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There Exists a Non-Recursively Enumerable Set $\{n \in N : \varphi(n)\}$ Such That the Formula $\varphi(n)$ Is Short and Can Be Easily Translated into a First-Order Formula Which Uses Only + and ·

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There Exists a Non-Recursively Enumerable Set $\{n \in \mathbb{N} : \varphi(n)\}$ Such That the Formula $\varphi(n)$ Is Short and Can Be Easily Translated into a First-Order Formula Which Uses Only $+$ and $\cdot$

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Abstract

We prove that the set

$$T = \Big\{n \in \mathbb{N} : \exists p, q \in \mathbb{N} \Big((2n = (p+q)(p+q+1)+2q) \wedge$$
$$\forall (x_0, \ldots, x_p) \in \mathbb{N}^{p+1} \, \exists (y_0, \ldots, y_p) \in \{0, \ldots, q\}^{p+1}$$
$$\big((\forall k \in \{0, \ldots, p\} \; (1 = x_k \Rightarrow 1 = y_k)) \wedge$$
$$(\forall i, j, k \in \{0, \ldots, p\} \; (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$
$$(\forall i, j, k \in \{0, \ldots, p\} \; (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k)))\Big)\Big\}$$

is not recursively enumerable. By using Gödel's β function, we prove that the formula that defines the set T can be easily translated into a first-order formula which uses only $+$ and $\cdot$. The same properties has the set

$$\Big\{n \in \mathbb{N} : \exists p, q \in \mathbb{N} \Big((2n = (p+q)(p+q+1)+2q) \wedge$$
$$\forall (x_0, \ldots, x_p) \in \mathbb{N}^{p+1} \, \exists (y_0, \ldots, y_p) \in \{0, \ldots, q\}^{p+1}$$
$$\big((\forall j, k \in \{0, \ldots, p\} \; (x_j + 1 = x_k \Rightarrow y_j + 1 = y_k)) \wedge$$
$$(\forall i, j, k \in \{0, \ldots, p\} \; (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k)))\Big)\Big\}$$

Keywords: computable function; eventual domination; Gödel's β function; limit-computable function; recursively enumerable set; undecidable decision problem

MSC: 03D25

This article is a shortened version of the article [6]. Semi-algorithms differ from algorithms, as they may not terminate.

Definition 1. *(cf. [4, pp. 233–235]). A computation in the limit of a function $f : \mathbb{N} \to \mathbb{N}$ is a semi-algorithm which takes as input a non-negative integer n and for every $m \in \mathbb{N}$ prints a non-negative integer $\xi(n, m)$ such that $\lim_{m \to \infty} \xi(n, m) = f(n)$.*

By Definition 1, a function $f : \mathbb{N} \to \mathbb{N}$ is computable in the limit when there exists an infinite computation which takes as input a non-negative integer n and prints a non-negative integer on each iteration and prints $f(n)$ on each sufficiently high iteration.

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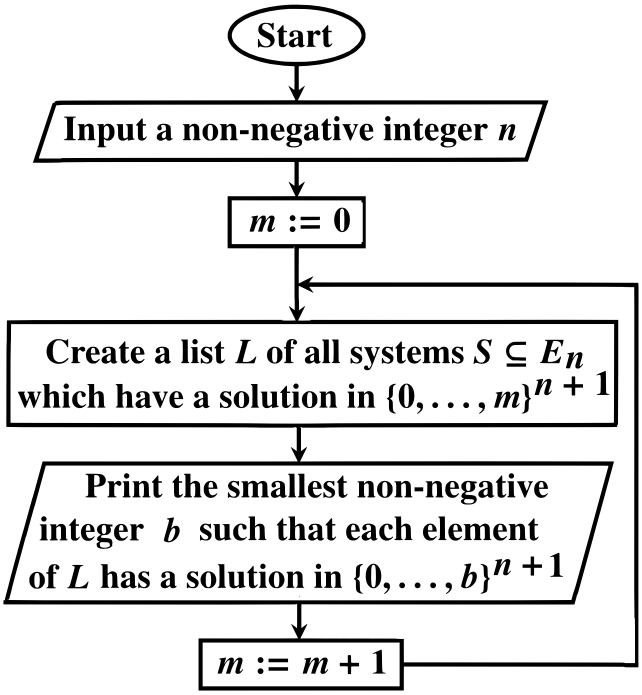
For $n \in \mathbb{N}$, let

$$E_n = \{1 = x_k,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i,j,k \in \{0,\dots,n\}\}$$

Theorem 1. *([3, p. 118]). There exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$.*

We present an alternative proof of Theorem 1. For $n \in \mathbb{N}$, $f(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then S has a solution in $\{0,\dots,b\}^{n+1}$. The function $f : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$, see [5]. The term *"dominated"* in the title of [5] means *"eventually dominated"*.

Theorem 2. *([5]). Flowchart 1 shows a semi-algorithm which computes $f(n)$ in the limit.*



Flowchart 1

A semi-algorithm which computes $f(n)$ in the limit

Definition 2. *An approximation of a tuple $(x_0,\dots,x_n) \in \mathbb{N}^{n+1}$ is a tuple $(y_0,\dots,y_n) \in \mathbb{N}^{n+1}$ such that*

$$(\forall k \in \{0,\dots,n\}\ (1 = x_k \Rightarrow 1 = y_k)) \wedge$$

$$(\forall i,j,k \in \{0,\dots,n\}\ (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$

$$(\forall i,j,k \in \{0,\dots,n\}\ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))$$

Observation 1. *For every $n \in \mathbb{N}$, there exists a set $A(n) \subseteq \mathbb{N}^{n+1}$ such that*

$$\mathrm{card}(A(n)) \leqslant 2^{\mathrm{card}(E_n)} = 2^{n+1+2\cdot(n+1)^3}$$

and every tuple $(x_0,\dots,x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $A(n)$.

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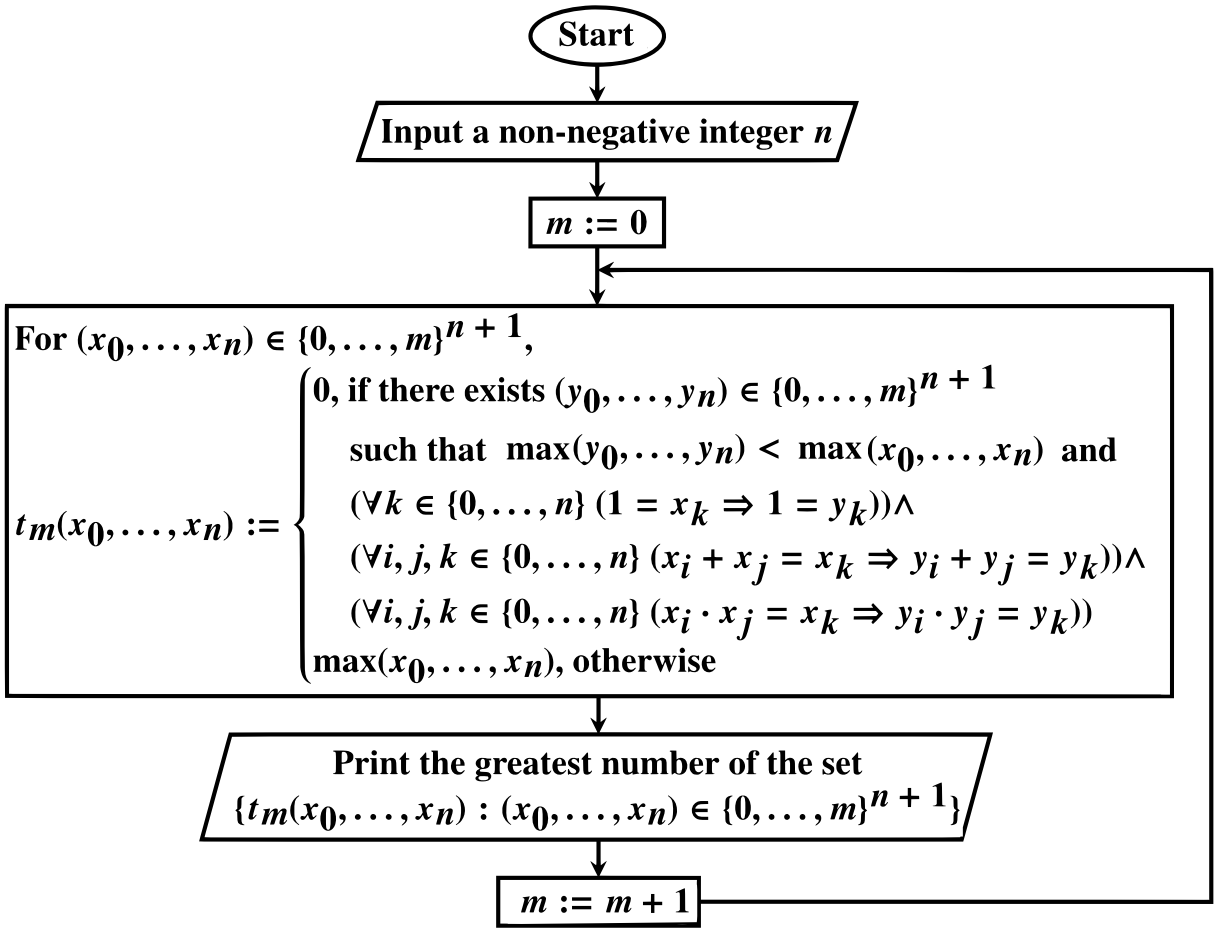
Flowchart 2 shows a simpler semi-algorithm which computes $f(n)$ in the limit.



Flowchart 2

A simpler semi-algorithm which computes $f(n)$ in the limit

Lemma 1. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 2 does not exceed the number printed by Flowchart 1.*

Proof. For every $(a_0, \ldots, a_n) \in \{0, \ldots, m\}^{n+1}$,

$$E_n \supseteq \{1 = x_k : (k \in \{0, \ldots, n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k : (i, j, k \in \{0, \ldots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k : (i, j, k \in \{0, \ldots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

Lemma 2. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 1 does not exceed the number printed by Flowchart 2.*

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $S \subseteq E_n$, if $(a_0, \ldots, a_n) \in \{0, \ldots, m\}^{n+1}$ and $(a_0, \ldots, a_n)$ solves S, then $(a_0, \ldots, a_n)$ solves the following system of equations:

$$\{1 = x_k : (k \in \{0, \ldots, n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k : (i, j, k \in \{0, \ldots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$

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$$\{x_i \cdot x_j = x_k : \ (i,j,k \in \{0,\dots,n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

Theorem 3. *For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the same number.*

Proof. It follows from Lemmas 1 and 2. □

Corollary 1. *For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the smallest $b \in \{0,\dots,m\}$ such that every tuple $(x_0,\dots,x_n) \in \{0,\dots,m\}^{n+1}$ possesses an approximation in $\{0,\dots,b\}^{n+1}$.*

Theorem 4. *For every $n \in \mathbb{N}$, $f(n)$ is the smallest $b \in \mathbb{N}$ such that every tuple $(x_0,\dots,x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $\{0,\dots,b\}^{n+1}$.*

Proof. It follows from Theorem 2 and Corollary 1. □

Theorem 5. *No algorithm takes as input non-negative integers n and m and decides whether or not*

$$\forall (x_0,\dots,x_n) \in \mathbb{N}^{n+1} \ \exists (y_0,\dots,y_n) \in \{0,\dots,m\}^{n+1}$$

$$((\forall k \in \{0,\dots,n\} \ (1 = x_k \Rightarrow 1 = y_k)) \wedge$$

$$(\forall i,j,k \in \{0,\dots,n\} \ (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$

$$(\forall i,j,k \in \{0,\dots,n\} \ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k)))$$

Proof. Since the function f is not computable, it follows from Theorem 4. □

Lemma 3. *([2]). The function*

$$\mathbb{N}^2 \ni (p,q) \to \frac{1}{2}(p+q)(p+q+1) + q \in \mathbb{N}$$

is bijective.

Theorem 6. *No algorithm takes as input a non-negative integer n and decides whether or not*

$$\exists p,q \in \mathbb{N} \ ((2n = (p+q)(p+q+1) + 2q)) \wedge$$

$$\forall (x_0,\dots,x_p) \in \mathbb{N}^{p+1} \ \exists (y_0,\dots,y_p) \in \{0,\dots,q\}^{p+1}$$

$$((\forall k \in \{0,\dots,p\} \ (1 = x_k \Rightarrow 1 = y_k)) \wedge$$

$$(\forall i,j,k \in \{0,\dots,p\} \ (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$

$$(\forall i,j,k \in \{0,\dots,p\} \ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))))$$

Proof. It follows from Theorem 5 and Lemma 3. □

Let

$$T = \{n \in \mathbb{N} : \ \exists p,q \in \mathbb{N} \ ((2n = (p+q)(p+q+1) + 2q) \wedge$$

$$\forall (x_0,\dots,x_p) \in \mathbb{N}^{p+1} \ \exists (y_0,\dots,y_p) \in \{0,\dots,q\}^{p+1}$$

$$((\forall k \in \{0,\dots,p\} \ (1 = x_k \Rightarrow 1 = y_k)) \wedge$$

$$(\forall i,j,k \in \{0,\dots,p\} \ (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$

$$(\forall i,j,k \in \{0,\dots,p\} \ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))))\}$$

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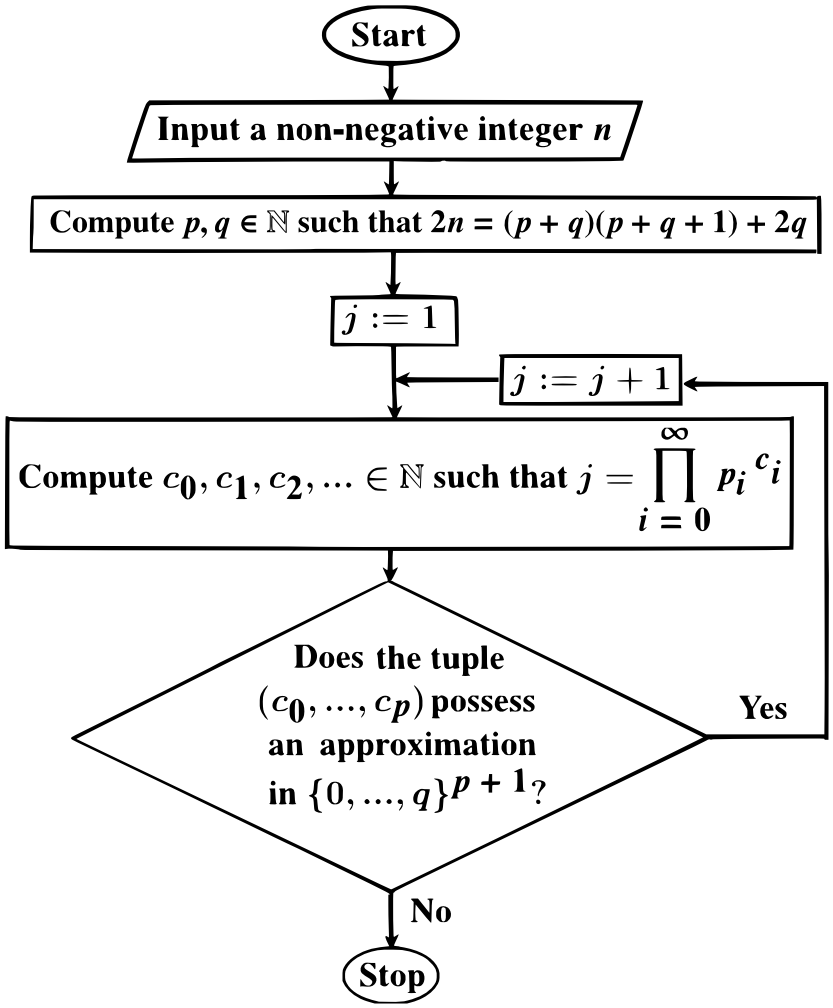
Theorem 7. *The set $\mathbb{N} \setminus T$ is recursively enumerable.*

Proof. For $i \in \mathbb{N}$, let p_i denote the i-th prime number. Flowchart 3 shows a semi-algorithm which takes as input $n \in \mathbb{N}$ and terminates if and only if $n \in \mathbb{N} \setminus T$.



Flowchart 3

A semi-algorithm which takes as input $n \in \mathbb{N}$ and terminates if and only if $n \in \mathbb{N} \setminus T$

□

Theorem 8. *The set T is not recursively enumerable.*

Proof. It follows from Theorems 6 and 7. □

Let $\beta : \mathbb{N}^3 \to \mathbb{N}$ denote Gödel's β function, see [1]. For $x_1, x_2, x_3 \in \mathbb{N}$, $\beta(x_1, x_2, x_3)$ equals the remainder after integer division of x_1 by $1 + (x_3 + 1) \cdot x_2$.

Lemma 4. *([1]). If $(d_0, \ldots, d_p) \in \mathbb{N}^{p+1}$, then $\exists b, c \in \mathbb{N}\ \forall l \in \{0, \ldots, p\}\ \beta(b, c, l) = d_l$.*

Theorem 9. *The formula that defines the set T can be easily translated into a first-order formula which uses only $+$ and $\cdot$.*

Proof. By Lemma 4, the set T consists of all $n \in \mathbb{N}$ such that

$$\forall u, v \in \mathbb{N}\ \exists a, b, p, q \in \mathbb{N}\ ((2n = (p+q)(p+q+1) + 2q) \wedge$$

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$$\forall i,j,k \in \{0,\ldots,p\}\ ((1=\beta(u,v,k) \Rightarrow 1=\min(\beta(a,b,k),q)) \wedge$$
$$(\beta(u,v,i)+\beta(u,v,j)=\beta(u,v,k) \Rightarrow \min(\beta(a,b,i),q)+\min(\beta(a,b,j),q)=\min(\beta(a,b,k),q)) \wedge$$
$$(\beta(u,v,i)\cdot\beta(u,v,j)=\beta(u,v,k) \Rightarrow \min(\beta(a,b,i),q)\cdot\min(\beta(a,b,j),q)=\min(\beta(a,b,k),q))))$$

The above formula can be easily translated into a first-order formula which uses only $+$ and $\cdot$. □

A more sophisticated proof shows that the set

$$W=\{n\in\mathbb{N}:\ \exists p,q\in\mathbb{N}\ ((2n=(p+q)(p+q+1)+2q)\wedge$$
$$\forall(x_0,\ldots,x_p)\in\mathbb{N}^{p+1}\ \exists(y_0,\ldots,y_p)\in\{0,\ldots,q\}^{p+1}$$
$$((\forall j,k\in\{0,\ldots,p\}\ (x_j+1=x_k \Rightarrow y_j+1=y_k))\wedge$$
$$(\forall i,j,k\in\{0,\ldots,p\}\ (x_i\cdot x_j=x_k \Rightarrow y_i\cdot y_j=y_k))))\}$$

is not recursively enumerable, see [6]. By using Gödel's β function, the formula that defines the set W can be easily translated into a first-order formula which uses only $+$ and $\cdot$.

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