Article

Not peer-reviewed version

Automorphisms Group and Radical Polynomial of Standard Trilinear Forms

Midoune Noureddine *

Posted Date: 6 February 2025

doi: 10.20944/preprints202502.0378.v1

Keywords: trivector; invariant; isotropy groups; radical polynomial; commutant

Article

Automorphisms Group and Radical Polynomial of Standard Trilinear Alternating Forms

Noureddine Midoune

Department of Mathematics, Laboratory of Pure and Applied Mathematics, M'sila University, M'sila 28000, Algeria; noureddine.midoune@univ-msila.dz

Abstract: Let V be a vector space of dimension n over a field K and let ω be a trivector of $\wedge^3 V$. For such trivectors, we can associate three invariants, the automorphism group $Aut(\omega)$, the radical polynominal $P(\omega)$ and the commutant $C(\omega)$. We use the classification of trivectors of rank ≤ 9, we give a general rule for the standard trivector ω_{3k} in dimension $3k$, the trivector with the transitive automorphism group and the trivector with an isotropic hyperplane ω_{2k+1} in dimension $2k+1$. We compute their radical polynomials and the sizes of the groups automorphisms. We demonstrate that there exists a vector space V and a trivector ω of $\wedge^3 V$ where $C(\omega)$ is not a Frobenius algebra and $dimV \leq 3\dim C(\omega)$. Finally, We give a classification of trivectors in dimension 8 over a finite field of characteristic 2 and its applications in the theory of codes.

Keywords: trivector; invariant; isotropy groups; radical polynomial; commutant

MSC: Primary 15A69; Secondary 15A75; 05A15; 15A18

1. Introduction

Let V be a $K-$vector space of dimension n and let $\wedge^3 V$ be the third-degree exterior power space of V over the field K. Any element ω of $\wedge^3 V$ is named trivector on V. By virtue of the canomical identification $\wedge^3 V \simeq (\wedge^3 V)^*$, there is no diffrence between trivectors and trilinear alternating form. The classification of trilinear alternating forms is the study of the action of $GL(V)$ on $\wedge^3 V$: $f.\omega = (\wedge^3 f)(\omega)$. For $n \leq 8$, this classification was completed for any fields and the number of orbits is finite. In dimension 9, the number of classes of trivectors over $\mathbb{C}$ is infinite (see [3]) and over $\mathbb{F}_2$ is finite (there are 317 classes, see [6]).

In this paper we examine the general cause for the trivector ω_{3k} in dimension $n = 3k, k \geq 1$, called standard trilinear alternating form, the form of the transitive automorphism group ω_L over L in which L is the extension of the field K and the trivector in dimension $n = 2k+1, k \geq 3$ with the isotropic hyperplane. We computed their groups of automorphisms and their radical polynomials. The main results are the Tables 1, 2, 3, 4 and 5 containing the trivectors ω_{3k} and ω_{2k+1} with the sizes of the group of automorphisms, the radical polynomials and the number of orbits of trivectors in dimension eight. The commutant of a trivector with a maximum rank of eight forms a Frobenius algebra with $dimV \geq 3\dim C(\omega)$ (See [10] Theorem 2.8, p.49), we show that this result is not true for $n \geq 9$.

The motivation behind this research stems from the graph theory, complexity and cryptography, in which they are interested in the alternating trilenear form equivalence (ATFE) problem and code loops, see [12] and [15]. We note that Frobenius algebras are significant in the algebraic approach.

2. Preliminaries

Let $\omega : V^3 \to K$ be a trilinear alternating form on a vector space V over a field $K, dimV = n$. The trivector ω satisfies the equality $\omega(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = sgn(\sigma)\omega(x_1, x_2, x_3)$ for every permutation $\sigma \in S_3$. Two forms ω_1 and ω_2 are equivalent, $\omega_1 \sim \omega_2$, if there exists a homomorphism bijective ϕ of V verifying:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 February 2025 doi:10.20944/preprints202502.0378.v1

$$\omega_2(\phi(x_1), \phi(x_2), \phi(x_3)) = \omega_1(x_1, x_2, x_3) \text{ for every } x_1, x_2, x_3 \in V$$

One of the possible approaches to the classification of trivectors is to use invariants. We recall three invariants of trivectors: the automorphism group $Aut(\omega)$, the radical polynomial $P(\omega)$ of ω introduced by J.Hora [6] and the commutant $C(\omega)$.

Definition 1. *The group of automorphisms of ω, $Aut(\omega)$, is defined by*

$$Aut(\omega) = \{\phi \in GL(V)/\phi.\omega = \omega\}$$

Example 1. *Let ω_3 be the trivector $\omega = e_1e_2e_3$, then $Aut(\omega_3) = SL_3(K)$.*

Definition 2. *The set $Rad(\omega) = \{v \in V, \omega(v, ., .) = 0\}$ is called the radical of ω and denoted by $Rad(\omega)$. If $Rad(\omega)$ is trivial ($Rad(\omega) = \{0\}$), then ω is called non degenerate.*

Fix $v \in V$ and define radical $Rad_\omega(v)$ of V as $Rad_\omega(v) = \{u \in V, \omega(u, v, .) = 0\}$. $Rad_\omega(v)$ is a subspace of V. The rank of $v \in V$, $r_\omega(v) = n - dimRad_\omega(v)$ is an even number.

Definition 3. *Let $K = \mathbb{F}_q$ be a finite field. The polynomial $P(\omega)$ defined by:*

$$P(\omega) = \sum_{v \in V} x^{r(v)} y^{n-r(v)}$$

or

$$P(\omega) = \sum_{i=0}^{n-1} \alpha_i x^i y^{n-i}$$

where $\alpha_i \in \mathbb{N}$ and $\sum_{i=0}^{n-1} \alpha_i = q^n$, is the radical polynomial of ω.

Definition 4. *Two vectors $u, v \in V^* = V - \{0\}$ are orthogonal $u \perp_\omega v$, if $u \in Rad(v)$. The subspaces V_1 and V_2 of V are orthogonal $V_1 \perp V_2$, if $v_1 \perp_\omega v_2$ for all $v_1 \in V_1$ and $v_2 \in V_2$.*

Definition 5. *We say that a non degenerate trilinear alternating form ω on V is decomposable if $V = W_1 \oplus W_2 \oplus ... \oplus W_m, m \geq 2$, and $W_i \perp W_j$ whenever $i \neq j$. The restrictions of ω to W_i are denoted by ω_i. $P(\omega)$ is compatible with the orthogonal decomposition:*

$$\text{if } \omega = \textstyle\sum_i \omega_i, \text{ then } P(\omega) = \prod_i P(\omega_i)$$

Example 2. *Let ω_6 be the trivector $\omega_6 = e_1e_2e_3 + e_4e_5e_6$. The radical polynomial of ω_6 is equal to*

$$P(\omega_6) = P(e_1e_2e_3) \times P(e_4e_5e_6) = (y^3 + 7x^2y)^2 = y^6 + 14x^2y^4 + 49x^4y^2$$

Definition 6. *The commutant of ω, $C(\omega)$ is*

$$C(\omega) = \{f \in End(V) : \omega(x, f(y), z) = \omega(x, y, f(z)), \text{for every } x, y, z \in V\}$$

Definition 7. *Let A be an algebra of finite dimension over a field K. We say that A is a F.algebra if $A \sim$ $\mathrm{Hom}(A, K)$ (isomorphic as A-modules).*

Note that if A is finite-dimensional $K-$algebra over a field K. Then A is a Frobenius algebra if and only if there is a non-degenerate symmetric bilinear form $\phi : A \times A \to K$ such that $\phi(ab, c) = \phi(a, bc)$ for all $a, b, c \in A$.

3. Invariants on Dimension $3k$

Standard Trilinear Form

Definition 8. *Let V be a 3k- dimensional vector space over K and let $B = \{e_1, e_2, e_3, ..., e_{3k-1}, e_{3k}\}$ be a fixed basis of V. A standard trilinear alternating form ω_{3k} can be expressed as*

$$\omega_{3k} = e_1e_2e_3 + e_4e_5e_6 + e_7e_8e_9 + ... + e_{3k-2}e_{3k-1}e_{3k} = \sum_{i=1}^{k} e_{3i-2}e_{3i-1}e_{3i}$$

Since ω_{3k} is a decomposable form, then $P(\omega_{3k})$ is compatible with the orthogonal decomposition and we have

$$P(\omega_{3k}) = \prod_i P(\omega_i) = (y^3 + 7x^2y)^k$$

Proposition 1. *Let $Aut(\omega_3)$ be the automorphism of ω_{3k}, then it satisfies the exact sequence*

$$1 \to [SL_3(K)]^k \to Aut(\omega_{3k}) \to S_k \to 1$$

i.e.

$$Aut(\omega_{3k}) \simeq [SL_3(K)]^k \rtimes S_k$$

If $K = \mathbb{F}_q, |Aut(\omega_{3k})| = k!q^{3k}(q^3-1)^k(q^2-1)^k$

Proof. The domain $R(\omega_{3k}) = V_1 \cup V_2 \cup ... \cup V_k = \cup_{i=1}^{k} V_i$ where $V_i =< e_{3i-2}, e_{3i-1}, e_{3i} >, i = 1, ..., k$ is invariant i.e. if $f \in Aut(\omega_{3k}), f(R(\omega_{3k})) \subset R(\omega_{3k})$. So that $f(V_i) \subset V_{\sigma(i)}$ for permutation $\sigma \in S_k$, we can define a groups homomorphism.

$\varphi : Aut(\omega_{3k}) \to S_k, \varphi(f) = \sigma$ where $\sigma = \begin{bmatrix} V_1 & V_2 & \dots & V_k \\ V_{\sigma(1)} & V_{\sigma(2)} & \dots & V_{\sigma(k)} \end{bmatrix}$

φ is surjective, we deduce that the sequence

$$1 \to ker\varphi \to Aut(\omega_{3k}) \to S_k \to 1$$

is exact.

Let $f \in ker\varphi$, then $f(V_i) \subset V_i, i = 1, .., k$ and from the equality $\wedge^3 f(\omega_{3k}) = \omega_{3k}$, hence $f(e_{3i-2}e_{3i-1}e_{3i}) = e_{3i-2}e_{3i-1}e_{3i}, i = 1, ..., k$

We can write the matrix of f as follows

$$M(f) = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{bmatrix}$$

with $det A_i = 1, i = 1, ..., k$ then $ker\varphi \simeq [SL_3(K)]^k$. □

The first Galois cohomology $H^1(G, Aut(\omega))$ where $G = Gal(\overline{K}/K)$, $\overline{K}$ is the algebraic closure of K, distinguishes forms over K and $\omega_{\overline{K}} \simeq \omega$.

We consider $\omega = \omega_{3k}$. If L is the extension of K, there exists a trivector $\omega_L \in \wedge^3 V$ such that $\omega_L \not\simeq \omega_{3k}$ and $\omega_L \otimes L \in \wedge^3(V \otimes_K L)$ is L-isomorphic to ω_{3k}. Let C be the set of orbits of the forms of ω_{3k}. Since $H^1(G, SL_3(K)) = 0$. The exact sequence of Galois cohomology sets gives us $C = H^1(G, S_k)$. Where $H^1(G, S_k) \simeq \{\text{Etale } K - \text{algebras of degree } k\}$.

We obtain the following lemma

Lemma 1. *The trivector ω_{3k} has a K-form ω_L with the automorphisms group $Aut(\omega_L)$ verify the following exact sequence:*

$$1 \to SL_3(L) \to Aut(\omega_L) \to \mathbb{Z}_k \to 1$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 February 2025 doi:10.20944/preprints202502.0378.v1

i.e.

$$Aut(\omega_L) \simeq SL_3(L) \rtimes \mathbb{Z}_k$$

If $K = \mathbb{F}_q$, $L = \mathbb{F}_{q^k}$ *and* $|Aut(\omega_L)| = kq^{3k}(q^{3k}-1)(q^{2k}-1)$.

Since $Aut(\omega_L)$ contain $SL_3(L)$ and $Aut(SL_n(\mathbb{F}_{q^k}))$ contain the finite cyclic group $Gal(\mathbb{F}_{q^k}/\mathbb{F}_q) \simeq \mathbb{Z}_k$, then $Aut(\omega_L)$ is semi-direct product $SL_3(L) \rtimes \mathbb{Z}_k$.

The only non trivial trivector over $\mathbb{F}_q$ with a transitive automorphisms group is the trivector arising from the three dimensional determinant over $\mathbb{F}_{q^k}$ (see J. Hora[6], page 11). So ω_L is the only trivector in dimension $3k$ with the transitive automorphisms group.

The trivector ω_L is the only trivector with equal ranks of all non zero vectors v, $rg(v) = 2k = 3k - dimRad(v), v \neq 0$. Its radical polynomial is equal to

$$P(\omega_L) = y^{3k} + (2^{2k}-1)x^{2k}y^k$$

We get the following Table 1.

Table 1. Automorphisms group and radical polynomial of ω_{3k}.

$dimV = 3k$, $k \geq 1$	Trivector	$P(\omega)$, $K = \mathbb{F}_2$	$Aut(\omega)$	$Aut(\omega)$ $K = \mathbb{F}_q$, $L = \mathbb{F}_{q^k}$
3	$\omega_3 = e_1e_2e_3$	$y^3 + 7x^2y$	$SL_3(K)$	$q^3(q^3-1)(q^2-1)$
6	$\omega_6 = e_1e_2e_3 + e_4e_5e_6$ ω_L	$(y^3+7x^2y)^2$ $y^6 + 63x^4y^2$	$[SL_3(K)]^2 \rtimes S_2$ $SL_3(L) \rtimes \mathbb{Z}_2$	$2q^6(q^3-1)^2(q^2-1)^2$ $2q^6(q^6-1)(q^4-1)$
9	$\omega_9 = e_1e_2e_3 + e_4e_5e_6 + e_7e_8e_9$ ω_L	$(y^3+7x^2y)^3$ $y^9 + 511x^6y^3$	$[SL_3(K)]^3 \rtimes S_3$ $SL_3(L) \rtimes \mathbb{Z}_3$	$6q^9(q^3-1)^3(q^2-1)^3$ $3q^9(q^9-1)(q^6-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3k$	$\omega_{3k} = e_1e_2e_3 + ... + e_{3k-2}e_{3k-1}e_{3k}$ ω_L	$(y^3+7x^2y)^k$ $y^{3k} + (2^{3k}-1)x^{2k}y^k$	$[SL_3(K)]^k \rtimes S_k$ $SL_3(L) \rtimes \mathbb{Z}_k$	$k!q^{3k}(q^3-1)^k(q^2-1)^k$ $kq^{3k}(q^{3k}-1)(q^{2k}-1)$

Remark 1. *If* $K = \mathbb{F}_2$, $L = \mathbb{F}_{2^3}$, *we have*
For $dimV = 3$, $\omega_3 = f_{3-1}$ *and* $|Aut(\omega_3)| = 168$
For $dimV = 6$, $\omega_6 = f_{6-1}$, $|Aut(\omega_6)| = 56448$ *and* $\omega_L = f_{6-3}$, $|Aut(\omega_L)| = 120960$
For $dimV = 9$, $\omega_9 = f_{9-007}$, $|Aut(\omega_9)| = 6.2^9(2^3-1)^3(2^2-1)^3 = 28449792$ *and* $\omega_L = f_{9-0004}$, $|Aut(\omega_L)| = 3.2^9(2^9-1)(2^6-1) = 49448448$. *(See J. Hora [6] Appendix A and B page 12-13).*

Remark 2. *The trivector* ω_9 *has three* K*-forms* ω_9, ω_L *and* $\omega_{L'}$ *in which* $L = K(t), t^3 = a$ *and* $L' = K' \times K, K' = K(\alpha)$ *where* $\alpha^2 = d(d \notin K^{*^2})$.

Proof. Let L be an extension of degree 3 of K and F is a space of dimension 3 over L. We consider a standard basis $\{e_1, e_2, e_3\}$ of F, the determinant form is defined by: $\varphi : \wedge^3 F \longrightarrow L$ where $\varphi(e_1 \wedge e_2 \wedge e_3) = 1$. The trace form is $Tr : L \longrightarrow K$, we put $\omega_L = Tr_L \circ \varphi : F^3 \longrightarrow K$, ω_L is a trivector of rank nine on $V = F$.
We take $L = K(t)$ with $t^3 = a$ in this case, the basis of V is
$\{e_1, e_2, e_3, e_4 = te_1, te_5 = te_2te_6 = te_3te_7 = t^2e_1, e_8 = t^2e_2, e_9 = t^3e_3\}$.
We can calculate: $\omega_L(e_1 \wedge e_2 \wedge e_3) = 3, \omega_L(e_4 \wedge e_5 \wedge e_6) = 3a, \omega_L(e_7 \wedge e_8 \wedge e_9) = 3a^2, \omega_L(e_1 \wedge e_5 \wedge e_9) = 3q, \omega_L(e_1 \wedge e_6 \wedge e_8) = -3a, \omega_L(e_2 \wedge e_4 \wedge e_9) = -3a, \omega_L(e_2 \wedge e_6 \wedge e_7) = 3a, \omega_L(e_9 \wedge e_4 \wedge e_8) = 3a, \omega_L(e_3 \wedge e_5 \wedge e_7) = -3a$ and $\omega_L(e_i \wedge e_j \wedge e_k) = 0$ otherwise. We obtain

$$\frac{1}{3}\omega_L = e_1e_2e_3 + ae_4e_5e_6 + a^2e_7e_8e_9 + ae_1e_5e_9 - ae_1e_6e_8 + ae_2e_4e_9 - ae_2e_6e_7 + ae_3e_4e_8 - ae_3e_5e_7$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 February 2025 doi:10.20944/preprints202502.0378.v1

and it follows that

$$\omega_L = e_1e_2e_3 + ae_4e_5e_6 + a^2e_7e_8e_9 + ae_1e_5e_9 - ae_1e_6e_8 + ae_2e_4e_9 - ae_2e_6e_7 + ae_3e_4e_8 - ae_3e_5e_7$$

If $L' = K' \times K$, $K' = K(\alpha)$ where $\alpha^2 = d(d \notin K^{*^2})$, is a quadratic extension of K, in this case, the basis of V is $\{e_1, e_2, e_3, e_4 = \alpha e_1, e_5 = \alpha e_2, e_6 = \alpha e_3, e_7, e_8, e_9\}$. Then,

$$\omega_L(e_1 \wedge e_2 \wedge e_3) = 3, \omega_L(e_7 \wedge e_8 \wedge e_9) = 3, \omega_L(e_1 \wedge e_2 \wedge e_5) = 3d, \omega_L(e_2 \wedge e_4 \wedge e_6) = -3d, \omega_L(e_3 \wedge e_4 \wedge e_5) = 3d,$$

and $\omega_L(e_i \wedge e_j \wedge e_k)$ otherwise.
We obtain

$$\frac{1}{3}\omega_L = e_1e_2e_3 + de_1e_5e_6 + de_2e_4e_6 + de_3e_4e_5 + e_7e_8e_9$$

and it follows that

$$\omega_{L'} = e_1e_2e_3 + de_1e_5e_6 + de_2e_4e_6 + de_3e_4e_5 + e_7e_8e_9 = \omega_{6,1,d} + e_7e_8e_9$$

If $L = K \times K \times K = K^3$ then $F = L^3 = K^9$, by the same method we obtain ω_9. □

If $K = \mathbb{F}_q$ finite field of order q, $L = \mathbb{F}_{q^3}$ and $L' = \mathbb{F}_{q^2} \times \mathbb{F}_q$.
We get the following Table 2.

Table 2. Automorphisms group and its size of ω_9

ω	Notation of Hora $K = \mathbb{F}_2$	$Aut(\omega)$ $K = \mathbb{F}_q$	$Aut(\omega)$	$Aut(\omega)$ $K = \mathbb{F}_2$
ω_9	f_7	$[SL_3(\mathbb{F}_q)]^3 \rtimes S_3$	$6q^9(q3-1)^3(q^2-1)^3$	28449792
ω_L	f_4	$SL_3(\mathbb{F}_{q^3}) \rtimes \mathbb{Z}_3$	$3q^9(q^9-1)(q^6-1)$	49448448
$\omega_{L'}$	f_{11}	$SL_3(\mathbb{F}_{q^2}) \times SL_3(\mathbb{F}_q) \rtimes \mathbb{Z}_2$	$2q^9(q^6-1)(q^4-1)\times$ $(q^3-1)(q^2-1)$	20321280

Remark 3. *In dimension 3k, the expression of the trivector ω_L seems to be a difficult problem (see Table 1.).*

4. Invariants on Dimension $2k+1$

4.1. General Rule for the Non Trivial Forms with an Isotropic Hyperplane

Let V be a vector space of dimension n, $n = 2k+1$ and let ω_{2k+1} be a trivector of rank $2k+1$. There exists a basis $B = \{e_1, e_2, e_3, ..., e_{2k}, e_{2k+1}\}$ such that

$$\omega_{2k+1} = e_1(e_2e_3 + e_4e_5 + ... + e_{2k}e_{2k+1})$$

It is the only trivector in dimension $2k+1, k \geq 2$ with an isotropic hyperplane (Hyperplane W such that the restriction on W is the zero form).

Proposition 2. *Let $Aut(\omega_3)$ be the automorphism of ω_{3k}, then it satisfies the exact sequence*

$$1 \to A' \to A \to K^* \to 1$$
$$1 \to K^{2k} \to A' \to Sp_{2k}(K) \to 1$$

i.e.

$$Aut(\omega_{2k+1}) \simeq Sp_{2k}(K) \times K^* \times K^{2k}$$

If $K = \mathbb{F}_q$, then $|Aut(\omega_{2k+1})| = q^{k^2+2k}(q-1)\prod_{i=1}^{k}(q^{2i}-1)$

Proof. We consider the set $V_1 = \{x \in V / x.\omega = 0\}$. If $x = \sum_{i=1}^{2k+1} \alpha_i e_i \in V_1$, we have $x = \alpha_1 e_1$, then $V_1 =< e_1 >$ and $\wedge^4 f(x\omega) = f(x) \wedge^3 f(\omega) = f(x)\omega = 0$, for $f \in A = Aut(\omega_{2k+1})$. Then $f(V_1) \subset V_1$ and there exists $\lambda \in K^* \; / \; f(e_1) = \lambda e_1$. The matrix of f with a basis $\{e_1, ..., e_{3k}\}$ must have the form $\begin{bmatrix} \lambda & X \\ 0 & B \end{bmatrix}$, this we define a group of homomorphism surjective $\varphi : A \to K^*, \varphi(f) = \lambda$.
Let $f \in A' = Ker\varphi$: $e_1 \wedge^2 f(e_2e_3 + ... + e_{2k}e_{2k+1}) = e_1(e_2e_3 + ... + e_{2k}e_{2k+1})$, which means $\wedge^2 f(e_2e_3 + ... + e_{2k}e_{2k+1}) = e_2e_3 + ... + e_{2k}e_{2k+1} + xe_1$ with $x \in V_2 =$ $< e_2, ..., e_{2k+1} >$. We consider $g : V_2 \longrightarrow V_2$ the linear application with the matrix $B : \wedge^2 g(e_2e_3 + ... + e_{2k}e_{2k+1}) = e_2e_3 + ... + e_{2k}e_{2k+1}$. The homomorphism $\psi : A' \to Sp_{2k}(K)$ defined by $\psi(f) = B$, is surjective and $ker\psi \simeq K^{2k}$. □

While observing the radical polynomial $P(\omega)$ of ω_5, ω_7 and ω_9, we deduce

The radical polynomial of ω_{2k+1} is equal to

$$P(\omega_{2k+1}) = y^{2k+1} + (2^{2k} - 1)x^2y^{2k-1} + 2^{2k}x^{2k}y$$

We get the following Table 3.

Table 3. Automorphisms group and radical polynomial of ω_{2k+1}

$dimV = 2k+1$, $k \geq 2$	Trivector	$P(\omega), K = \mathbb{F}_2$	$Aut(\omega)$ K arbitrary field	$Aut(\omega)$ $K = \mathbb{F}_q$
5	$\omega_5 = e_1(e_2e_3 + e_4e_5)$	$y^5 + 15x^2y^3 + 16x^4y$	$Sp_4(K) \times K^* \times K^4$	$q^8(q-1)\prod_{i=1}^{2}(q^{2i}-1)$
7	$\omega_7 = e_1(e_2e_3 + e_4e_5 + e_6e_7)$	$y^7 + 63x^2y^5 + 64x^6y$	$Sp_6(K) \times K^* \times K^6$	$q^{15}(q-1)\prod_{i=1}^{3}(q^{2i}-1)$
9	$\omega_9 = e_1(e_2e_3 + e_4e_5 + e_6e_7 + e_8e_9)$	$y^9 + 255x^2y^7 + 256x^8y$	$Sp_8(K) \times K^* \times K^8$	$q^{24}(q-1)\prod_{i=1}^{4}(q^{2i}-1)$
⋮	⋮	⋮	⋮	⋮
$2k+1$	$\omega_{2k+1} = e_1(e_2e_3 + ... + e_{2k}e_{2k+1})$	$y^{2k+1} + (2^{2k}-1)x^2y^{2k-1} + 2^{2k}x^{2k}y$	$Sp_{2k}(K) \times K^* \times K^{2k}$	$q^{k^2+2k}(q-1)\prod_{i=1}^{k}(q^{2i}-1)$

Remark 4. *If $K = \mathbb{F}_2$ finite field of order 2, we have:*
For $dimV = 5$, $\omega_5 = f_{5-2}$ and $|Aut(\omega_5)| = 11520$
For $dimV = 7$, $\omega_7 = f_{7-4}$ and $|Aut(\omega_7)| = 92897280$
For $dimV = 9$, $\omega_9 = f_{9-1}$ and $|Aut(\omega_9)| = 2^{24}(2-1)\prod_{i=1}^{4}(2^{2i}-1) = 12128668876800$, it is the class with the largest group of automorphisms. (See J. Hora [6] Appendix A and B page 12-13).

4.2. General Rule for the Form of the Type $\omega_{2k+1} = e_1e_2e_3 + e_3e_4e_5 + e_5e_6e_7 + ... + e_{2k-1}e_{2k}e_{2k+1}$

Let V be a vector space of dimension n, $n = 2k+1$ and let ω_{2k+1} be a trivector of rank $2k+1$. There exists a basis $B = \{e_1, e_2, e_3, ..., e_{2k}, e_{2k+1}\}$ such that $\omega_{2k+1} = e_1e_2e_3 + e_3e_4e_5 + e_5e_6e_7 + ... + e_{2k-1}e_{2k}e_{2k+1}$

Proposition 3. *Let $Aut(\omega_3)$ be the automorphism of ω_{3k}, then it satisfies the exact sequence*

$$1 \to A' \to A \to \mathbb{Z}_2 \to 1$$

$$1 \to A'' \to A' \to (K^*)^{k-1} \to 1$$

$$1 \to K^{3k+1} \to A'' \to SL_2(K) \times SL_2(K) \to 1$$

i.e.

$$A = Aut(\omega_{2k+1}) = (SL_2(K))^2 \times (K^*)^{k-1} \times K^{3k+1} \times \mathbb{Z}_2$$

If $K = \mathbb{F}_q$, $|Aut(\omega_{2k+1})| = 2q^{3k+3}(q^2-1)^2(q-1)^{k-1}$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 February 2025 doi:10.20944/preprints202502.0378.v1

Proof. We observe the domain (linear span) $R =< e_3, e_5, e_7, ..., e_{2k+1} = Ke_3 \cup Ke_5 \cup ... \cup Ke_{2k-1}$ is invariant. We obtain

$$1 \to A' \to A \xrightarrow{\varphi} \mathbb{Z}_2 \to 1$$

$$\varphi(f) = \begin{cases} 1 & \text{if } f(e_3) \subset e_3, f(e_5) \subset e_5,..., f(e_{2k+1}) \subset e_{2k+1} \\ 0 & \text{otherwise} \end{cases}$$

Then $A' = \{f/f \subset A, f(e_3) \subset e_3, f(e_5) \subset e_5, ..., f(e_{2k+1}) \subset e_{2k+1}\}$ therefore, $\wedge^{k+1} f$ stabilizer each of the two subspaces $F_1 = \{e_3, e_5, e_7, .., e_{2k-1}, e_{2k}, e_{2k+1}\}$ and
$F_2 = \{e_1, e_2, e_3, e_5, e_7, ..., e_{2k-1}\}$. We can define a group homomorphisms

$$\psi : A' \longrightarrow (K^*)^{k-1}, \psi(f) = (f(e_3)/e_3, f(e_5)/e_5, ..., f(e_{2k-1})/e_{2k-1})$$

Then

$$1 \to A'' \to A' \to (K^*)^{k-1} \to 1$$

If $f \in A'' = ker\psi$, f acts on e_1e_2 and $e_{2k}e_{2k+1}$ as $SL_2(K) \times SL_2(K)$.

We obtain the exact sequence

$$1 \to K^{3k+1} \to A'' \to SL_2(K) \times SL_2(K) \to 1$$

If f is the identity over e_1, e_2, e_{2k} and e_{2k+1} (modulo $P = \{e_3, e_5, ..., e_{2k-1}\}$),
$f(e_i) = e_i + v_i, i \in \{1, 2, 2k, 2k+1\}$ where $v_i \in P$, then
$f(e_4) = e_4 + v_4$, $f(e_6) = e_6 + v_6$,...,$f(e_{2k-2}) = e_{2k-2} + v_{2k-2}$
A'' is an additive group isomorphic to K^{3k+1} $\square$

We get the following Table 4.

Table 4. Automorphisms group and its size of ω_{2k+1}

$dimV = 2k+1$ $k \geq 3$	trivector	$Aut(\omega)$, K arbitrary field	$\lvert Aut(\omega)\rvert$, $K = \mathbb{F}_q$
7	ω_7	$[SL_2(K)]^2 \times (K^*)^2 \times K^{10} \times \mathbb{Z}_2$	$2q^{12}(q^2-1)^2(q-1)^2$
9	ω_9	$[SL_2(K)]^2 \times (K^*)^3 \times K^{13} \times \mathbb{Z}_2$	$2q^{15}(q^2-1)^2(q-1)^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k+1$	ω_{2k+1}	$[SL_2(K)]^2 \times (K^*)^{k-1} \times K^{3k+1} \times \mathbb{Z}_2$	$2q^{3k+3}(q^2-1)^2(q-1)^{k-1}$

Remark 5. ***Remark 4.*** *If $K = \mathbb{F}_2$, we have*
For $dimV = 7$, $\omega_7 = f_{7-1}$, $|Aut(\omega_7)| = 73728$
For $dimV = 9$, $\omega_9 = f_{9-23}$, $|Aut(\omega_9)| = 2.2^{15}(2^2-1)^2(2-1)^3 = 589824$ (See J. Hora [6] Appendix A and B page 12-13).

5. Commutant of a Trivector and Frobenius Algebra

Proposition 4. *There exists a vector space V of dimension nine and $\omega \in \wedge^3 V$ in which $C(\omega)$ is not a F.algebra and $\dim V < 3 \dim C(\omega)$.*

Proof. Let ω be a trivector of rank nine (See [3] Table 1):

$$\omega = e_1e_2e_9 + e_1e_3e_5 + e_1e_4e_6 + e_2e_3e_7 + e_2e_4e_8 + e_3e_4e_9$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 February 2025 doi:10.20944/preprints202502.0378.v1

We consider f, an element of the commutant and M_B, the matrix of f, where B is the standard basis, then by direct competitions we have:

$$\begin{aligned}
\omega(f(e_1), e_2, e_9) &= \omega(e_1, f(e_2), e_9) = \omega(e_1, e_2, f(e_9)) = a_{11} = a_{22} = a_{99} \\
\omega(f(e_1), e_3, e_5) &= \omega(e_1, f(e_3), e_5) = \omega(e_1, e_3, f(e_5)) = a_{11} = a_{33} = a_{55} \\
\omega(f(e_1), e_4, e_6) &= \omega(e_1, f(e_4), e_6) = \omega(e_1, e_4, f(e_6)) = a_{11} = a_{44} = a_{66} \\
\omega(f(e_2), e_3, e_7) &= \omega(e_2, f(e_3), e_7) = \omega(e_2, e_4, f(e_7)) = a_{22} = a_{33} = a_{77} \\
\omega(f(e_2), e_4, e_8) &= \omega(e_2, f(e_4), e_8) = \omega(e_2, e_4, f(e_8)) = a_{22} = a_{44} = a_{88} \\
\omega(f(e_3), e_4, e_9) &= \omega(e_3, f(e_4), e_9) = \omega(e_3, e_4, f(e_9)) = a_{33} = a_{44} = a_{99}
\end{aligned}$$

Then,

$$a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = a_{88} = a_{99} = \alpha$$

Also we have,

$$\begin{aligned}
\omega(f(e_1), e_2, e_3) &= \omega(e_1, f(e_2), e_3) = \omega(e_1, e_2, f(e_3)) \\
&\Rightarrow a_{71} = -a_{52} = a_{93} = \beta \\
\omega(f(e_1), e_2, e_4) &= \omega(e_1, f(e_2), e_4) = \omega(e_1, e_2, f(e_4)) \\
&\Rightarrow a_{81} = -a_{62} = a_{94} = \gamma \\
\omega(f(e_2), e_3, e_4) &= \omega(e_2, f(e_3), e_4) = \omega(e_2, e_3, f(e_4)) \\
&\Rightarrow a_{92} = -a_{83} = a_{74} = \tau \\
\omega(f(e_1), e_3, e_4) &= \omega(e_1, f(e_3), e_4) = \omega(e_1, e_3, f(e_4)) \\
&\Rightarrow a_{91} = -a_{63} = a_{54} = \mu
\end{aligned}$$

$$\omega(f(e_i), e_j, e_k) = \omega(e_i, f(e_j), e_k) = \omega(e_i, e_j, f(e_k)) = 0 \Rightarrow a_{ij} = 0, \text{ for the other cases.}$$

$M_B(f)$ has the form

$$M_B(f) = \begin{pmatrix}
\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta & 0 & \mu & \alpha & 0 & 0 & 0 & 0 \\
0 & -\gamma & -\mu & 0 & 0 & \alpha & 0 & 0 & 0 \\
\beta & 0 & 0 & \tau & 0 & 0 & \alpha & 0 & 0 \\
\gamma & 0 & -\tau & 0 & 0 & 0 & 0 & \alpha & 0 \\
\mu & \tau & \beta & \gamma & 0 & 0 & 0 & 0 & \alpha
\end{pmatrix},$$

thus $C(\omega) = K \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_4$ with $V_i = K\varepsilon_i, i = 1, \ldots, 4$ and the matrices of ε_1, ε_2, ε_3 and ε_4 are presented by

$$M_B(\varepsilon_1) = \left(\begin{array}{c|c}
0_{4\times 4} & 0_{4\times 5} \\
\hline
\begin{matrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} & 0_{5\times 5}
\end{array}\right),$$

 doi:10.20944/preprints202502.0378.v1

$$M_B(\varepsilon_2) = \left(\begin{array}{c|c} 0_{4\times 4} & 0_{4\times 5} \\ \hline \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} & 0_{5\times 5} \end{array}\right),$$

$$M_B(\varepsilon_3) = \left(\begin{array}{c|c} 0_{4times4} & 0_{4\times 5} \\ \hline \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} & 0_{5\times 5} \end{array}\right),$$

and

$$M_B(\varepsilon_4) = \left(\begin{array}{c|c} 0_{3\times 4} & 0_{3\times 5} \\ \hline \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & 0_{6\times 5} \end{array}\right).$$

By computation we prove that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 satisfy:

$\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = \varepsilon_4^2 = \varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_1\varepsilon_4 = \varepsilon_2\varepsilon_1 = \varepsilon_3\varepsilon_1 = \varepsilon_4\varepsilon_1 = \varepsilon_2\varepsilon_3 = \varepsilon_2\varepsilon_4 = \varepsilon_3\varepsilon_2 = \varepsilon_4\varepsilon_2 = \varepsilon_3\varepsilon_4 = \varepsilon_4\varepsilon_3 = 0$.

If $C(\omega)$ is a Frobenius algebra, there exists a non-degenerate symmetric bilinear form $\phi : C(\omega) \times C(\omega) \to K$ in which $\phi(fg,h) = \phi(f,gh)$ where $f,g,h \in C(\omega)$. We put $C = M_B(\phi) = (c_{ij})$ the matrix of ϕ in the basis $B = \{1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, then

$$
\begin{aligned}
\phi(\varepsilon_1,\varepsilon_2) &= \phi(1.\varepsilon_1,\varepsilon_2) = \phi(1,\varepsilon_1\varepsilon_2) = \phi(\varepsilon_3.\varepsilon_1,\varepsilon_2) = \phi(\varepsilon_3,\varepsilon_1\varepsilon_2)\\
&= \phi(\varepsilon_4.\varepsilon_1,\varepsilon_2) = \phi(\varepsilon_4,\varepsilon_1\varepsilon_2) = 0 \text{ imply that } c_{23} = 0\\
\phi(\varepsilon_2,\varepsilon_1) &= \phi(1.\varepsilon_2,\varepsilon_1) = \phi(1,\varepsilon_2\varepsilon_1) = \phi(\varepsilon_3.\varepsilon_2,\varepsilon_1) = \phi(\varepsilon_3,\varepsilon_2\varepsilon_1)\\
&= \phi(\varepsilon_4.\varepsilon_2,\varepsilon_1) = \phi(\varepsilon_4,\varepsilon_1\varepsilon_2) = 0 \text{ imply that } c_{32} = 0\\
\phi(\varepsilon_1,\varepsilon_3) &= \phi(1.\varepsilon_1,\varepsilon_3) = \phi(1,\varepsilon_1\varepsilon_3) = \phi(\varepsilon_2.\varepsilon_1,\varepsilon_3) = \phi(\varepsilon_2,\varepsilon_1\varepsilon_3)\\
&= \phi(\varepsilon_4.\varepsilon_1,\varepsilon_3) = \phi(\varepsilon_4,\varepsilon_1\varepsilon_3) = 0 \text{ imply that } c_{24} = 0\\
\phi(\varepsilon_3,\varepsilon_1) &= \phi(1.\varepsilon_3,\varepsilon_1) = \phi(1,\varepsilon_3\varepsilon_1) = \phi(\varepsilon_2.\varepsilon_3,\varepsilon_1) = \phi(\varepsilon_2,\varepsilon_3\varepsilon_1)\\
&= \phi(\varepsilon_4.\varepsilon_3,\varepsilon_1) = \phi(\varepsilon_4,\varepsilon_3\varepsilon_1) = 0 \text{ imply that } c_{42} = 0\\
\phi(\varepsilon_1,\varepsilon_4) &= \phi(1.\varepsilon_1,\varepsilon_4) = \phi(1,\varepsilon_1\varepsilon_4) = \phi(\varepsilon_2.\varepsilon_1,\varepsilon_4) = \phi(\varepsilon_2,\varepsilon_1\varepsilon_4)\\
&= \phi(\varepsilon_3.\varepsilon_1,\varepsilon_4) = \phi(\varepsilon_3,\varepsilon_1\varepsilon_4) = 0 \text{ imply that } c_{25} = 0\\
\phi(\varepsilon_4,\varepsilon_1) &= \phi(1.\varepsilon_4,\varepsilon_1) = \phi(1,\varepsilon_4\varepsilon_1) = \phi(\varepsilon_2.\varepsilon_4,\varepsilon_1) = \phi(\varepsilon_2,\varepsilon_4\varepsilon_1)\\
&= \phi(\varepsilon_3.\varepsilon_4,\varepsilon_1) = \phi(\varepsilon_3,\varepsilon_4\varepsilon_1) = 0 \text{ imply that } c_{52} = 0\\
\phi(\varepsilon_2,\varepsilon_3) &= \phi(1.\varepsilon_2,\varepsilon_3) = \phi(1,\varepsilon_2\varepsilon_3) = \phi(\varepsilon_1.\varepsilon_2,\varepsilon_3) = \phi(\varepsilon_1,\varepsilon_2\varepsilon_3)\\
&= \phi(\varepsilon_4.\varepsilon_2,\varepsilon_3) = \phi(\varepsilon_4,\varepsilon_2\varepsilon_3) = 0 \text{ imply that } c_{34} = 0\\
\phi(\varepsilon_3,\varepsilon_2) &= \phi(1.\varepsilon_3,\varepsilon_2) = \phi(1,\varepsilon_3\varepsilon_2) = \phi(\varepsilon_1.\varepsilon_3,\varepsilon_2) = \phi(\varepsilon_1,\varepsilon_3\varepsilon_2)\\
&= \phi(\varepsilon_4.\varepsilon_3,\varepsilon_2) = \phi(\varepsilon_4,\varepsilon_3\varepsilon_2) = 0 \text{ imply that } c_{43} = 0\\
\phi(\varepsilon_2,\varepsilon_4) &= \phi(1.\varepsilon_2,\varepsilon_4) = \phi(1,\varepsilon_2\varepsilon_4) = \phi(\varepsilon_1.\varepsilon_2,\varepsilon_4) = \phi(\varepsilon_1,\varepsilon_2\varepsilon_4)\\
&= \phi(\varepsilon_3.\varepsilon_2,\varepsilon_4) = \phi(\varepsilon_3,\varepsilon_2\varepsilon_4) = 0 \text{ imply that } c_{35} = 0\\
\phi(\varepsilon_4,\varepsilon_2) &= \phi(1.\varepsilon_4,\varepsilon_2) = \phi(1,\varepsilon_4\varepsilon_2) = \phi(\varepsilon_1.\varepsilon_4,\varepsilon_2) = \phi(\varepsilon_1,\varepsilon_4\varepsilon_2)\\
&= \phi(\varepsilon_3.\varepsilon_4,\varepsilon_2) = \phi(\varepsilon_3,\varepsilon_4\varepsilon_2) = 0 \text{ imply that } c_{53} = 0\\
\phi(\varepsilon_3,\varepsilon_4) &= \phi(1.\varepsilon_3,\varepsilon_4) = \phi(1,\varepsilon_3\varepsilon_4) = \phi(\varepsilon_1.\varepsilon_3,\varepsilon_4) = \phi(\varepsilon_1,\varepsilon_3\varepsilon_4)\\
&= \phi(\varepsilon_2.\varepsilon_3,\varepsilon_4) = \phi(\varepsilon_2,\varepsilon_3\varepsilon_4) = 0 \text{ imply that } c_{45} = 0\\
\phi(\varepsilon_4,\varepsilon_3) &= \phi(1.\varepsilon_4,\varepsilon_3) = \phi(1,\varepsilon_4\varepsilon_3) = \phi(\varepsilon_1.\varepsilon_4,\varepsilon_3) = \phi(\varepsilon_1,\varepsilon_4\varepsilon_3)\\
&= \phi(\varepsilon_2.\varepsilon_4,\varepsilon_3) = \phi(\varepsilon_2,\varepsilon_4\varepsilon_3) = 0 \text{ imply that } c_{54} = 0\\
\phi(\varepsilon_1,\varepsilon_1) &= \phi(1.\varepsilon_1,\varepsilon_1) = \phi\left(1,\varepsilon_1^2\right) = 0 \text{ imply that } c_{22} = 0\\
\phi(\varepsilon_2,\varepsilon_2) &= \phi(1.\varepsilon_2,\varepsilon_2) = \phi\left(1,\varepsilon_2^2\right) = 0 \text{ imply that } c_{33} = 0\\
\phi(\varepsilon_3,\varepsilon_3) &= \phi(1.\varepsilon_1,\varepsilon_3) = \phi\left(1,\varepsilon_3^2\right) = 0 \text{ imply that } c_{44} = 0\\
\phi(\varepsilon_4,\varepsilon_4) &= \phi(1.\varepsilon_4,\varepsilon_4) = \phi\left(1,\varepsilon_4^2\right) = 0 \text{ imply that } c_{55} = 0
\end{aligned}
$$

The matrix C is represented by

$$
C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15}\\ c_{21} & 0 & 0 & 0 & 0\\ c_{31} & 0 & 0 & 0 & 0\\ c_{41} & 0 & 0 & 0 & 0\\ c_{51} & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

The rank of $\phi = 2$, different from 5, hence this contradicts the non-degeneracy of ϕ. We conclude that $C(\omega)$ is not a F.algebra and $\dim C(\omega) = 5$. We deduce that $\dim V < 3 \dim C(\omega)$. $\square$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 February 2025 doi:10.20944/preprints202502.0378.v1

6. Classification of Trivectors in Dimension 8 over a Finite Field of Characteristic 2

For $n = 8$, the classification of trivectors over finite fields, except for characteristic 2 or 3, and over a finite field of two elements has been done in [7] and [5] respectively. More recently a classifications have appeared for a finite field of characteristic 3 [9]. We give the following classification over $\mathbb{F}_{2^m}$.

Theorem 1. *Let V be a vector space of dimension eight over a finite field $\mathbb{F}_q$ of characteristic 2, $q = 2^m$. If m is odd, there are 20 inequivalent trivectors in $\wedge^3 V$ which are of a full rank.*

Table 5. The cardinality of the automorphisms groups $Aut(\omega_i)$ for trivectors ω_i of rank 8 over $\mathbb{F}_{2^m}$. $d \in (\mathbb{F}_q^*)^2$, $a \notin (\mathbb{F}_q^*)^3$

$\omega_{8,i}$	Trivector	Notation of Hora	$\lvert Aut(\omega_{8,i})\rvert$ over $\mathbb{F}_q$	$\lvert Aut(\omega_{8,i})\rvert$ over $\mathbb{F}_2$
$\omega_{8,1}$	$e_1(e_2e_3 + e_4e_5) + e_6e_7e_8$	f_{8-01}	$q^{11}(q^4-1)(q^3-1)(q^2-1)^2(q-1)$	1935360
$\omega_{8,2}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7) + e_5e_6e_8$	f_{8-02}	$q^{18}(q^2-1)^2(q-1)^2$	2359296
$\omega_{8,3}$	$e_1(e_3e_4 + e_5e_6) + e_2(e_3e_5 + e_7e_8)$	f_{8-03}	$q^{11}(q^2-1)^2(q-1)^2$	18432
$\omega_{8,4}$	$e_1(e_2e_3 + e_4e_5) + e_6(e_2e_7 + e_4e_8)$	f_{8-04}	$2q^{14}(q^2-1)^2(q-1)^2$	294912
$\omega_{8,4,d}$	$e_1(e_2e_5 + e_3e_7) + e_4(e_2e_5 + e_5e_7 + de_2e_3)+$ $e_6(e_3e_5) + e_5(e_3e_4) + e_5(e_3e_4)$	f_{8-12}	$2q^{14}(q^4-1)(q^2-1)$	1474560
$\omega_{8,5}$	$e_1(e_2e_3 + e_4e_5) + e_6(e_2e_3 + e_7e_8)$	f_{8-20}	$6q^9(q^2-1)^3(q-1)$	82944
$\omega_{8,5,d}$	$e_3(e_1e_2 + e_4e_7 + e_6e_8) + e_5(e_1e_4 + e_8e_2 + de_6e_7)$	f_{8-19}	$2q^9(q^4-1)(q^2-1)(q-1)$	46080
$\omega_{8,5,a}$	$e_1(ae_3e_4 + ae_5e_6 + e_7e_8) + e_2(e_3e_5 + e_4e_7 + e_6e_8)$	f_{8-13}	$3q^9(q^6-1)(q-1)$	96768
$\omega_{8,6}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7) + e_8(e_4e_3 + e_5e_6)$	f_{8-07}	$q^{14}(q^2-1)(q-1)^2$	49152
$\omega_{8,7}$	$e_1(e_2e_3 + e_4e_6 + e_5e_7) + e_2(e_5e_6 + e_7e_8)$	f_{8-05}	$q^{17}(q^2-1)(q-1)^2$	393216
$\omega_{8,8}$	$e_1(e_2e_8 + e_3e_6 + e_4e_7) + e_6e_7e_8 + e_3e_4e_5$	f_{8-06}	$q^{10}(q^2-1)(q-1)^2$	3072
$\omega_{8,9}$	$e_1[e_2(e_3+e_4) + e_5e_6] + e_3e_5e_7 + e_4e_6e_8$	f_{8-18}	$6q^9(q-1)^3$	3072
$\omega_{8,9,d}$	$e_1(e_3e_4 + de_5e_6) + e_2(e_3e_6 + e_4e_5 + e_5e_6)+$ $e_3(e_1e_2 + e_6e_8)$	f_{8-17}	$2q^9(q^2-1)(q-1)$	3072
$\omega_{8,9,a}$	$e_2[(e_3+e_4)e_1 + e_7e_6] + e_8[a(e_6+e_7)e_1 + e_3e_4]$ $+e_5(e_3e_7 + e_6e_4)$	f_{8-14}	$3q^9(q^3-1)$	10752
$\omega_{8,10}$	$e_1(e_2e_8 + e_6e_7) + e_2e_3e_5 + e_3e_4e_6 + e_4e_5e_7$	f_{8-16}	$2q^7(q^2-1)(q-1)^2$	768
$\omega_{8,10,d}$	$e_2(e_7e_8 + de_1e_3) + e_5(e_1e_4 + e_6e_3) + e_4e_6e_7$	f_{8-08}	$2q^7(q^2-1)^2$	2304
$\omega_{8,11}$	$e_1(e_3e_7 + e_5e_4 + e_8e_2) + e_8(e_4e_3 + e_6e_7) + e_2e_4e_6$	f_{8-09}	$q^{13}(q^2-1)(q-1)$	24576
$\omega_{8,12}$	$e_1[(e_4-e_7)(e_3-e_8) + e_5e_7] + e_2(e_3e_4 + e_5e_6)+$ $e_6e_7e_8$	f_{8-10}	$q^6(q^2-1)(q-1)$	192
$\omega_{8,13}$	$e_1[e_5(e_3-e_7) + e_8e_4] + e_2(e_3e_4 + e_5e_6) + e_6e_7e_8$	f_{8-11}	$2q^3(q^3-1)(q^2-1)$	336
$\omega_{8,13,d}$	$e_1(e_2e_5 + e_3e_7) + e_4(e_2e_5 + e_5e_7 + de_2e_3)+$ $e_6(e_3e_5) + e_5(e_3e_4) + e_7(e_3e_4 + e_8e_1)$	f_{8-15}	$2q^3(q^3+1)(q^2-1)$	432

Remark 6. *This classification was done over $\mathbb{F}_2$ (see table 2 page 3468 in [5] or see Appendix A page 12 in [6]).*

7. Weight Varieties of a Non-Degenerate Form

We can use the classification of trivectors in the theory of codes (See [13]).

Some undefined terms can be found in [13, page 426-429].

Similar arguments applied in [13] are used for determining the varieties $X_1(\omega_{8,i})$ and $X_2(\omega_{8,i})$ for some trivectors and we have:

Proposition 5. *The varieties $X_1(\omega_{8,i})$ and $X_2(\omega_{8,i})$ for $7 \le i \le 11$ are given by:*

$\omega_{8,i}$	$X_1(\omega_{8,i})$	$X_2(\omega_{8,i})$
$\omega_{8,7}$	$\{x_3 = x_4 = x_5 = x_7 = x_1x_8 + x_6^2 = 0\}$	$\{x_5 = x_7 = 0\}$
$\omega_{8,8}$	$\{x_1 = x_2 = x_4 = x_5 = x_6 = x_7 = x_8 = 0\} \cup \{x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0\}$	$x_5 = x_8 = x_4x_6 - x_2x_7 = 0\} \cup \{x_4 = x_7 = x_5^2 - x_1x_8 = 0\}$
$\omega_{8,9}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_8 = 0\}$	$\{x_3x_5 + x_2x_6 + x_7x_8 = x_1 = x_4 = 0\} \cup \{x_1x_4 + x_8^2 = x_1x_3 - x_6x_8 = x_1x_2 + x_5x_8 = x_4x_6 + x_3x_8 = x_4x_5 - x_2x_8 = x_3x_5 + x_2x_6 = 0\}$
$\omega_{8,10}$	$\{x_1 = x_4 = x_5 = x_6 = x_7 = x_8 = x_2 - x_3 = 0\} \cup \{x_1 = x_4 = x_5 = x_6 = x_7 = x_8 = x_2 + x_3 = 0\} \cup \{x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0\}$	$\{x_5 - x_6 = x_7 - x_8 = x_4x_6 - x_2x_7 + x_3x_7 = 0\} \cup \{x_5 + x_6 = x_7 + x_8 = x_4x_6 - x_2x_7 - x_3x_7 = 0\} \cup \{x_7 = x_4 = x_5^2 - x_6^2 - x_1x_8 = 0\}$
$\omega_{8,11}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0\}$	$\{x_3 = x_5 = x_1x_4 - x_7^2 = 0\}$

8. Conclusions

In this paper, by using the invariants of the trivectors, we deduce the general rule for some trivectors. As a future work, one can calculate the cardinalities of $X_1(\omega_{8,i})$ and $X_2(\omega_{8,i})$ and use them to fully determine the spectrum of $C(3,8)$.

References

1. A. M. COHEN and A. G. HELMINCK: Trilinear alternating forms on a vector space of dimension 7,*Communications in algebra.*, **16** (1988) 1–25.
2. D. DJOKOVIC : Classification of trivectors of an eight dimensional real vector space, *Linear Multilinear Algebr.*, **13** (1983) 3–39..
3. A.G. Elashvili and E.B. Vinberg. *Classification des trivecteurs d'un espace de dimension 9 (en russe)*. Trudy sem.Vekt ,Tenz. Anlisu XVIII **55** (1978), 197–233.
4. G. B. GUREVITCH : Foundations of the Theory of Algebraic Invariants, *P. Noordhoff Ltd.,Groningen, the Netherlands* (1964).
5. J. HORA and P. PUDLÁK: Classification of 8-dimensional trilinear alternating forms over $GF(2)$, *Commun Algebra.*, **34** (2015) 3459–3471.
6. J. HORA and P. PUDLÁK:Classification of 9-dimensional trilinear alternating forms over $GF(2)$, *Finite Fields and Their Applications.*, **70** (2021) 101788.
7. N. MIDOUNE and L. NOUI: Trilinear alternating forms on a vector space of dimension 8 over a finite field, *Linear and Multilinear Algebra.*, **61** (2013) 15–21.
8. N. Midoune, L. Noui. *K-forms of 2-step splitting trivectors*. Int. J. Algebra **2** (2008), 369–382.
9. N. Midoune , M.A. Rakdi . *Trilinear alternating forms and related CMLs and GECs*. International Journal of Group Theory **12** (2023), 227–235.
10. L. Noui and Ph. Revoy. *Formes multilinéaires alternées*. Linear and Multilinear Algebra **1** (1994), 43–69. Blaise Pascal 1: 43-69.
11. L. NOUI: Transvecteur de rang 8 sur un corps algébriquement clos,*Comptes Rendus de l'Académie des Sciences-Series I-Mathematics* , **324** (1997) 611–614.
12. E.O'Brien ,P. Vojtěchovskỳ *Code loops in dimension at most 8*. Journal of Algebra 2017, **473**, 607–626.
13. M.A. Rakdi , N. Midoune. *Weights of the Fq-forms of 2-step splitting trivectors of rank 8 over a finite field*. Carpathian Mathematical Publications **11** (2019), 422–430.
14. J. Schouten. *Klassifizierung der alternierenden Gröszen dritten Grades in 7 Dimensionen*. Rendiconti del Circolo Matematico di Palermo (1884-1940) **55**(1931), 137–156.

 doi:10.20944/preprints202502.0378.v1

15. G. Tang, D. H. Duong, A. Joux, T. Plantard, Y. Qiao, and W. Susilo. Practical post-quantum signature schemes from isomorphism problems of trilinear forms. In O. Dunkelman and S. Dziembowsk, editors,*Advances in Cryptology - EURO- Crypt 2022 - 41st Annual International Conference on the Theory and Applications of Cryptographic Techniques, Trondheim, Norway, May 30 - June 3, 2022, Proceedings*, Part III, volume 13277 of *Lecture Notes in Computer Science*, pages 582-612, Springer, 2022.
16. Borovoi, Mikhail, Willem A. de Graaf, and Hông Vân Lê. "Classification of real trivectors in dimension nine." Journal of Algebra 603 (2022), 118-163.