\left|\frac{\|\phi(\alpha)\|}{\|\phi(\beta)\|}-1\right|\right\}\right\} \leq \sup_{\alpha\in\Omega}\|\phi(\alpha)\|.$$

Proof. Let $\beta \in \Omega$ be fixed. Then

$$\begin{aligned}\left\|\int_\Omega \phi(\alpha)\,d\mu(\alpha)\right\| &= \left\|\int_\Omega \frac{\|\phi(\beta)\|\phi(\alpha)}{\|\phi(\alpha)\|}\,d\mu(\alpha)+\int_\Omega\left(1-\frac{\|\phi(\beta)\|}{\|\phi(\alpha)\|}\right)\phi(\alpha)\,d\mu(\alpha)\right\|\\ &\leq \max\left\{\left\|\int_\Omega \frac{\|\phi(\beta)\|\phi(\alpha)}{\|\phi(\alpha)\|}\,d\mu(\alpha)\right\|,\left\|\int_\Omega\left(1-\frac{\|\phi(\beta)\|}{\|\phi(\alpha)\|}\right)\phi(\alpha)\,d\mu(\alpha)\right\|\right\}\\ &= \max\left\{\|\phi(\beta)\|\left\|\int_\Omega \frac{\phi(\alpha)}{\|\phi(\alpha)\|}\,d\mu(\alpha)\right\|,\left\|\int_\Omega\left(1-\frac{\|\phi(\beta)\|}{\|\phi(\alpha)\|}\right)\phi(\alpha)\,d\mu(\alpha)\right\|\right\}\\ &\leq \max\left\{\|\phi(\beta)\|\left\|\int_\Omega \frac{\phi(\alpha)}{\|\phi(\alpha)\|}\,d\mu(\alpha)\right\|,\sup_{\alpha\in\Omega}\left\|\left(1-\frac{\|\phi(\beta)\|}{\|\phi(\alpha)\|}\right)\phi(\alpha)\right\|\right\}\\ &= \max\left\{\|\phi(\beta)\|\left\|\int_\Omega \frac{\phi(\alpha)}{\|\phi(\alpha)\|}\,d\mu(\alpha)\right\|,\sup_{\alpha\in\Omega}\big|\|\phi(\alpha)\|-\|\phi(\beta)\|\big|\right\}\\ &= \|\phi(\beta)\|\max\left\{\left\|\int_\Omega \frac{\phi(\alpha)}{\|\phi(\alpha)\|}\,d\mu(\alpha)\right\|,\sup_{\alpha\in\Omega}\left|\frac{\|\phi(\alpha)\|}{\|\phi(\beta)\|}-1\right|\right\}.\end{aligned}$$

By varying β and taking infimum in the previous inequality gives

$$\left\|\int_\Omega \phi(\alpha)\,d\mu(\alpha)\right\| \leq \inf_{\beta\in\Omega}\left\{\|\phi(\beta)\|\sup\left\{\left\|\int_\Omega \frac{\phi(\alpha)}{\|\phi(\alpha)\|}\,d\mu(\alpha)\right\|,\sup_{\alpha\in\Omega}\left|\frac{\|\phi(\alpha)\|}{\|\phi(\beta)\|}-1\right|\right\}\right\}.$$

□

3. Conclusions

1. In 1992, Hudzik and Landes improved centuries old triangle inequality in normed linear spaces.
2. In 2006, Dragomir extended Hudzik-Landes inequality for more than two vectors.
3. In 2007, Kato, Saito and Tamura extended Hudzik-Landes inequality without knowing the work of Dragomir.
4. In this article, we extended centuries old ultra-norm inequality.

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