THE GENERATING FUNCTION OF THE CATALAN NUMBERS AND LOWER TRIANGULAR INTEGER MATRICES

FENG QI

Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia, 028043, China; Institute of Mathematics, Henan Polytechnic University, Jiaozuo, Henan, 454010, China; Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300387, China

XIAO-LONG QIN

Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, China

YONG-HONG YAO

Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, China; Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, China

Abstract. In the paper, by the Faà di Bruno formula, several identities for the Bell polynomials of the second kind, and an inversion theorem, the authors simplify coefficients of two families of nonlinear ordinary differential equations for the generating function of the Catalan numbers and discover inverses of fifteen closely related lower triangular integer matrices.

1. Motivation

The Catalan numbers C_n for $n \ge 0$ form a combinatorial sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted

---

E-mail addresses: qifeng618@gmail.com, qifeng618@hotmail.com, qxlxajh@163.com, yaoyonghong@aliyun.com.

2010 *Mathematics Subject Classification.* Primary 05A15; Secondary 11B65, 11B75, 11B83, 15A09, 15B36, 34A05, 34A34, 40E99.

Key words and phrases. simplification; coefficient; nonlinear ordinary differential equation; generating function; Catalan number; inverse matrix; lower triangular integer matrix; Faà di Bruno formula; Bell polynomial of the second kind; inversion theorem.

This paper was typeset using $\mathcal{AMS}$-LaTeX.

separately? whose solution is the Catalan number C_{n-2}". See the monographs [5, 43]. The Catalan numbers C_n can be generated by

$$G(x)=\frac{2}{1+\sqrt{1-4x}}=\sum_{n=0}^{\infty}C_nx^n=1+x+2x^2+5x^3+14x^4+42x^5+\cdots \tag{1.1}$$

and can be explicitly and alternatively expressed as

$$C_n=\frac{1}{n+1}\binom{2n}{n}={}_2F_1(1-n,-n;2;1)=\frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)},$$

where the classical Euler gamma function $\Gamma(z)$ can be found in [3, 8, 12, 19, 29] and the generalized hypergeometric series

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}$$

is defined [1, 3, 8] for complex numbers $a_i\in\mathbb{C}$ and $b_i\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, for positive integers $p,q\in\mathbb{N}$, and in terms of the rising factorials

$$(x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)\cdots(x+n-1), & n\ge1;\\ 1, & n=0.\end{cases}$$

For more information on the Catalan numbers C_n and their recent developments, please refer to the monographs [2, 5, 43], the papers [6, 7, 9, 10, 13, 14, 15, 16, 20, 21, 22, 23, 28, 33, 34, 35, 36, 37, 41, 42, 44, 46, 47, 48], and the closely related references therein.

In [4, Theorem 2.1], Kims established recursively and inductively that the family of differential equations

$$G^{(n)}(x)=\sum_{i=1}^{n}a_i(n)(1-4x)^{-(2n-i)/2}G^{i+1}(x),\quad n\in\mathbb{N} \tag{1.2}$$

has a solution $G(x)=\frac{2}{1+\sqrt{1-4x}}$, where $a_1(n)=2^{n-1}(2n-3)!!$ and

$$a_i(n)=2^{n-i}i!\sum_{k_{i-1}=0}^{n-i}\sum_{k_{i-2}=0}^{n-i-k_{i-1}}\cdots\sum_{k_1=0}^{n-i-k_{i-1}-\cdots-k_2}\left(2n-2\sum_{j=1}^{i-1}k_j-2i-1\right)!!\\ \times\prod_{\ell=1}^{i-1}\left\langle 2n-2\sum_{j=\ell+1}^{i-1}k_j-2i-1+\ell;2\right\rangle_{k_\ell} \tag{1.3}$$

with $\langle x;\alpha\rangle_0=1$ and $\langle x;\alpha\langle_n=x(x-\alpha)\cdots[x-(n-1)\alpha]$ for $n\in\mathbb{N}$.

In [4, Theorem 3.1], by similar argument as in the proof of [4, Theorem 2.1], they found that the family of differential equations

$$n!G^{n+1}(x)=\sum_{i=0}^{\lfloor n/2\rfloor}b_i(n)(1-4x)^{n/2-i}G^{(n-i)}(x),\quad n\in\mathbb{N} \tag{1.4}$$

has a solution $G(x)=\frac{2}{1+\sqrt{1-4x}}$, where $\lfloor t\rfloor$ denotes the floor function whose value is the largest integer less than or equal to t, the coefficients $b_0(n)=1$ and

$$b_i(n)=(-2)^iS_{n+1-2i,i},\quad 1\le i\le\left\lfloor\frac{n}{2}\right\rfloor \tag{1.5}$$

with $S_{n,1}=n+(n-1)+\dots+1$ and

$$S_{n,j}=nS_{n+1,j-1}+(n-1)S_{n,j-1}+\dots+1S_{2,j-1},\quad j\ge2.$$

In [4, Theorems 2.2 and 3.2] and [4, Remark], they also used the coefficients $a_i(n)$ and $b_i(n)$ respectively defined in (1.3) and (1.5) to express their other results in [4]. In other words, the quantities $a_i(n)$ and $b_i(n)$ are the core of the paper [4].

It is obvious that the coefficients $a_i(n)$ and $b_i(n)$ respectively defined in (1.3) and (1.5) can not be easily remembered, possibly understood, and simply computed.

The aim of this paper is the same one as in the papers [11, 14, 17, 18, 24, 25, 26, 27, 30, 31, 32, 38, 39, 40, 41, 45] and closely related references therein. Concretely speaking, our aim in this paper is to discover simple, significant, meaningful, easily remembered, possibly understood, readily computed expressions for the coefficients $a_i(n)$ and $b_i(n)$ in the families (1.2) and (1.4) respectively. Consequently and equivalently, we further derive inverses of fifteen closely related lower triangular integer matrices.

2. Lemmas

To reach our aim in this paper, we recall the following lemmas.

Lemma 2.1 ([2, pp. 134 and 139])**.** *The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind* $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *by*

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big) \tag{2.1}$$

for $n\ge0$*, where the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by* $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *for* $n\ge k\ge0$*, are defined by*

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n}i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

Lemma 2.2 ([2, p. 135])**.** *For* $n\ge k\ge0$*, we have*

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}), \tag{2.2}$$

where a *and* b *are any complex numbers.*

Lemma 2.3 ([23, Theorem 5.17] and [35, Theorem 1.2])**.** *For* $n\ge k\ge0$*, we have*

$$\mathrm{B}_{n,k}((-1)!!,1!!,3!!,\dots,[2(n-k)-1]!!)=[2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}, \tag{2.3}$$

where the double factorial of negative odd integers $-(2n+1)$ *is defined by*

$$(-2n-1)!!=\frac{(-1)^n}{(2n-1)!!}=(-1)^n\frac{2^nn!}{(2n)!},\quad n=0,1,\dots.$$

Lemma 2.4 ([14, Theorem 12.1] and [41, Theorem 4.3 and Remark 6.2])**.** *For* $n\ge k\ge1$*, let* $\{s_k\}_{k\in\mathbb{N}}$ *and* $\{S_k\}_{k\in\mathbb{N}}$ *be two sequences which are independent of* n*. Then*

$$s_n=\sum_{k=1}^{n}\binom{k}{n-k}S_k\quad\text{if and only if}\quad(-1)^nnS_n=\sum_{k=1}^{n}\binom{2n-k-1}{n-1}(-1)^kks_k.$$

Peer-reviewed version available at Journal of Taibah University for Science 2019; doi:10.1080/16583655.2019.1663782

3. Main results and their proofs

Now we are in a position to state our main results and to prove them simply.

Theorem 3.1. *For $n\in\mathbb{N}$, the nth derivative and the powers of the generating function $G(x)$ defined in (1.1) satisfy*

$$G^{(n)}(x)=\frac{(n-1)!}{(1-4x)^n}\sum_{k=1}^{n}k\binom{2n-k-1}{n-1}(1-4x)^{k/2}G^{k+1}(x). \tag{3.1}$$

Proof. This proof is a slight modification of the first part in the second proof of [35, Theorem 1.1].

Taking $f(u)=\frac{2}{1+u}$ and $u=h(x)=\sqrt{1-4x}$ in the formula (2.1) and utilizing the identity (2.2) yield

$$\begin{aligned}
G^{(n)}(x)&=2\sum_{k=0}^{n}(-1)^k\frac{k!}{(1+u)^{k+1}}\mathrm{B}_{n,k}\biggl(-\frac{2}{(1-4x)^{1/2}},-\frac{2^2}{(1-4x)^{3/2}},\\
&\qquad\dots,-\frac{2^{n-k+1}[2(n-k+1)-3]!!}{(1-4x)^{[2(n-k+1)-1]/2}}\biggr)\\
&=2\sum_{k=0}^{n}(-1)^k\frac{k!}{\bigl(1+\sqrt{1-4x}\,\bigr)^{k+1}}(-1)^k2^n\frac{(1-4x)^{k/2}}{(1-4x)^n}\\
&\quad\times\mathrm{B}_{n,k}\bigl((-1)!!,1!!,\dots,[2(n-k)-1]!!\bigr)\\
&=\frac{2^{n+1}}{(1-4x)^n}\sum_{k=0}^{n}\frac{k!\bigl(\sqrt{1-4x}\,\bigr)^k}{\bigl(1+\sqrt{1-4x}\,\bigr)^{k+1}}\mathrm{B}_{n,k}\bigl((-1)!!,1!!,\dots,[2(n-k)-1]!!\bigr)
\end{aligned}$$

for $n\in\mathbb{N}$. Further making use of the formula (2.3) and simplifying arrive at

$$\begin{aligned}
G^{(n)}(x)&=\frac{2^{n+1}}{(1-4x)^n}\sum_{k=1}^{n}k![2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}\frac{(1-4x)^{k/2}}{\bigl(1+\sqrt{1-4x}\,\bigr)^{k+1}}\\
&=\frac{2^{n+1}}{(1-4x)^n}\sum_{k=1}^{n}k\frac{(2n-k-1)!}{2^{n-k}(n-k)!}\frac{(1-4x)^{k/2}}{\bigl(1+\sqrt{1-4x}\,\bigr)^{k+1}}\\
&=\frac{(n-1)!}{(1-4x)^n}\sum_{k=1}^{n}k\binom{2n-k-1}{n-k}(1-4x)^{k/2}G^{k+1}(x)\\
&=\frac{(n-1)!}{(1-4x)^n}\sum_{k=1}^{n}k\binom{2n-k-1}{n-1}(1-4x)^{k/2}G^{k+1}(x).
\end{aligned}$$

The proof of Theorem 3.1 is complete. □

Remark 3.1. Comparing (1.2) with (3.1) derives

$$a_k(n)=(n-1)!\binom{2n-k-1}{n-1}=\frac{(2n-k-1)!}{(n-k)!},\quad n\ge k\ge1.$$

This expression is quite simpler, more easily remembered, more possibly understood, more readily computed, more significant, and more meaningful than the one in (1.3)!!!

Theorem 3.2. *For $n \in \mathbb{N}$, the power to n and the derivatives of the generating function $G(x)$ defined in* (1.1) *satisfy*

$$G^{n+1}(x) = \frac{(-1)^n}{(1-4x)^{n/2}} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \binom{k}{n-k} (1-4x)^k G^{(k)}(x). \tag{3.2}$$

Proof. The derivative formula (3.1) can be rearranged as

$$(-1)^n n\biggl[\frac{(4x-1)^n}{n!} G^{(n)}(x)\biggr] = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} (-1)^k k\bigl[(-1)^k (1-4x)^{k/2} G^{k+1}(x)\bigr], \quad n \in \mathbb{N}.$$

Considering Lemma 2.4 leads straightforwardly to

$$(-1)^n (1-4x)^{n/2} G^{n+1}(x) = \sum_{k=1}^{n} \binom{k}{n-k} \frac{(4x-1)^k}{k!} G^{(k)}(x), \quad n \in \mathbb{N}.$$

The proof of Theorem 3.2 is complete. □

Remark 3.2. Comparing (1.4) with (3.2) reveals

$$b_i(n) = (-1)^i \frac{n!}{(n-i)!} \binom{n-i}{i}, \quad 0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor.$$

This expression is rather simpler, more easily remembered, more possibly understood, more readily computed, more significant, and more meaningful than the one in (1.5)!!!

4. Inverses of lower triangular integer matrices

Every inversion theorem in combinatorics corresponds to a lower triangular invertible matrix and its inverse. Conversely, every lower triangular invertible matrix and its inverse correspond to an inversion theorem. Generally, it is not easy to compute the inverse of a lower triangular invertible matrix.

Lemma 2.4 is equivalent to that the lower triangular integer matrices $A_n = \bigl(a_{i,j}\bigr)_{1 \le i,j \le n}$ and $B_n = \bigl(b_{i,j}\bigr)_{1 \le i,j \le n}$ with

$$a_{i,j} = \begin{cases} 0, & i < j \\ \dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases} \quad \text{and} \quad b_{i,j} = \begin{cases} 0, & 1 \le i < j \\ (-1)^{i-j} \dfrac{j}{i} \dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other. See [14, Thorem 8.1] and [41, Theorem 4.1] also.

Theorem 4.1. *For $n \in \mathbb{N}$, the lower triangular integer matrices $P_n = \bigl(p_{i,j}\bigr)_{1 \le i,j \le n}$ and $Q_n = \bigl(q_{i,j}\bigr)_{1 \le i,j \le n}$ with*

$$p_{i,j} = \begin{cases} 0, & i < j \\ (-1)^{i-j} \dfrac{i}{j} \dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases} \quad \text{and} \quad q_{i,j} = \begin{cases} 0, & 1 \le i < j \\ \dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other.

Peer-reviewed version available at Journal of Taibah University for Science 2019; doi:10.1080/16583655.2019.1663782

Proof. This follows from rewriting the equations (3.1) and (3.2) as

$$\frac{(1-4x)^n}{(n-1)!}G^{(n)}(x)=\sum_{k=1}^{n}\binom{2n-k-1}{n-1}\bigl[k(1-4x)^{k/2}G^{k+1}(x)\bigr]$$

and

$$n(1-4x)^{n/2}G^{n+1}(x)=\sum_{k=1}^{n}(-1)^{n-k}\frac{n}{k}\binom{k}{n-k}\frac{(1-4x)^k}{(k-1)!}G^{(k)}(x).$$

This also follows from rearranging Lemma 2.4 as

$$(-1)^n n s_n=\sum_{k=1}^{n}(-1)^{n-k}\frac{n}{k}\binom{k}{n-k}\bigl[(-1)^k kS_k\bigr]$$

if and only if

$$(-1)^n nS_n=\sum_{k=1}^{n}\binom{2n-k-1}{n-1}(-1)^k ks_k,$$

The proof of Theorem 4.1 is complete. □

By similar argument as in the proof of Theorem 4.1, we derive the following conclusions about lower triangular integer matrices and their inverses.

Theorem 4.2. *For $n\in\mathbb{N}$, the lower triangular integer matrices $U_n=\bigl(u_{i,j}\bigr)_{1\le i,j\le n}$ and $V_n=\bigl(v_{i,j}\bigr)_{1\le i,j\le n}$ with*

$$u_{i,j}=\begin{cases}0, & i<j\\ (-1)^i i\dbinom{j}{i-j}, & j\le i\le 2j\\ 0, & i>2j\end{cases}$$

and

$$v_{i,j}=\begin{cases}0, & 1\le i<j\\ (-1)^j j\dbinom{2i-j-1}{i-1}, & i\ge j\ge 1\end{cases}$$

are inversive to each other.

Theorem 4.3. *For $n\in\mathbb{N}$, the lower triangular integer matrices $C_n=\bigl(c_{i,j}\bigr)_{1\le i,j\le n}$ and $D_n=\bigl(d_{i,j}\bigr)_{1\le i,j\le n}$ with*

$$c_{i,j}=\begin{cases}0, & i<j\\ (-1)^j i\dbinom{j}{i-j}, & j\le i\le 2j\\ 0, & i>2j\end{cases}$$

and

$$d_{i,j}=\begin{cases}0, & 1\le i<j\\ (-1)^i j\dbinom{2i-j-1}{i-1}, & i\ge j\ge 1\end{cases}$$

are inversive to each other.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 November 2017 doi:10.20944/preprints201711.0120.v1
Peer-reviewed version available at Journal of Taibah University for Science 2019; doi:10.1080/16583655.2019.1663782

Theorem 4.4. *For $n \in \mathbb{N}$, the lower triangular integer matrices $Y_n = \left(y_{i,j}\right)_{1 \le i,j \le n}$ and $Z_n = \left(z_{i,j}\right)_{1 \le i,j \le n}$ with*

$$y_{i,j} = \begin{cases} 0, & i < j \\ \dfrac{(-1)^i}{j}\dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases}$$

and

$$z_{i,j} = \begin{cases} 0, & 1 \le i < j \\ \dfrac{(-1)^j}{i}\dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other.

Theorem 4.5. *For $n \in \mathbb{N}$, the lower triangular integer matrices $E_n = \left(e_{i,j}\right)_{1 \le i,j \le n}$ and $F_n = \left(f_{i,j}\right)_{1 \le i,j \le n}$ with*

$$e_{i,j} = \begin{cases} 0, & i < j \\ \dfrac{(-1)^j}{j}\dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases}$$

and

$$f_{i,j} = \begin{cases} 0, & 1 \le i < j \\ \dfrac{(-1)^i}{i}\dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other.

Theorem 4.6. *For $n \in \mathbb{N}$, the lower triangular integer matrices $G_n = \left(g_{i,j}\right)_{1 \le i,j \le n}$ and $H_n = \left(h_{i,j}\right)_{1 \le i,j \le n}$ with*

$$g_{i,j} = \begin{cases} 0, & i < j \\ (-1)^i\dfrac{i}{j}\dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases}$$

and

$$h_{i,j} = \begin{cases} 0, & 1 \le i < j \\ (-1)^j\dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other.

Theorem 4.7. *For $n \in \mathbb{N}$, the lower triangular integer matrices $K_n = \left(k_{i,j}\right)_{1 \le i,j \le n}$ and $L_n = \left(\ell_{i,j}\right)_{1 \le i,j \le n}$ with*

$$k_{i,j} = \begin{cases} 0, & i < j \\ (-1)^j\dfrac{i}{j}\dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases}$$

Peer-reviewed version available at Journal of Taibah University for Science 2019; doi:10.1080/16583655.2019.1663782

and

$$\ell_{i,j}=\begin{cases}0, & 1\le i<j\\ (-1)^i\dbinom{2i-j-1}{i-1}, & i\ge j\ge 1\end{cases}$$

are inversive to each other.

Theorem 4.8. *For $n\in\mathbb{N}$, the lower triangular integer matrices $M_n=\big(m_{i,j}\big)_{1\le i,j\le n}$ and $N_n=\big(n_{i,j}\big)_{1\le i,j\le n}$ with*

$$m_{i,j}=\begin{cases}0, & i<j\\ (-1)^j\dbinom{j}{i-j}, & j\le i\le 2j\\ 0, & i>2j\end{cases}$$

and

$$n_{i,j}=\begin{cases}0, & 1\le i<j\\ (-1)^i\dfrac{j}{i}\dbinom{2i-j-1}{i-1}, & i\ge j\ge 1\end{cases}$$

are inversive to each other.

Theorem 4.9. *For $n\in\mathbb{N}$, the lower triangular integer matrices $O_n=\big(o_{i,j}\big)_{1\le i,j\le n}$ and $R_n=\big(r_{i,j}\big)_{1\le i,j\le n}$ with*

$$o_{i,j}=\begin{cases}0, & i<j\\ (-1)^i\dbinom{j}{i-j}, & j\le i\le 2j\\ 0, & i>2j\end{cases}$$

and

$$r_{i,j}=\begin{cases}0, & 1\le i<j\\ (-1)^j\dfrac{j}{i}\dbinom{2i-j-1}{i-1}, & i\ge j\ge 1\end{cases}$$

are inversive to each other.

Theorem 4.10. *For $n\in\mathbb{N}$, the lower triangular integer matrices $T_n=\big(t_{i,j}\big)_{1\le i,j\le n}$ and $W_n=\big(w_{i,j}\big)_{1\le i,j\le n}$ with*

$$t_{i,j}=\begin{cases}0, & i<j\\ (-1)^{i-j}i\dbinom{j}{i-j}, & j\le i\le 2j\\ 0, & i>2j\end{cases}\quad\text{and}\quad w_{i,j}=\begin{cases}0, & 1\le i<j\\ j\dbinom{2i-j-1}{i-1}, & i\ge j\ge 1\end{cases}$$

are inversive to each other.

Theorem 4.11. *For $n\in\mathbb{N}$, the lower triangular integer matrices $\Lambda_n=\big(\lambda_{i,j}\big)_{1\le i,j\le n}$ and $\Theta_n=\big(\theta_{i,j}\big)_{1\le i,j\le n}$ with*

$$\lambda_{i,j}=\begin{cases}0, & i<j\\ (-1)^{i-j}\dfrac{1}{j}\dbinom{j}{i-j}, & j\le i\le 2j\\ 0, & i>2j\end{cases}$$

and

$$\theta_{i,j} = \begin{cases} 0, & 1 \le i < j \\ \dfrac{1}{i}\dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other.

Theorem 4.12. *For $n \in \mathbb{N}$, the lower triangular integer matrices $\Phi_n = (\phi_{i,j})_{1 \le i,j \le n}$ and $\Psi_n = (\psi_{i,j})_{1 \le i,j \le n}$ with*

$$\phi_{i,j} = \begin{cases} 0, & i < j \\ \dfrac{1}{j}\dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases}$$

and

$$\psi_{i,j} = \begin{cases} 0, & 1 \le i < j \\ (-1)^{i-j}\dfrac{1}{i}\dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other.

Theorem 4.13. *For $n \in \mathbb{N}$, the lower triangular integer matrices $\mathcal{A}_n = (\alpha_{i,j})_{1 \le i,j \le n}$ and $\mathcal{B}_n = (\beta_{i,j})_{1 \le i,j \le n}$ with*

$$\alpha_{i,j} = \begin{cases} 0, & i < j \\ i\dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases} \quad \text{and} \quad \beta_{i,j} = \begin{cases} 0, & 1 \le i < j \\ (-1)^{i-j} j\dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other.

Theorem 4.14. *For $n \in \mathbb{N}$, the lower triangular integer matrices $\mathcal{P}_n = (\mu_{i,j})_{1 \le i,j \le n}$ and $\mathcal{Q}_n = (\nu_{i,j})_{1 \le i,j \le n}$ with*

$$\mu_{i,j} = \begin{cases} 0, & i < j \\ (-1)^{i-j}\dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases} \quad \text{and} \quad \nu_{i,j} = \begin{cases} 0, & 1 \le i < j \\ \dfrac{j}{i}\dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other.

Theorem 4.15. *For $n \in \mathbb{N}$, the lower triangular integer matrices $\mathcal{T}_n = (\tau_{i,j})_{1 \le i,j \le n}$ and $\mathcal{K}_n = (\kappa_{i,j})_{1 \le i,j \le n}$ with*

$$\tau_{i,j} = \begin{cases} 0, & i < j \\ \dfrac{i}{j}\dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases} \quad \text{and} \quad \kappa_{i,j} = \begin{cases} 0, & 1 \le i < j \\ (-1)^{i-j}\dbinom{2i-j-1}{i-1}, & i \ge j \ge 1 \end{cases}$$

are inversive to each other.

Peer-reviewed version available at *Journal of Taibah University for Science* **2019**; doi:10.1080/16583655.2019.1663782

5. Acknowledgements

The authors are grateful to Dr. Professor Taekyun Kim (Kwangwoon University, South Korea) for his sending an electronic copy of his paper [4] to the first author on November 15, 2017 through e-mail.

References

[1] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 32, Stechert-Hafner, New York, 1964.

[2] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.

[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Eighth edition, Revised from the seventh edition, Elsevier/Academic Press, Amsterdam, 2015; Available online at `https://doi.org/10.1016/B978-0-12-384933-5.00013-8`.

[4] D. S. Kim and T. Kim, *A new approach to Catalan numbers using differential equations*, Russ. J. Math. Phys. **24** (2017), no. 4, 487–496; Available online at `https://doi.org/10.1134/S1061920817040070`.

[5] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, Oxford, 2009.

[6] F.-F. Liu, X.-T. Shi, and F. Qi, *A logarithmically completely monotonic function involving the gamma function and originating from the Catalan numbers and function*, Glob. J. Math. Anal. **3** (2015), no. 4, 140–144; Available online at `https://doi.org/10.14419/gjma.v3i4.5187`.

[7] M. Mahmoud and F. Qi, *Three identities of the Catalan–Qi numbers*, Mathematics **4** (2016), no. 2, Article 35, 7 pages; Available online at `https://doi.org/10.3390/math4020035`.

[8] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010; Available online at `http://dlmf.nist.gov/`.

[9] F. Qi, *An improper integral with a square root*, Preprints **2016**, 2016100089, 8 pages; Available online at `https://doi.org/10.20944/preprints201610.0089.v1`.

[10] F. Qi, *Asymptotic expansions, complete monotonicity, and inequalities of the Catalan numbers*, ResearchGate Technical Report (2015), available online at `https://doi.org/10.13140/RG.2.1.4371.6321`.

[11] F. Qi, *Explicit formulas for the convolved Fibonacci numbers*, ResearchGate Working Paper (2016), available online at `https://doi.org/10.13140/RG.2.2.36768.17927`.

[12] F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat **27** (2013), no. 4, 601–604; Available online at `https://doi.org/10.2298/FIL1304601Q`.

[13] F. Qi, *Parametric integrals, the Catalan numbers, and the beta function*, Elem. Math. **72** (2017), no. 3, 103–110; Available online at `https://doi.org/10.4171/EM/332`.

[14] F. Qi, *The inverse of a matrix and several identities related to the Catalan numbers and the Chebyshev polynomials of the second kind*, ResearchGate Presentation (2017), available online at `https://doi.org/10.13140/RG.2.2.22832.46081/1`. (Chinese)

[15] F. Qi, A. Akkurt, and H. Yildirim, *Catalan numbers, k-gamma and k-beta functions, and parametric integrals*, J. Comput. Anal. Appl. **25** (2018), no. 6, 1036–1042.

[16] F. Qi and P. Cerone, *Some properties of the Fuss–Catalan numbers*, Preprints **2017**, 2017080056, 14 pages; Available online at `https://doi.org/10.20944/preprints201708.0056.v1`.

[17] F. Qi and B.-N. Guo, *A diagonal recurrence relation for the Stirling numbers of the first kind*, Appl. Anal. Discrete Math. **12** (2018), in press; Available online at `https://doi.org/10.2298/AADM170405004Q`.

[18] F. Qi and B.-N. Guo, *Explicit formulas and recurrence relations for higher order Eulerian polynomials*, Indag. Math. **28** (2017), no. 4, 884–891; Available online at `https://doi.org/10.1016/j.indag.2017.06.010`.

[19] F. Qi and B.-N. Guo, *Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function*, Rev. R. Acad. Cienc. Exactas

Fís. Nat. Ser. A Math. RACSAM **111** (2017), no. 2, 425–434; Available online at `https://doi.org/10.1007/s13398-016-0302-6`.

[20] F. Qi and B.-N. Guo, *Integral representations of Catalan numbers and their applications*, Mathematics **5** (2017), no. 3, Article 40, 31 pages; Available online at `https://doi.org/10.3390/math5030040`.

[21] F. Qi and B.-N. Guo, *Logarithmically complete monotonicity of a function related to the Catalan–Qi function*, Acta Univ. Sapientiae Math. **8** (2016), no. 1, 93–102; Available online at `https://doi.org/10.1515/ausm-2016-0006`.

[22] F. Qi and B.-N. Guo, *Logarithmically complete monotonicity of Catalan–Qi function related to Catalan numbers*, Cogent Math. (2016), **3**:1179379, 6 pages; Available online at `https://doi.org/10.1080/23311835.2016.1179379`.

[23] F. Qi and B.-N. Guo, *Some properties and generalizations of the Catalan, Fuss, and Fuss–Catalan numbers*, Chapter 5 in Mathematical Analysis and Applications: Selected Topics, 35 pages, edited by M. Ruzhansky, H. Dutta, and R. P. Agarwal, Wiley, 2018, ISBN 9781119414346.

[24] F. Qi and B.-N. Guo, *Some properties of the Hermite polynomials and their squares and generating functions*, Preprints **2016**, 2016110145, 14 pages; Available online at `https://doi.org/10.20944/preprints201611.0145.v1`.

[25] F. Qi and B.-N. Guo, *Viewing some ordinary differential equations from the angle of derivative polynomials*, Preprints **2016**, 2016100043, 12 pages; Available online at `https://doi.org/10.20944/preprints201610.0043.v1`.

[26] F. Qi, D. Lim, and B.-N. Guo, *Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM (2018), in press; Available online at `https://doi.org/10.1007/s13398-017-0427-2`.

[27] F. Qi, D. Lim, and B.-N. Guo, *Some identities relating to Eulerian polynomials and involving Stirling numbers*, Preprints **2017**, 2017080004, 10 pages; Available online at `https://doi.org/10.20944/preprints201708.0004.v1`.

[28] F. Qi, M. Mahmoud, X.-T. Shi, and F.-F. Liu, *Some properties of the Catalan–Qi function related to the Catalan numbers*, SpringerPlus (2016), **5**:1126, 20 pages; Available online at `https://doi.org/10.1186/s40064-016-2793-1`.

[29] F. Qi, C. Mortici, and B.-N. Guo, *Some properties of a sequence arising from geometric probability for pairs of hyperplanes intersecting with a convex body*, Comput. Appl. Math. **37** (2018), in press; Available online at `https://doi.org/10.1007/s40314-017-0448-7`.

[30] F. Qi, D.-W. Niu, and B.-N. Guo, *Simplification of coefficients in differential equations associated with higher order Frobenius–Euler numbers*, Preprints **2017**, 2017080017, 7 pages; Available online at `https://doi.org/10.20944/preprints201708.0017.v1`.

[31] F. Qi, D.-W. Niu, and B.-N. Guo, *Simplifying coefficients in differential equations associated with higher order Bernoulli numbers of the second kind*, Preprints **2017**, 2017080026, 6 pages; Available online at `https://doi.org/10.20944/preprints201708.0026.v1`.

[32] F. Qi, D.-W. Niu, and B.-N. Guo, *Some identities for a sequence of unnamed polynomials connected with the Bell polynomials*, Preprints **2017**, 2017080044, 10 pages; Available online at `https://doi.org/10.20944/preprints201708.0044.v1`.

[33] F. Qi, X.-T. Shi, and P. Cerone, *A unified generalization of the Catalan, Fuss, and Fuss–Catalan numbers and the Catalan–Qi function*, ResearchGate Working Paper (2015), available online at `https://doi.org/10.13140/RG.2.1.3198.6000`.

[34] F. Qi, X.-T. Shi, and F.-F. Liu, *An integral representation, complete monotonicity, and inequalities of the Catalan numbers*, ResearchGate Technical Report (2015), available online at `https://doi.org/10.13140/RG.2.1.3754.4806`.

[35] F. Qi, X.-T. Shi, F.-F. Liu, and D. V. Kruchinin, *Several formulas for special values of the Bell polynomials of the second kind and applications*, J. Appl. Anal. Comput. **7** (2017), no. 3, 857–871; Available online at `https://doi.org/10.11948/2017054`.

[36] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *Schur-convexity of the Catalan–Qi function related to the Catalan numbers*, Tbilisi Math. J. **9** (2016), no. 2, 141–150; Available online at `https://doi.org/10.1515/tmj-2016-0026`.

[37] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *The Catalan numbers: a generalization, an exponential representation, and some properties*, J. Comput. Anal. Appl. **23** (2017), no. 5, 937–944.

[38] F. Qi, J.-L. Wang, and B.-N. Guo, *Notes on a family of inhomogeneous linear ordinary differential equations*, Preprints **2017**, 2017040026, 5 pages; Available online at https://doi.org/10.20944/preprints201704.0026.v1.

[39] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying and finding nonlinear ordinary differential equations*, ResearchGate Working Paper (2017), available online at https://doi.org/10.13140/RG.2.2.28855.32166.

[40] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying differential equations concerning degenerate Bernoulli and Euler numbers*, Trans. A. Razmadze Math. Inst. **171** (2017), no. 3, in press; Available online at https://doi.org/10.1016/j.trmi.2017.08.001.

[41] F. Qi, Q. Zou, and B.-N. Guo, *Some identities and a matrix inverse related to the Chebyshev polynomials of the second kind and the Catalan numbers*, Preprints **2017**, 2017030209, 25 pages; Available online at https://doi.org/10.20944/preprints201703.0209.v2.

[42] X.-T. Shi, F.-F. Liu, and F. Qi, *An integral representation of the Catalan numbers*, Glob. J. Math. Anal. **3** (2015), no. 3, 130–133; Available online at https://doi.org/10.14419/gjma.v3i3.5055.

[43] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, New York, 2015; Available online at https://doi.org/10.1017/CBO9781139871495.

[44] L. Yin and F. Qi, *Several series identities involving the Catalan numbers*, Preprints **2017**, 2017030029, 11 pages; Available online at https://doi.org/10.20944/preprints201703.0029.v1.

[45] J.-L. Zhao, J.-L. Wang, and F. Qi, *Derivative polynomials of a function related to the Apostol–Euler and Frobenius–Euler numbers*, J. Nonlinear Sci. Appl. **10** (2017), no. 4, 1345–1349; Available online at https://doi.org/10.22436/jnsa.010.04.06.

[46] R. R. Zhou and W. Chu, *Identities on extended Catalan numbers and their q-analogs*, Graphs Combin. **32** (2016), no. 5, 2183–2197; Available online at https://doi.org/10.1007/s00373-016-1694-y.

[47] Q. Zou, *Analogues of several identities and supercongruences for the Catalan–Qi numbers*, J. Inequal. Spec. Funct. **7** (2016), no. 4, 235–241.

[48] Q. Zou, *The q-binomial inverse formula and a recurrence relation for the q-Catalan–Qi numbers*, J. Math. Anal. **8** (2017), no. 1, 176–182.

URL: https://qifeng618.wordpress.com