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Article

Entropy Considerations in Stochastic Electrodynamics

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Abstract: The use of entropy concepts in the field of stochastic electrodynamics is briefly reviewed here. Entropy calculations to date that have been fully carried out are discussed in two main situations: first, where electric dipole oscillators interact with zero-point, or zero-point plus Planckian, or Rayleigh-Jeans radiation, and second where only these radiation fields exist within a cavity. The emphasis here will be on the first more difficult situation where both charged particles and radiation fields are present and interacting. Unlike the usual exposition on entropy in clasical statistical mechanics involving probabilistic notions of phase space occupation, the calculations to date for both particles and fields or for fields alone, follow the caloric entropy method where the notions of heat flow, adiabatic surfaces, and isothermal conditions, are utilized. Probability notions certainly still enter into the calculations, as the fields and charged particles interact stochastically together, following Maxwellian electrodynamics. Examples of phase-space calculations for harmonic oscillators and classical hydrogen atoms are carried out, emphasizing how much farther caloric entropy calculations have been successfully utilized.

Keywords: stochastic electrodynamics; classical physics; harmonic oscillator; entropy; thermodynamics; statistical mechanics; electromagnetic radiation

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1. Introduction

This article discusses aspects of entropy calculations in the theory of nature usually referred to as "stochastic electrodynamics" (SED). This theory involves only classical physics, where by this we mean electrodynamics described by the microscopic classical Maxwell's equations plus the relativistic version of Newton's equation of motion for a charged point particle, or, in other words, the Lorentz-Dirac equation [1,2]. However, it is recognized in SED that to properly describe nature requires that electromagnetic fields and particle motion must allow for a fluctuating behavior even at temperature $T = 0$. For electromagnetic fields, this consists of nonzero fluctuating classical radiation at temperature $T = 0$, or "zero-point" (ZP) radiation, that satisfies Maxwell's equations. These ZP fields serve as the homogeneous or source free boundary conditions for Maxwell's equations, present even when radiation sources of charges and currents equal zero, just as occurs when classical thermal radiation fields exist in a cavity with no free charges present.

ZP radiation obeys a number of interesting physical properties. Two of them are: (1) the spectrum must be Lorentz invariant [3,4], so that all inertial frames see the same spectrum, and (2) that the fundamental definition of $T = 0$ must be obeyed by ZP radiation of no heat flow during reversible thermodynamic operations [5–10]. In addition, ZP radiation obeys a number of other interesting and important properties that are discussed more in some of the reviews of SED in Refs. [11–15].

The classical theory of SED is able to successfully describe a range of natural phenomena in agreement with quantum mechanical (QM) theory. The results are surprising in that much of quantum phenomena can be understood qualitatively and quantitatively with this inclusion of ZP behavior of fields and particles. As an example of the comparison between SED and QM and quantum electrodynamics (QED), Ref. [16] showed that for the simple harmonic oscillator (SHO) system, SED agreed with QED for $T \geq 0$ (provided QM/QED operator orders were symmetrized), and it agreed with QM in the "resonance approximation" of small charge discussed in Ref. [16]. The complicated fully retarded, valid at all distances, van der Waals forces between atoms, as modelled by electric dipole SHOs, also share this agreement [17–19], as do Casimir forces between continuum materials. These

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agreements hold for all temperature conditions. Even the "atomic collapse" problem of Rutherford's classical "satellite model" seems to be resolved once ZP radiation is taken into account. The qualitative mechanism was first proposed in Ref. [12] and has since been shown in numerical simulations in Refs. [20–23]. However, interestingly enough, ionization problems, rather than atomic collapse, are now the concern [21–23]. Possibilities of more accurate relativistic calculations [24–26] and consideration of numerical based "chaotic effects" when the classical electron's orbit becomes large, may be points that could rectify this situation.

For general reviews of SED, the following references should be of help: Refs: [11–15]. The possibility has been raised by researchers that SED may be a more fundamental theory than QM, in that QM may be derivable from SED, but not vice-versa. However, there are many unsolved problems in SED including a full understanding of hydrogen, line spectra and a deeper understanding of excited states (Refs. [27–29] may have some bearing here eventually), diffraction and interference patterns of charged particles, and creation and annihilation operations of charged particles. For some of these, qualitative and sometimes deeper explanations exist, such as for the wavelike behavior of diffraction and interference of particles [12,30], photon-like behavior [15,31], and superfluid behavior [15,32]. As emphasized by Boyer in Refs. [15,30], SED provides a classical physics description with the recognition that ZP electromagnetic fields need to be included, resulting in a stochastic classical physics theory that greatly widens the physical phenomena that are addressable, including SHO behavior, Casimir forces, van de Waals forces, oscillator specific heats, blackbody radiation, diamagnetism, and effects of acceleration through the vacuum, all of which agree with QM results.

The present article examines how entropy effects have been included in the analysis of classical electrodynamic systems in SED. In an early SED article in 1969 [33], Boyer presented physical arguments that there was a need to distinguish between what he referred to as "caloric entropy", S_{cal}, and "probabilistic entropy", S_{prob}. When zero-point energy is included, he argued that the two approaches yield different results. The most detailed analyses in SED have dealt with the former, caloric entropy, which is what we will concentrate on in this article.

As for an outline, Sec. II will discuss general concepts of these two entropies. The remainder of the article deals with calculations involving S_{cal}. Section III will turn to thermodynamic processes involving displacement operations and temperature changes for interacting electric dipole SHOs bathed in ZP plus thermal radiation. Section IIIA covers the "all distance" case between SHO electric dipoles, while Sec. IIIB turns to the shorter distance scenario, which results in more recognizable formulae. Section IIIC briefly discusses the thermodynamics of radiation within cavities that can change in size and shape. The article ends with "concluding remarks" in Sec. IV.

Before proceeding with these discussions, a brief outline of the main results of the thermodynamic operations analyzed in Refs. [5–9] is as follows. Let $\rho(\omega, T)$ be the classical electromagnetic radiation spectrum in thermal equilibrium with the systems discussed in Refs. [5–10], where

$$\frac{1}{8\pi}\left\langle \mathbf{E}_{\text{T}}^2 + \mathbf{B}_{\text{T}}^2 \right\rangle = \int_0^\infty d\omega \rho(\omega, T) \ , \tag{1}$$

and $\mathbf{E}_{\text{T}} + \mathbf{B}_{\text{T}}$ are the corresponding electric and magnetic electromagnetic radiation fields that constitute the thermodynamic equilibrium radiation fields (ZP fields included) at temperature T. The angular brackets in Eq. (1) represent an ensemble average over the radiation fields. Here, ω is the angular frequency associated with these fields. Each of Refs. [5–10] contain a demonstration that for no heat to flow at temperature $T = 0$ during (slow) reversible displacement operations of the discussed systems, then $\rho(\omega, T = 0)$ must be proportional to ω^3. Expressing $\rho(\omega, T)$ in terms of $U(\omega, T)$, the average energy per normal mode at temperature T and frequency ω, then

$$\rho(\omega, T) = \frac{\omega^2}{\pi^2 c^3} U(\omega, T) \ , \tag{2}$$

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where $\frac{\omega^2}{\pi^2 c^3}$ in Eq. (2) is the number of normal modes per unit volume and per unit angular frequency interval. Hence, at $T = 0$,

$$U(\omega, T = 0) = K\omega \ , \tag{3}$$

where K is a constant. References [5–9] concern electrodynamic systems interacting via either van der Waals force or Casimir forces. To obtain the correct results for these situations requires that the constant scaling factor K in Eq. (3) must be given by

$$K = \frac{\hbar}{2} \ . \tag{4}$$

The name "zero point radiation" has to do with the thermodynamic radiation fields at the absolute temperature $T = 0$. It should also be noted that Refs. [5–10] usually dealt with the function $h^2(\omega, T)$ where

$$\rho(\omega, T) \equiv \frac{\omega^2}{c^3} h^2(\omega, T) \ , \tag{5}$$

rather than the function $U(\omega, T)$ in Eq. (2). Either is fine, but $U(\omega, T)$ has a more relatable physical meaning.

2. Two forms of entropy

2.1. Caloric form of entropy S_{cal} in classical physics

As described in Secs. 7-7 through 7-10 in Ref. [34], the concept of S_{cal} is a well defined quantity due to the first and second laws of thermodynamics, with the second law being particularly important here. The second law is what ensures that dS_{cal} is an exact differential and that the absolute temperature Kelvin scale exists and is well defined. Energy conservation holds that the change ΔU_{int} in internal energy of a system is equal to the heat energy Q that flows into the system plus the work W done on the system:

$$\Delta U_{\text{int}} = Q + W \ . \tag{6}$$

We will be considering ensembles of similarly prepared systems. Taking the ensemble average, represented by angular brackets $\langle \ \rangle$, and considering for now small changes, reduces to the first law of thermodynamics:

$$d\langle U_{\text{int}} \rangle = ð\langle Q \rangle + ð\langle W \rangle \ . \tag{7}$$

Here, $ðQ$ and $ðW$ are inexact differentials, dependent on the path of the thermodynamic process and not on the endpoints. In this article and in the articles cited here, $ð\langle Q \rangle$ will be the small change in the ensemble average of the heat radiation that flows into the system of interest, while $ð\langle W \rangle$ will be the same for work done on the system. The symbol ð represents an inexact differential that depends on the path chosen for the system to evolve. As known in mathematics, the sum of two inexact differentials can sum to an exact differential, as is the case here for $d\langle U_{\text{int}} \rangle$.

From the second law of thermodynamics, an integrating factor of $1/\lambda$ can be proven to exist for the heat flow $ð\langle Q \rangle_{\text{R}}$ into the system during any reversible process, so that

$$dS_{\text{cal}} = \frac{ð\langle Q \rangle_{\text{R}}}{\lambda} \ , \tag{8}$$

is an exact differential of the "caloric" entropy function S_{cal}. Here, dS_{cal} is the difference between two entropy surfaces, where the symbol "R" means that reversible thermodynamic processes are being considered. As discussed in Ref. [34], the differential relationship for the function of state dS_{cal} exists because of the second law of thermodynamics. For the systems discussed in Sec. IIIA and IIIB in this article, where there are N electric dipole SHOs in 3D space, S_{cal} will be a function of $3N$ independent thermodynamic coordinates for the $N \times 3$ position coordinates of the oscillators, plus

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one more coordinate, namely, the temperature of the system, so $3N+1$ independent thermodynamic coordinates.

Moreover, not only does an integrating factor exist for the $\eth Q$ of any system, but this integrating factor λ can be expressed as

$$\lambda = \phi(T) f(S_{\text{cal}}) \ , \tag{9}$$

where $\phi(T)$ is as yet an undetermined function of temperature, but the same function for any system, times a function of S_{cal}. Because of Eqs. (8) and (9), the ratio of the reversible isothermal heat flow at temperature T_1, $\Delta\langle Q\rangle_{\text{R},T_1}$, divided by the reversible isothermal heat flow at temperature T_2, $\Delta\langle Q\rangle_{\text{R},T_2}$, where both isothermal curves traverse between the same entropy surfaces $S_{\text{cal},1}$ and $S_{\text{cal},2}$, is given by:

$$\frac{\Delta\langle Q\rangle_{\text{R},T_1}}{\Delta\langle Q\rangle_{\text{R},T_2}} = \frac{\phi(T_1)\int_{S_{\text{cal},1}}^{S_{\text{cal},2}} f(S_{\text{cal}}) dS_{\text{cal}}}{\phi(T_2)\int_{S_{\text{cal},1}}^{S_{\text{cal},2}} f(S_{\text{cal}}) dS_{\text{cal}}} = \frac{\phi(T_1)}{\phi(T_2)} \ . \tag{10}$$

Using the choice of $\phi(T) = T$, so that

$$\frac{\phi(T_1)}{\phi(T_2)} = \frac{T_1}{T_2} \ , \tag{11}$$

results in the absolute temperature Kelvin scale and also results in the concept of being at a "zero point" temperature with no heat flow at $T=0$ in Eq. (10). Note that systems can fluctuate in energy at $T=0$, but the ensemble average of heat flow at $T=0$ is zero during this reversible thermodynamics process, since

$$\Delta\langle Q\rangle_{\text{R},T=0} = \phi(T=0)\int_{S_{\text{cal},1}}^{S_{\text{cal},2}} f(S_{\text{cal}}) dS_{\text{cal}} = 0 \ . \tag{12}$$

The calculations in Sec. III, yet to come here, proceed from this overview of S_{cal}. In the case of electric dipole SHOs in Sec. IIIA and IIIB, these calculations follow from the electromagnetic ZP fluctuations and how they propagate to the fluctuations of the SHOs. In Sec. IIIC, just field fluctuations are considered within cavities of electromagnetic radiation. The ensemble averages of the heat flow, the internal energy, and the work done, are all carried out with respect to the impact of the radiation fluctuations on the net systems that will be discussed.

2.2. Probabilistic based entropy S_{prob} in classical physics

2.2.1. Main points of S_{prob} and SED

Here we will use the suggestion by Boyer in Ref. [33] to refer to the "probabilistic entropy," S_{prob}, as being related to the quantity Ω of the number of energy microstates of a system that provides the same macrostate, such that

$$S_{\text{prob}} = k_{\text{B}} \ln \Omega \ , \tag{13}$$

where k_{B} is Boltzmann's constant. In the original developments of statistical mechanics by Boltzmann, Maxwell, Gibbs, and others, the notions of S_{prob} and Ω occurred before the start of QM. As originally conceived, the measure or size of the phase space of a mechanical system at a constant value of energy E, was key to determining the probability of the system being in this net "energy state." All phase space points with the same energy value were presumed to be equally probable to occur. A key example of such a mechanical system was a point particle with mass m linearly bound by position in 3D space to an equilibrium point. If this simple classical mechanical system was in equilibrium with a heat reservoir at a constant temperature T, then all phase space points for particles (6D phase space

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with 3D in position and 3D in momentum) with the same energy were assumed to be equally probable when computing $S_{\rm prob}$ in statistical mechanics for classical physics.

Once one alters the SHOs to being N electric dipole oscillators, each interacting electromagnetically with each other as well as with ZP radiation or ZP plus Planckian radiation (ZPP), the situation clearly becomes significantly more complicated regarding the probability and "counting" of phase space points with equal energy in this $6N$ dimensional phase space. This leads us to recognize one important point in SED, that to date there is no direct connection between $S_{\rm prob}$ in Eq. (13) to the ensemble average of heat flow and work done, as was discussed for $S_{\rm cal}$ in Sec. IIA. $S_{\rm prob}$ may not satisfy Eq. (8) with $dS_{\rm prop}$ substituted for $dS_{\rm cal}$ in Eq. (8), especially with the zero point fluctuations needing to be taken into account. In contrast, when combining QM with statistical mechanics notions, the situation is quite different. Counting of equal energy microstates is well defined in this situation and derivations are available and taught on how to calculate internal energy and work done; the use of the partition function for particles energies is a common tool to simplify this process.

The point trying to be made here is that in SED, except in the simplest of situations (such as a single SHO), the phase space idea has not been used to calculate ensemble averages of internal energy, heat flow, and work done. This does not mean that probability values for equilibrium situations, ensemble averages, etc., have not been calculated in SED. They certainly have. References [35–37] provide a method and examples of such calculations, starting from the probability notions of ZPP radiation and as propagating to the interaction behavior of charged point particles. As early as in 1963 [38] the probability distribution for a single particle in a SHO binding potential was calculated with ZP radiation present. More complicated systems, such as described next in Sec. III, have not been tackled in any sort of detail using phase space notions, with part of the problem being how to include phase space notions with ZP fields included in the analysis.

As an aside here, the original ideas of Boltzmann and others for "counting" or including phase space states of measure/size Ω in classical physics, will be illustrated here, before turning to $S_{\rm cal}$ notions in Sec. III. The expressions follow the early classical ideas of these quantities, although we are not aware of these calculations being examined in the past in any sort of detail. We will consider three cases for illustration, namely, the 1D SHO phase space, then the phase space for N independent 1D SHOs, which includes the case of a single 3D SHO, and finally the phase space for the nonrelativistic classical hydrogen model. In each case Ω will be computed as a function of energy of the system, as if the single system was in thermodynamic equilibrium with a heat reservoir.

2.2.2. $\Omega(E)$ for a 1D SHO

Let the infinitesimal phase space "area" for a 1D SHO be denoted by $\Delta x \Delta p_x$, where Δx and Δp_x are an infinitesimal length and momentum in phase space, respectively, and where for our nonrelativistic consideration, $p_x = mv_x = m\dot{x}$. (This example is discussed on pp. 55-56 in Ref. [39], but not carried out in detail.) The "number" of phase space regions of size $\Delta x \Delta p_x$ within the energy interval of E to $E + \delta E$ is then given by

$$\Omega_{\rm SHO}(E) = \frac{\int dx \int dp_x \delta\left[E - \left(\frac{1}{2m}p_x^2 + \frac{1}{2}kx^2\right)\right]}{\Delta x \Delta p_x} \delta E \ , \tag{14}$$

where $\delta\left[E - \left(\frac{1}{2m}p_x^2 + \frac{1}{2}kx^2\right)\right]$ is the Dirac delta function, being used to select the SHO energies of value E. This quantity equals zero except when

$$p_x = \pm\sqrt{2m\left(E - \frac{1}{2}kx^2\right)} = \pm\sqrt{2mE - mkx^2} \ . \tag{15}$$

We'll use the relationship

$$\delta\left[E - \left(\frac{1}{2m}p_x^2 - \frac{1}{2}kx^2\right)\right] = \frac{\delta\left\{p_x - \left[2mE - mkx^2\right]^{1/2}\right\}}{\left|\frac{\partial E}{\partial p_x}\right|} + \frac{\delta\left\{p_x + \left[2mE - mkx^2\right]^{1/2}\right\}}{\left|\frac{\partial E}{\partial p_x}\right|} \ , \tag{16}$$

where

$$\left|\frac{\partial E}{\partial p_x}\right| = \left|\frac{\partial}{\partial p_x}\left(\frac{1}{2m}p_x^2 + \frac{1}{2}kx^2\right)\right| = \left|\frac{p_x}{m}\right| \ . \tag{17}$$

Integrating in Eq. (14) first over p_x results in:

$$\int_{-\infty}^{+\infty} dp_x \delta\left[E - \left(\frac{1}{2m}p_x^2 + \frac{1}{2}kx^2\right)\right] = \frac{2}{\frac{1}{m}(2mE - mkx^2)^{1/2}} \ . \tag{18}$$

Continuing with the integration over x, we first note that the potential energy $\frac{1}{2}kx^2$ cannot be larger than E, since, from the outset, $E = \frac{1}{2m}p_x^2 + \frac{1}{2}kx^2 \geq 0$. Thus, the largest x can be is when p_x is zero and x reaches its maximum at the extremes at the ends of the ellipse $\left(\frac{1}{2m}p_x^2 + \frac{1}{2}kx^2\right)$ at

$$x = \pm\left(\frac{2E}{k}\right)^{1/2} \ . \tag{19}$$

Consequently,

$$\int dx \int dp_x \delta\left[E - \left(\frac{1}{2m}p_x^2 + \frac{1}{2}kx^2\right)\right] = 2\sqrt{\frac{m}{k}} \int_{-\left(\frac{2E}{k}\right)^{1/2}}^{+\left(\frac{2E}{k}\right)^{1/2}} dx \frac{1}{\left(\frac{2E}{k} - x^2\right)^{1/2}} \ . \tag{20}$$

Letting $A = +\left(\frac{2E}{k}\right)^{1/2}$, the integral

$$\int_{-A}^{A} dx \frac{dx}{(A^2 - x^2)^{1/2}} \ ,$$

with substitution

$$\frac{x}{A} = \sin u \ \text{, so, } dx = A(\cos u)du \ ,$$

becomes

$$\int_{-A}^{+A} \frac{dx}{(A^2 - x^2)^{1/2}} = \frac{1}{A}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{A(\cos u)du}{(1 - \sin^2 u)^{1/2}} = \pi \ . \tag{21}$$

Hence,

$$\int dx \int dp_x \delta\left[E - \left(\frac{1}{2m}p_x^2 + \frac{1}{2}kx^2\right)\right] = 2\pi\sqrt{\frac{m}{k}} \ , \tag{22}$$

and

$$\Omega_{\text{SHO}}(E) = \frac{\int dx \int dp_x \delta\left[E - \left(\frac{1}{2m}p_x^2 + \frac{1}{2}kx^2\right)\right]}{\Delta x \Delta p_x}\delta E = \frac{2\pi\left(\frac{m}{k}\right)^{1/2}}{\Delta x \Delta p_x}\delta E \ . \tag{23}$$

Thus, for the 1D SHO, the somewhat surprising result is that the "number" of phase space regions of size $\Delta x \Delta p_x$ within the energy interval of E to $E + \delta E$, is independent of E, so for any value of E,

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the number of microstates $\Omega_{\text{SHO}}(E)$ does not change. This serves as a counter example to the usual notion that Ω increases rapidly with E [39].

2.2.3. $\Omega(E)$ for N 1D SHOs

Let the infinitesimal phase space for N 1D SHOs be denoted by $\Delta x_1 \Delta p_1 \ldots \Delta x_N \Delta p_N$. The "number" of phase space regions of size $\Delta x_1 \Delta p_1 \ldots \Delta x_N \Delta p_N$ within the energy interval of E to $E + \delta E$ is then given by

$$\Omega_{\text{N-1D-SHOs}}(E) = \frac{\int dx_1 \int dp_1 \ldots \int dx_N \int dp_N \delta\left[E - \sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right]}{\Delta x_1 \Delta p_1 \ldots \Delta x_N \Delta p_N} \delta E \ . \tag{24}$$

It should be noted that when $N = 3$, the result will be the same as for three 1D oscillators from the formula above, the same as for a single 3D isotropic SHO.

Since the order does not matter, we will integrate over the momentum coordinates first, starting with p_1. Contributions to

$$\int_{-\infty}^{\infty} dp_1 \delta\left[E - \sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right]$$

occur when

$$p_1 = \pm\left(2mE - \sum_{i=2}^{N} p_i^2 - \sum_{i=1}^{N} mkx_i^2\right)^{1/2} \ . \tag{25}$$

Note that this means that only the two real roots of this square root function are selected, since p_1 ranges only from $-\infty \le p_1 \le \infty$ over real values, and for this to occur, the quantity

$$2mE - \sum_{i=2}^{N} p_i^2 - \sum_{i=1}^{N} mkx_i^2$$

on the RHS in ( ) brackets of Eq. (25) must be positive, or zero, but clearly non-negative, since

$$2mE \ge \sum_{i=2}^{N} p_i^2 + \sum_{i=1}^{N} mkx_i^2 \ , \tag{26}$$

which will restrict the range of integrations for each of the variables, p_i for $i = 2, \ldots, N$, and x_i for $i = 1, \ldots, N$.

Continuing,

$$\delta\left[E - \sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right] = \frac{\delta\left[p_1 - \left(2mE - \sum_{i=2}^{N} p_i^2 - \sum_{i=1}^{N} mkx_i^2\right)^{1/2}\right]}{\left|\frac{\partial E}{\partial p_1}\right|} + \frac{\delta\left[p_1 + \left(2mE - \sum_{i=2}^{N} p_i^2 - \sum_{i=1}^{N} mkx_i^2\right)^{1/2}\right]}{\left|\frac{\partial E}{\partial p_1}\right|} \ . \tag{27}$$

Here,

$$\left|\frac{\partial E}{\partial p_1}\right| = \left|\frac{\partial}{\partial p_1}\sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right| = \left|\frac{p_1}{m}\right| \ . \tag{28}$$

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Hence:

$$\int_{-\infty}^{+\infty} dp_1 \delta\left[E - \sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right] = \frac{2m}{\left(2mE - \sum_{i=2}^{N} p_i^2 - \sum_{i=1}^{N} mkx_i^2\right)^{1/2}} \; . \tag{29}$$

In the remaining integrations we will have integration limits of:

$$2mE \geq \sum_{i=2}^{N} p_i^2 + \sum_{i=1}^{N} mkx_i^2 \;, \tag{30}$$

which will restrict the range of integrations for each of the variables, p_i for $i = 2, \ldots, N$, and x_i for $i = 1, \ldots, N$. The appropriate real range of p_2 is bounded by:

$$-\left(2mE - \sum_{i=3}^{N} p_i^2 - \sum_{i=1}^{N} mkx_i^2\right)^{1/2} \leq p_2 \leq \left(2mE - \sum_{i=3}^{N} p_i^2 - \sum_{i=1}^{N} mkx_i^2\right)^{1/2} \; . \tag{31}$$

Since p_2 ranges only over real values above, then

$$2mE - \sum_{i=3}^{N} p_i^2 - \sum_{i=1}^{N} mkx_i^2$$

in Eq. (31) must be positive, or zero, but clearly non-negative , so

$$2mE \geq \sum_{i=3}^{N} p_i^2 + \sum_{i=1}^{N} mkx_i^2 \;, \tag{32}$$

which will restrict the range of integrations for each of the variables, p_i for $i = 3, \ldots, N$, and x_i for $i = 1, \ldots, N$.

Hence, with

$$p_{2,\mathrm{m}} \equiv \left(2mE - \sum_{i=1}^{N} mkx_i^2 - \sum_{i=3}^{N} p_i^2\right)^{1/2} \;, \tag{33}$$

then

$$\begin{aligned} &\int_{-p_{2,\mathrm{m}}}^{p_{2,\mathrm{m}}} dp_2 \int dp_1 \delta\left[E - \sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right] \\ &= 2m \int_{-p_{2,\mathrm{m}}}^{p_{2,\mathrm{m}}} dp_2 \frac{1}{\left[\left(2mE - \sum_{i=1}^{N} mkx_i^2 - \sum_{i=3}^{N} p_i^2\right) - (p_2)^2\right]^{1/2}} \;, \end{aligned} \tag{34}$$

which results in the same integral as in Eq. (21).

Consequently,

$$\int_{-p_{2,\mathrm{m}}}^{p_{2,\mathrm{m}}} dp_2 \int_{-\infty}^{\infty} dp_1 \delta\left[E - \sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right] = 2m\pi \; . \tag{35}$$

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Continuing, we now have that:

$$\begin{aligned}&\int dx_1 \int dp_1 \ldots \int dx_N \int dp_N \delta\left[E - \sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right] \\ &= 2m\pi\left(\int dx_1 \int dx_2 \ldots \int dx_N \int dp_3 \int dp_4 \ldots \int dp_N\right)_{\text{Such that } 2mE \geq \sum_{i=3}^{N} p_i^2 + \sum_{i=1}^{N} mkx_i^2} . \end{aligned} \tag{36}$$

Let us call this quantity Q and let us make coordinates to be of the same dimensions, so, let $p_i' = (mk)^{1/2}x_i$ for $i = 1, \ldots, N$. We then have that

$$\begin{aligned} Q &= 2m\pi\left(\int dx_1 \int dx_2 \ldots \int dx_N \int dp_3 \int dp_4 \ldots \int dp_N\right)_{\text{Such that } 2mE \geq \sum_{i=3}^{N} p_i^2 + \sum_{i=1}^{N} mkx_i^2} \\ &= \frac{2m\pi}{(mk)^{N/2}}\left(\int dp_1' \int dp_2' \ldots \int dp_N' \int dp_3 \int dp_4 \ldots \int dp_N\right)_{\text{Such that } 2mE \geq \sum_{i=3}^{N} p_i^2 + \sum_{i=1}^{N} (p_i')^2} . \end{aligned} \tag{37}$$

The integrations within the ( ) parentheses form a sphere in $2N-2$ dimensional space. The n-dimensional volume of a Euclidean ball of radius R in n-dimensional Euclidean space is [40]

$$V_n(R) = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}R^n \ , \tag{38}$$

where Γ is Leonhard Euler's gamma function. It satisfies $\Gamma(n) = (n-1)!$ if n is a positive integer. We have a Euclidean ball of radius $R = (2mE)^{1/2}$ in $(2N-2)$-dimensional Euclidean space. Thus,

$$Q = \frac{2m\pi}{(mk)^{N/2}} \frac{\pi^{\frac{(2N-2)}{2}}}{\Gamma\left(1+\frac{(2N-2)}{2}\right)}\left[(2mE)^{1/2}\right]^{(2N-2)} = \frac{2m\pi}{(mk)^{N/2}} \frac{\pi^{(N-1)}}{(N-1)!}(2mE)^{(N-1)} \ , \tag{39}$$

since

$$\Gamma\left(1+\frac{(2N-2)}{2}\right) = \Gamma(N) = (N-1)! \tag{40}$$

Summarizing:

$$\begin{aligned} Q &= \int dx_1 \int dp_1 \ldots \int dx_N \int dp_N \delta\left[E - \sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right] \\ &= \left(\int dx_1 \int dx_2 \ldots \int dx_N \int dp_3 \int dp_4 \ldots \int dp_N 2m\pi\right)_{\text{Such that } 2mE \geq \sum_{i=3}^{N} p_i^2 + \sum_{i=1}^{N} mkx_i^2} . \\ &= \frac{2m\pi}{(mk)^{N/2}} \frac{\pi^{(N-1)}}{(N-1)!}(2mE)^{(N-1)} \ . \end{aligned} \tag{41}$$

Consequently,

$$\begin{aligned} \Omega_{\text{N-1D-SHOs}}(E) &= \frac{\int dx_1 \int dp_1 \ldots \int dx_N \int dp_N \delta\left[E - \sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2 + \frac{1}{2}kx_i^2\right)\right]}{\Delta x_1 \Delta p_1 \ldots \Delta x_N \Delta p_N}\delta E \\ &= Q \times \frac{\delta E}{\Delta x_1 \Delta p_1 \ldots \Delta x_N \Delta p_N} \\ &= 2^N \pi^N E^{(N-1)}\left(\frac{m}{k}\right)^{\frac{N}{2}} \frac{1}{(N-1)!} \times \frac{\delta E}{\Delta x_1 \Delta p_1 \ldots \Delta x_N \Delta p_N} \ . \end{aligned} \tag{42}$$

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Thus the number of microstates in $E \to E + \delta E$ scales as $E^{(N-1)}$. As an aside we note that the electromagnetic radiation in a cavity can be expressed as a sum of energies of mathematical form similar to the material SHOs discussed above. (See pp. 13-15 in Ref. [41].)

When $N = 1$, we obtain

$$2\pi\left(\frac{m}{k}\right)^{\frac{1}{2}} \times \frac{\delta E}{\Delta x_1 \Delta p_1} \ ,$$

agreeing with our earlier result in Eq. (23). When $N = 3$, we get the result for a single isotropic 3D SHO with

$$\Omega_{\text{3D-SHOs}}(E) = 2^3\pi^3 E^2\left(\frac{m}{k}\right)^{\frac{3}{2}}\frac{1}{2} \times \frac{\delta E}{\Delta x_1 \Delta p_1 \Delta x_2 \Delta p_2 \Delta x_3 \Delta p_3} \ . \tag{43}$$

2.2.4. $\Omega(E)$ for the nonrelativistic classical hydrogen model

We are now interested in:

$$\Omega_{\text{H}}(E) = \frac{\int d^3x \int d^3p\,\delta\left[E - \left(\frac{\mathbf{p}^2}{2m} - \frac{e^2}{|\mathbf{x}|}\right)\right]}{\Delta x \Delta p_x \Delta y \Delta p_y \Delta z \Delta p_z}\delta E \ . \tag{44}$$

We will focus on only the elliptical orbits, or the "bound orbits." For any elliptical orbit, the energy is $E = -\frac{e^2}{2a}$, where a is the semimajor axis. Let us first integrate over the momentum variables:

$$\int d^3p\,\delta\left[E - \left(\frac{\mathbf{p}^2}{2m} - \frac{e^2}{|\mathbf{x}|}\right)\right] = \int_0^\infty dp \int_0^\pi d\theta_p\, p \sin\theta_p \int_0^{2\pi} p d\phi_p \delta\left[E - \left(\frac{\mathbf{p}^2}{2m} - \frac{e^2}{r}\right)\right] \ . \tag{45}$$

Replacing E with $-\frac{e^2}{2a}$,

$$\delta\left[E - \left(\frac{\mathbf{p}^2}{2m} - \frac{e^2}{r}\right)\right] = \frac{\delta\left\{p - \left[2m\left(\frac{e^2}{r} - \frac{e^2}{2a}\right)\right]^{1/2}\right\}}{|p/m|} + \frac{\delta\left\{p + \left[2m\left(\frac{e^2}{r} - \frac{e^2}{2a}\right)\right]^{1/2}\right\}}{|p/m|} \ . \tag{46}$$

p is always positive in this spherical coordinate momentum integration. Here

$$p = \left[2m\left(\frac{e^2}{r} - \frac{e^2}{2a}\right)\right]^{1/2} \ , \tag{47}$$

will contribute in the integration of Eq. (45) over positive p, if

$$\left[2m\left(\frac{e^2}{r} - \frac{e^2}{2a}\right)\right]^{1/2} \geq 0 \ , \tag{48}$$

or when

$$r \leq 2a \ . \tag{49}$$

The second delta function in Eq. (46) will not contribute in Eq. (45).

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$$\begin{aligned}
&\int_0^\infty dp \int_0^\pi d\theta_p p \sin\theta_p \int_0^{2\pi} p d\phi_p \delta\left[E - \left(\frac{p^2}{2m} - \frac{e^2}{r}\right)\right] \\
&= m \int_0^\pi d\theta_p \sin\theta_p \int_0^{2\pi} d\phi_p \left\{ \begin{array}{c} \left[2m\left(\frac{e^2}{r} - \frac{e^2}{2a}\right)\right]^{1/2} \text{ if } r \leq 2a \\ 0 \text{ if } r > 2a \end{array} \right\} \\
&= m4\pi \times \left\{ \begin{array}{c} \left[2m\left(\frac{e^2}{r} - \frac{e^2}{2a}\right)\right]^{1/2} \text{ if } r \leq 2a \\ 0 \text{ if } r > 2a \end{array} \right\} . \qquad (50)
\end{aligned}$$

Consequently:

$$\begin{aligned}
&\int d^3x \int d^3p \delta\left[E - \left(\frac{\mathbf{p}^2}{2m} - \frac{e^2}{|\mathbf{x}|}\right)\right] \\
&= \int_0^\infty dr \int_0^\pi d\theta r \sin\theta \int_0^{2\pi} r d\phi \times \left(m4\pi \times \left\{ \begin{array}{c} \left[2m\left(\frac{e^2}{r} - \frac{e^2}{2a}\right)\right]^{1/2} \text{ if } r \leq 2a \\ 0 \text{ if } r > 2a \end{array} \right\} \right) \\
&= m4\pi \int_0^{2a} dr \int_0^\pi d\theta r \sin\theta \int_0^{2\pi} r d\phi \left[2m\left(\frac{e^2}{r} - \frac{e^2}{2a}\right)\right]^{1/2} \\
&= m(4\pi)^2 \int_0^{2a} drr^2 \left[2m\left(\frac{e^2}{r} - \frac{e^2}{2a}\right)\right]^{1/2} \qquad (51) \\
&= m(4\pi)^2 e(2m)^{1/2} \int_0^{2a} drr^2 \left[\left(\frac{1}{r} - \frac{1}{2a}\right)\right]^{1/2} . \qquad (52)
\end{aligned}$$

Here, with the use of the Mathematica software tool,

$$\int_0^{2a} drr^2 \left[\left(\frac{1}{r} - \frac{1}{2a}\right)\right]^{1/2} = \frac{a^{\frac{5}{2}}\pi}{2^{\frac{3}{2}}} , \qquad (53)$$

resulting in

$$\int d^3x \int d^3p \delta\left[E - \left(\frac{\mathbf{p}^2}{2m} - \frac{e^2}{|\mathbf{x}|}\right)\right] = 8m^{\frac{3}{2}}\pi^3 e a^{\frac{5}{2}} . \qquad (54)$$

Substituting in $a = \frac{e^2}{2|E|}$, then

$$\int d^3x \int d^3p \delta\left[E - \left(\frac{\mathbf{p}^2}{2m} - \frac{e^2}{|\mathbf{x}|}\right)\right] = 2^{1/2}\pi^3 m^{\frac{3}{2}} e^6 \frac{1}{|E|^{\frac{5}{2}}} , \qquad (55)$$

and

$$\Omega_{\mathrm{H}}(E) = 2^{1/2}\pi^3 m^{\frac{3}{2}} e^6 \frac{1}{|E|^{\frac{5}{2}}} \frac{\delta E}{\Delta x \Delta p_x \Delta y \Delta p_y \Delta z \Delta p_z} . \qquad (56)$$

Since $E = -\frac{e^2}{2a}$ for elliptical orbits with semimajor axis a, as a increases, E decreases and $\Omega_{\mathrm{H}}(E)$ increases, as might be expected.

2.2.5. Some summary points on S_{prob}

In conventional statistical mechanics with QM systems, Ω for a system in thermodynamic equilibrium with a heat reservoir (constant temperature T) is arrived at by "counting" the quantized energy

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microstates of a system. In classical physics, the microstates are the "size" of the net phase space divided by a small phase space size, such as $\Delta x \Delta p_x$ for the 1D SHO, such that all phase space points in this phase space have an energy between E and $E + \delta E$. Three such examples were analyzed here to illustrate the dependence of the size of Ω as a function of E. Although illustrative of the basic ideas, these examples do not bring us closer to handling the types of systems in SED that will be discussed in Sec. III using S_{cal} . The methods in Refs. [35–37] could be used to properly propagate the stochastic nature of the ZP or ZPP fields to the particle motion. For a single SHO, or multiple SHOs that are not interacting, the calculations would not be very difficult and have in fact been done for a single SED some time ago [38]. However, when N electric dipole SHOs are interacting, as is the case in the next section, this method seems quite prohibitive.

3. Calculations in SED using S_{cal}

3.1. N 3D electric dipole SHOs bathed in ZPP classical electromagnetic radiation

3.1.1. All distances and temperature van der Waals conditions

An arbitrary, but fixed number N, of electric dipole oscillators were studied in Refs. [5,6,19] that were in interaction with stochastic "incident" electromagnetic radiation fields. Specific incident and thermal radiation examples were specifically examined for possibility of satisfying various demands such as "no heat flow" during reversible displacement operations, the third law of thermodynamics, and restrictions on specific heats. ZP and ZPP radiation satisfied these constraints. The positions of the oscillators are arbitrary. The resonant frequencies could readily have been made arbitrary but for convenient purposes were all held to the same value of ω_0. All distances were allowed and interactions between electric dipole oscillators were executed by the van der Waals force expressions valid for all distances [17–19,42]. The ensemble average of the full retarded force between oscillators was obtained in Ref. [19], which was used in Ref. [5] for determining the "work done" when slowly displacing the oscillators. Calculations for ensemble averages of changes in internal energy were carried out for a large volume V surrounding the oscillators, where the surface of this volume was far from any of the oscillators. By finding the change in ensemble averages of internal energy and work done during (slow) reversible displacements of the oscillators from each other, the ensemble average of the heat flow due to radiation across the surface bounding the volume V was deduced.

The expectation value of the internal energy within V was calculated in Ref. [5] via the kinetic and potential energy changes of the oscillators, but also the far more complicated electromagnetic field energy (cgs units)

$$\left\langle \frac{1}{8\pi} \int_V d^3x \left(\mathbf{E}^2_{\text{Total}} + \mathbf{B}^2_{\text{Total}} \right) \right\rangle ,$$

where the total fields were due to the "incident" radiation plus the radiated fields from each of the N oscillators. Later sections of Refs. [5,6] considered the specific "incident radiation" cases of thermodynamic candidates of ZP, ZPP, and Rayleigh-Jeans (RJ) radiation. Because of the square of the fields in the above expression, cross terms existed for the oscillator fields and the oscillator-incident fields. These represented the longest of the calculations, published as an appendix in Ref. [5]. For isothermal reversible thermodynamic displacements, the only expectation value of a nonzero spectrum of radiation that resulted in no heat radiated in or out of V, was found to be the ZP radiation spectrum of Eq. (3) [5].

A rather surprising result was that the radiated energy from the oscillators, as if they were single oscillators radiating, increased in accordance to the size of V, but was cancelled out by a similar term, a cross term, between the incident fields and the dipole fields. A similar cancellation occurred between the cross term field energy of the oscillators and the cross term field energy of the oscillator fields and incident fields. No approximation was made that the interaction energy was either small (or

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large) between dipoles, which is often made in statistical mechanics assumptions of particle interaction versus interaction with a heat reservoir.

In this manner, the expectation value of the work done and change in internal energy was found in Ref. [5] for a large volume V enclosing the oscillators and their displacement positions. The proof of no heat flow, for nonzero "incident" radiation, during reversible displacement operations for N oscillators, was found valid only when a ZP radiation spectrum existed, coinciding with the definition of $T = 0$. To show this, the "steady state" solution to the radiation fields acting on the dipole oscillators was calculated, including the interaction in this solution to all dipoles acting on each other. The stochastic nature of the fields then propagated to the stochastic behavior of the SHOs.

Reference [6] turned to taking the heat flow calculations to obtain dS_{cal} from Eq. (8) during reversible displacement operations for isothermal conditions of constant T. This process was then combined with taking into account heat flows when the dipole positions were held fixed, but the temperature within V of both fields and dipoles was changed reversibly by considering accumulated infinitesimal interactions with heat reservoirs of slowly changing temperature. The net result was an expression for S_{cal} (Eq. 59 in Ref. [6]) that was a function of $3N$ +1 thermodynamic coordinates ($3\times N$ position coordinates plus temperature T). The result enables one to analyze arbitrary reversible processes for this system, including a cycle such as the Carnot cycle of two processes done isothermally, bordered by two adiabatic processes (no heat flow). The latter is done by changing the position of the oscillators while also changing T, to ensure that dS_{cal} remains zero. Adiabatic surfaces in $3 \times N$ thermodynamic coordinate space can then be constructed (see for example Fig. 7-8 in Ref. [34]).

3.1.2. Unretarded van der Waals condition and resonant SHO approximation

The expressions in Refs. [5,6] were quite lengthy. To simplify the calculations and still see if no heat, or no change in S_{cal}, occurred in reversible position displacement operations at $T = 0$, Ref. [7] took the calculations from [5] and made two approximations: (1) the small charge limit was made so that a "resonant oscillator approximation" could be invoked, and (2) the unretarded van der Waals approximation of $\omega_0 R/c \ll 1$, where ω_0 was the resonant frequency of the oscillator and R was a typical distance between oscillators. This resulted in the following sums, looking much more like QM results for sums of QM SHO energy terms. The ensemble average of the potential energy of the oscillators was found in Ref. [7] to be (see Eq. (5)

$$\langle U_{\text{PE}} \rangle \approx \sum_{A=1}^{N} \sum_{i=1}^{3} \frac{\pi^2}{2} h_{\text{in}}^2(\tilde{\omega}_{Ai}) \frac{\omega_0^2}{\tilde{\omega}_{Ai}^2} \quad , \tag{57}$$

while the corresponding kinetic energy was the sum

$$\langle U_{\text{KE}} \rangle \approx \sum_{A=1}^{N} \sum_{i=1}^{3} \frac{\pi^2}{2} h_{\text{in}}^2(\tilde{\omega}_{Ai}) \quad , \tag{58}$$

where A sums over the N oscillators, i sums over the three x, y, z degrees of freedom of each oscillator, ω_0 is the natural resonant frequency of each oscillator, and $\tilde{\omega}_{Ai}$ represents the eigenfrequencies of the system once electromagnetic interactions are taken into account. Finally, the unretarded electromagnetic dipole-dipole interaction energy was found to be

$$\left\langle U_{\text{EM},D-D}^{\text{ur}} \right\rangle \approx \sum_{A=1}^{N} \sum_{i=1}^{3} \frac{\pi^2}{2} h_{\text{in}}^2(\tilde{\omega}_{Ai}) \frac{\tilde{\omega}_{Ai}^2 - \omega_0^2}{\tilde{\omega}_{Ai}^2} \quad . \tag{59}$$

The result for the net of these became:

$$\left\langle U_{\text{PE}} + U_{\text{KE}} + U_{\text{EM},D-D}^{\text{ur}} \right\rangle \approx \sum_{A=1}^{N} \sum_{i=1}^{3} \pi^2 h_{\text{in}}^2(\tilde{\omega}_{Ai}) \quad . \tag{60}$$

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Making the same approximations for the "work done" when slowly displacing the oscillators from each other resulted in

$$\langle W \rangle = \sum_{A=1}^{N} \sum_{i=1}^{3} \int_{\tilde{\omega}_{Ai,\mathrm{I}}}^{\tilde{\omega}_{Ai,\mathrm{II}}} d\tilde{\omega}_{Ai} \pi^2 \frac{h_{\mathrm{in}}^2(\tilde{\omega}_{Ai})}{\tilde{\omega}_{Ai}} \quad . \tag{61}$$

From Eqs. (7), (60), (61):

$$\Delta\left(U_{\mathrm{PE}} + U_{\mathrm{KE}} + U_{\mathrm{EM},D-D}^{\mathrm{ur}}\right) - \langle W \rangle = \sum_{A=1}^{N} \sum_{i=1}^{3} \int_{\tilde{\omega}_{Ai,\mathrm{I}}}^{\tilde{\omega}_{Ai,\mathrm{II}}} d\tilde{\omega}_{Ai} \pi^2 \left[\frac{\partial h_{\mathrm{in}}^2(\tilde{\omega}_{Ai})}{\partial \tilde{\omega}_{Ai}} - \frac{h_{\mathrm{in}}^2(\tilde{\omega}_{Ai})}{\tilde{\omega}_{Ai}} \right] \quad . \tag{62}$$

This equals zero when $h_{\mathrm{in}}^2(\tilde{\omega}_{Ai})$ is proportional to $\tilde{\omega}_{Ai}$, no matter the positions of the oscillators. Since

$$U(\omega, T) = \pi^2 h^2(\omega, T) \tag{63}$$

from Eqs. (2) and (5), we again arrive at $U(\omega, T = 0) = K\omega$. The value of $K = \frac{\hbar}{2}$ then is found by the needed connection to the van der Waals force used throughout the calculations.

Although still not exactly simple, the analysis in Ref. [7] is far more succinct and less involved than in Ref. [5]. Moreover, it is good to see that even when the approximations of unretarded van der Waals expressions, and the resonant approximation of the oscillators are made, that the final $T = 0$ result still holds.

3.1.3. Dynamics involving cavities and fields, using S_{cal}

The analysis outlined here in the previous two subsections for deducing the needed ZP radiation spectral form at $T = 0$ is important in that both radiation fields and charged particle interactions were included, yet still the result of $U(\omega, T = 0) = K\omega$ was deduced. However, this result also still holds up if one only considers radiation fields between two parallel plates or within a cavity. References [8] and [9] examine this situation where the walls of the cavity are deformed in shape and size. Casimir forces are assumed to interact between the "plates" or walls of the cavity, which enters into the "work done" as the walls are displaced. Reference [8] focuses more on operations like a piston in a cylinder, as in Wien's displacement theorem analysis, while Ref. [9] considers more general operations like simple movements or deformations to the walls. In both cases, though, the result still arises that in order to have no heat flow in or out of the cavity, then $U(\omega, T = 0) = K\omega$, with $K = \frac{\hbar}{2}$ being the needed value to fit the form of Casimir forces. In addition, Sec. VII in Ref. [5] calculates S_{cal} for the radiation between two plates. The results for Refs. [8,9] were deduced from the demands of the second law of thermodynamics and the thermodynamic definition of $T = 0$ as applied to radiation in a cavity.

4. Concluding remarks

This article reviewed the situation for S_{prob} and S_{cal}, recognizing along with Ref. [33] that there is a difference in these quantities since ZP fluctuations need to be taken into account. Use of S_{cal} however can be carried out for very complicated systems, with results that make good sense. This quantity is calculated by ensemble averages over either fields (in cavities) as in Refs. [8,9], or over oscillator energies plus field energies as in Refs. [5,6]. Making unretarted van der Waals approximations and the resonant oscillator approximation as in Ref. [7] greatly simplifies the expressions, but still results in the same condition at $T = 0$ of the ZP radiation spectrum, where $\rho(T = 0)$ is proportional to ω^3.

The $T = 0$ condition of Lorentz invariance that yields this same result is in some ways more fundamental; each inertial reference frame should "see" the same ZP spectrum. But in another sense, the "no heat flow" at $T = 0$ for reversible displacement operations also holds in just cavities of radiation, thereby also serving as a fundamental stipulation. Moreover, this result needs to hold for an arbitrary number of dipole oscillators, so fields plus oscillators must obey this result. If the analysis was possible for hydrogen or other atoms, we would also expect the same result to hold.

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