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Not peer-reviewed version

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Posted Date: 19 May 2025

doi: 10.20944/preprints202505.1432.v1

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On the Generalized Dirichlet Divisor Problem

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Abstract: Using more advanced results on the growth exponent for Riemann zeta–function and accurate numerical estimations, we obtain better upper bounds for α_k ($9 \leqslant k \leqslant 20$) on the generalized Dirichlet divisor problem. This gives a minor improvement upon the recent result of Trudgian and Yang.

Keywords: Dirichlet divisor problem; Riemann zeta–function; large value theorem

MSC: 11M06; 11N37; 11N56

1. Introduction

Let $k \geqslant 2$ denotes an integer and $d_k(n)$ is the divisor function that represents the number of ways n may be written as a product of exactly k factors. The generalized Dirichlet divisor problem consists of the estimation of the function

$$\Delta_k(x) = \sum_{n \leqslant x} d_k(n) - xP_{k-1}(\log x), \tag{1}$$

where P_{k-1} is an explicit polynomial of degree $k-1$. Clearly we have $\Delta_k(x) = o(x)$. We then define α_k as the least exponent for which

$$\Delta_k(x) \ll x^{\alpha_k + \varepsilon}. \tag{2}$$

In 1916, Hardy [1] first proved a lower bound that $\alpha_k \geqslant \frac{1}{2} - \frac{1}{2k}$ for all $k \geqslant 2$. The generalized Dirichlet divisor problem conjecture states that $\alpha_k = \frac{1}{2} - \frac{1}{2k}$ holds for all $k \geqslant 2$, and this conjecture implies the Lindelöf hypothesis. Now, the best upper bounds for α_k ($k \leqslant 8$) are

$$\alpha_2 \leqslant 0.3144831759741, \qquad \alpha_3 \leqslant \frac{43}{96}, \qquad \alpha_k \leqslant \frac{3k-4}{4k} \text{ for } 4 \leqslant k \leqslant 8$$

by Li and Yang [2], Kolesnik [3] and Heath–Brown [4] (and Ivić [5]) respectively. Ivić also gave upper bounds with $k \geqslant 9$ in his book. For results with large k, one can see works of Heath–Brown [6] and Bellotti and Yang [7]. We also refer the readers to the blueprint of the new project ANTEDB organized by Tao, Trudgian and Yang.

In 1989, Ivić and Ouellet [8] refined the technique used in and gave better bounds for α_k with $k \geqslant 9$. In [5], Ivić connected this problem with the function $m(\sigma)$ defined as follows: For any fixed $\frac{1}{2} < \sigma < 1$ we define $m(\sigma)$ as the supremum of all numbers $m \geqslant 4$ such that

$$\int_1^T |\zeta(\sigma + it)|^m dt \ll T^{1+\varepsilon}. \tag{3}$$

In order to obtain good bounds for α_k, one need to get lower bounds for $m(\sigma)$. Ivić and Ouellet [8] used a large value theorem and growth exponents for Riemann zeta–function to bound $m(\sigma)$. Specially, for $10 \leqslant k \leqslant 20$ they got

$$\alpha_{10} \leqslant 0.675, \qquad \alpha_{11} \leqslant 0.6957, \qquad \alpha_{12} \leqslant 0.7130, \qquad \alpha_{13} \leqslant 0.7306,$$

$$\alpha_{14} \leqslant 0.7461, \quad \alpha_{15} \leqslant 0.75851, \quad \alpha_{16} \leqslant 0.7691, \quad \alpha_{17} \leqslant 0.7785,$$
$$\alpha_{18} \leqslant 0.7868, \quad \alpha_{19} \leqslant 0.7942, \quad \alpha_{20} \leqslant 0.8009.$$

In 2024, Trudgian and Yang [9] mentioned a series of new bounds for α_k. They combined the method of Ivić and Ouellet [8] with their new growth exponents for Riemann zeta–function to obtain those bounds.

Theorem 1 ([9], Theorem 2.9). *We have*

$$\begin{aligned}
&\alpha_9 \leqslant 0.64720, && \alpha_{10} \leqslant 0.67173, && \alpha_{11} \leqslant 0.69156, && \alpha_{12} \leqslant 0.70818, \\
&\alpha_{13} \leqslant 0.72350, && \alpha_{14} \leqslant 0.73696, && \alpha_{15} \leqslant 0.74886, && \alpha_{16} \leqslant 0.75952, \\
&\alpha_{17} \leqslant 0.76920, && \alpha_{18} \leqslant 0.77792, && \alpha_{19} \leqslant 0.78581, && \alpha_{20} \leqslant 0.79297, && \alpha_{21} \leqslant 0.79951.
\end{aligned}$$

In this paper, we use the essentially same methods to give a very minor improvement on their results.

Theorem 2. *We have*

$$\begin{aligned}
&\alpha_9 \leqslant 0.638889, && \alpha_{10} \leqslant 0.663329, && \alpha_{11} \leqslant 0.684349, && \alpha_{12} \leqslant 0.701768, \\
&\alpha_{13} \leqslant 0.717611, && \alpha_{14} \leqslant 0.732262, && \alpha_{15} \leqslant 0.745070, && \alpha_{16} \leqslant 0.756380, \\
&\alpha_{17} \leqslant 0.766588, && \alpha_{18} \leqslant 0.775721, && \alpha_{19} \leqslant 0.783939, && \alpha_{20} \leqslant 0.791374.
\end{aligned}$$

2. Growth Exponents for Riemann Zeta–Function

In this section we list the new growth exponents for Riemann zeta–function proved by Trudgian and Yang [9], which is the most powerful and important input in the proof of Theorem 2 (and also Theorem 1).

Lemma 1 ([9], Theorem 2.4). *We have*

$$\mu(\sigma) \leqslant \begin{cases}
\frac{31}{36} - \frac{3}{7}\sigma, & \frac{1}{2} \leqslant \sigma \leqslant \frac{88225}{153852}, \\
\frac{220633}{620612} - \frac{62831}{155153}\sigma, & \frac{88225}{153852} \leqslant \sigma \leqslant \frac{521}{796}, \\
\frac{1333}{3825} - \frac{1508}{3825}\sigma, & \frac{521}{796} \leqslant \sigma \leqslant \frac{53141}{76066}, \\
\frac{405}{1202} - \frac{227}{601}\sigma, & \frac{53141}{76066} \leqslant \sigma \leqslant \frac{454}{641}, \\
\frac{779}{2590} - \frac{423}{1295}\sigma, & \frac{454}{641} \leqslant \sigma \leqslant \frac{3473692}{4856993}, \\
\frac{1610593}{5622410} - \frac{861996}{2811205}\sigma, & \frac{3473692}{4856993} \leqslant \sigma \leqslant \frac{52209}{69128}, \\
\frac{157319}{560830} - \frac{251324}{841245}\sigma, & \frac{52209}{69128} \leqslant \sigma \leqslant \frac{1389}{1736}, \\
\frac{2841}{10316} - \frac{754}{2579}\sigma, & \frac{1389}{1736} \leqslant \sigma \leqslant \frac{587779}{702192}, \\
\frac{1691}{6554} - \frac{890}{3277}\sigma, & \frac{587779}{702192} \leqslant \sigma \leqslant \frac{7441}{8695}, \\
\frac{29}{130} - \frac{3}{13}\sigma, & \frac{7441}{8695} \leqslant \sigma \leqslant \frac{277}{300}.
\end{cases}$$

3. Ivić Large Value Theorem

Now we provide the large value theorem used by Ivić and Ouellet [8].

Lemma 2 (, Lemma 1). *Let $t_1, \dots, t_R$ be real numbers such that $T \leqslant t_r \leqslant 2T$ for $r = 1, \dots, R$ and $|t_r - t_s| \geqslant (\log T)^4$ for $1 \leqslant r \neq s \leqslant R$. If*

$$T^{\varepsilon} < V \leqslant \left| \sum_{m \sim M} a_m m^{-\sigma - it_r} \right|$$

 doi:10.20944/preprints202505.1432.v1

where $a_m \ll M^\varepsilon$ *for* $m \sim M$, $1 \ll M \ll T^C$, *then*

$$R \ll T^\varepsilon \left(M^{2-2\sigma} V^{-2} + TV^{-f(\sigma)} \right),$$

where

$$f(\sigma) = \begin{cases} \frac{2}{3-4\sigma}, & \frac{1}{2} < \sigma \leqslant \frac{2}{3}, \\ \frac{10}{7-8\sigma}, & \frac{2}{3} \leqslant \sigma \leqslant \frac{11}{14}, \\ \frac{34}{15-16\sigma}, & \frac{11}{14} \leqslant \sigma \leqslant \frac{13}{15}, \\ \frac{98}{31-32\sigma}, & \frac{13}{15} \leqslant \sigma \leqslant \frac{57}{62}, \\ \frac{5}{1-\sigma}, & \frac{57}{62} \leqslant \sigma \leqslant 1-\varepsilon. \end{cases}$$

4. Proof of Theorem 2

We shall use the method of Ivić and Ouellet [8] to prove Theorem 2. It was shown in [, Chapter 8] that to obtain bounds for $m(\sigma)$ it suffices to obtain bounds of the form

$$R \ll T^{1+\varepsilon} V^{-m(\sigma)}, \tag{4}$$

where R is the number of points $t_r (1 \leqslant r \leqslant R)$ such that $|t_r| \leqslant T$, $|t_r - t_s| \geqslant (\log T)^4$ for $1 \leqslant r \neq s \leqslant R$ and $|\zeta(\sigma + it_r)| \geqslant V > 0$ for any given V. Moreover, by [, (8.97)] we know that

$$R \ll T^\varepsilon \left(TV^{-2f(\sigma)} + T^{\frac{4-4\sigma}{1+2\sigma}} V^{\frac{-12}{1+2\sigma}} + T^{\frac{4(1-\sigma)(\kappa+\lambda)}{((2-4\lambda)\sigma-1+2\kappa-2\lambda)}} V^{\frac{-4(1+2\kappa+2\lambda)}{((2-4\lambda)\sigma-1+2\kappa-2\lambda)}} \right), \tag{5}$$

where (κ, λ) is an exponent pair. We shall use $(\kappa, \lambda) = \left(\frac{3}{40}, \frac{31}{40}\right)$ in the rest of our paper for the sake of convenience.

Now, for every $\frac{1}{2} \leqslant \sigma \leqslant \frac{277}{300}$ we define $c(\sigma)$ is the piecewise function given by Lemma 1. Clearly $c(\sigma)$ is an upper bound for $\mu(\sigma)$. By () and the definitions of $f(\sigma)$ and $c(\sigma)$, we can easily calculate the corresponding $m(\sigma)$ for some σ between $\frac{1}{2}$ and $\frac{277}{300}$. Numerical calculation gives that

$$\begin{aligned} &m(0.638889) > 9, && m(0.663329) > 10, && m(0.684349) > 11, && m(0.701768) > 12, \\ &m(0.717611) > 13, && m(0.732262) > 14, && m(0.745070) > 15, && m(0.756380) > 16, \\ &m(0.766588) > 17, && m(0.775721) > 18, && m(0.783939) > 19, && m(0.791374) > 20 \end{aligned}$$

and Theorem 2 is now proved.

Acknowledgments: The author would like to thank the Analytic Number Theory Exponent Database (ANTEDB) project.

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