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Partial Trace Reduction Without the Born Rule

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Partial Trace Reduction Without the Born Rule

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Abstract: Reduction of the density operator for quantum subsystems is derived without the use of the Born Rule.

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1. Introduction

Consider a composite quantum system consisting of the tensor product of two Hilbert spaces. A general quantum state of this system is entangled between the two subsystems. Nevertheless, if the subsystems are decoupled at a point in time, meaning the Hamiltonian becomes a tensor product of Hamiltonians for the subsystems, the physics for one subsystem is completely described by the reduced density matrix

$$\rho^r_{ab} = \sum_\alpha \rho_{a\alpha,b\alpha} \tag{1}$$

where ρ is the density matrix for the entangled state. Each composite index includes a roman letter for the target subsystem and a greek letter for the discarded subsystem. This operation is known as the partial trace.

The partial trace is normally justified by showing that statistics for measurements made on the target subsystem are unchanged [1], thus implicitly assuming the Born rule. For quantum foundational questions, including the measurement problem, the Born rule might not be postulated. It is therefore desirable to justify the partial trace without appeal to the statistical interpretation of the density matrix.

To this end, we will show that trace reduction is the only linear operation which has the correct decoupling behavior. If the Hamiltonian is a tensor product then the resulting evolution is a tensor product unitary transformation $U = U^T \otimes U^R$, where T and R reference the target and discarded (reduced) Hilbert spaces. Decoupling means the reduction operation is invariant under U^R and preserves the action of U^T.

2. Theorem

Let $\mathfrak{S}(\mathcal{H})$ be the set of all positive unit-trace operators on a complex Hilbert space $\mathcal{H}$. If $\mathcal{H} = \mathcal{H}_T \otimes \mathcal{H}_R$ is a tensor product Hilbert space and $f : \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{H}_T)$ is a linear map satisfying

$$f((U^T \otimes U^R)\rho(U^{T\dagger} \otimes U^{R\dagger})) = U^T f(\rho) U^{T\dagger} \tag{2}$$

for all $\rho \in \mathfrak{S}(\mathcal{H})$, unitary U^T acting in $\mathcal{H}_T$, and unitary U^R acting in $\mathcal{H}_R$, then

$$f(\rho) = tr^R(\rho) \tag{3}$$

for all $\rho \in \mathfrak{S}(\mathcal{H})$.

Proof:

f is linear and its components can be written

$$f(\rho)^{ab} = \sum_{c\gamma d\delta} M^{ab}_{c\gamma,d\delta} \rho_{d\delta,c\gamma} \tag{4}$$

for some matrix M^{ab}. Our strategy will be to choose U and ρ and use (2) to derive relations among components of M. Our U and ρ matrices will be trivial in all but two values of each index, so we will

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start with the EPR universe where $\mathcal{H}_T$ and $\mathcal{H}_R$ each represent a single spin. Pauli matrices are used for notational convenience. For the test density matrix, we choose

$$\begin{aligned} \rho_{c\gamma,d\delta} &= T_{cd}\delta_{\gamma\delta}\delta_{\delta 1} \\ T &= \begin{pmatrix} a & z \\ z^* & 1-a \end{pmatrix}, \end{aligned} \tag{5}$$

where z is implicitly restricted so the eigenvalues are non-negative. Using (5) and $U^T = I^T$, (2) becomes

$$\sum_{c\gamma d\delta} M^{ab}_{c\gamma,d\delta} U^R_{\delta 1} T_{dc} U^{R\dagger}_{1\gamma} = \sum_{cd} M^{ab}_{c1d1} T_{dc} \,. \tag{6}$$

Choosing $U^R = \sigma^x$ and varying a and z independently in (6) leads to

$$M^{ab}_{c1,d1} = M^{ab}_{c2,d2} \,. \tag{7}$$

Choosing $U^R = (\sigma^x + \sigma^z)/\sqrt{2}$ leads to

$$M^{ab}_{c1,d2} = -M^{ab}_{c2,d1} \,. \tag{8}$$

Choosing $U^R = (\sigma^y + \sigma^z)/\sqrt{2}$ leads to

$$M^{ab}_{c1,d2} = M^{ab}_{c2,d1} = 0 \tag{9}$$

so we can write

$$M^{ab}_{c\gamma,d\delta} = N^{ab}_{cd}\delta_{\gamma\delta} \,, \tag{10}$$

and (4) becomes

$$f(\rho)^{ab} = \sum_{cd} N^{ab}_{cd}\rho^r_{dc} \,. \tag{11}$$

Using (5), (11), (1), and $U^R = I^R$, (2) becomes

$$\sum_{cdef} N^{ab}_{cd} U^T_{de} T_{ef} U^{T\dagger}_{fc} = \sum_{cdef} U^T_{ae} N^{ef}_{cd} T_{dc} U^{T\dagger}_{fb} \,. \tag{12}$$

Going forward we set $a = 1$ and $z = 0$. Choosing $U^T = \sigma^x$ and $U^T = \sigma^y$ sequentially in (12) gives 8 equations, which combined lead to

$$\begin{aligned} N^{22}_{22} &= N^{11}_{11} \equiv A \\ N^{11}_{22} &= N^{22}_{11} \equiv B \\ N^{12}_{11} &= N^{21}_{11} = 0 \\ N^{12}_{22} &= N^{21}_{22} = 0 \,. \end{aligned} \tag{13}$$

Choosing $U^T = (\sigma^x + \sigma^z)/\sqrt{2}$ and $U^T = (\sigma^y + \sigma^z)/\sqrt{2}$ sequentially gives 8 equations, which combined lead to

$$\begin{aligned} N^{12}_{12} &= N^{21}_{21} = 0 \\ N^{12}_{21} &= N^{21}_{12} = A - B \\ N^{11}_{12} &= N^{11}_{21} = 0 \\ N^{22}_{12} &= N^{22}_{21} = 0 \,. \end{aligned} \tag{14}$$

Choosing $U^T = (I + \sigma^x)/\sqrt{2}$ leads to $B = 0$ and we can write

$$N^{ab}_{cd} = A\delta_{ad}\delta_{bc} \tag{15}$$

and

$$f(\rho) = Atr^R(\rho) \ . \tag{16}$$

Requiring unit-trace implies $A = 1$. The proof for the general case proceeds the same way for each pair of indices. For each pair, we expand the U choices to include the identity in other indices and we expand the ρ choices to be zero in other indices.

Finally we note that (3) satisfies (2) directly. This completes the proof.

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Reference

1. Nielsen, M.A.; Chuang, I.L. *Quantum computation and quantum information*; Cambridge University Press, 2010.