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Posted Date: 25 February 2025

doi: 10.20944/preprints202502.2042.v1

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On Quaternionic Analysis and a Certain Generalized Fractal-Fractional ψ-Fueter Operator

José Oscar González-Cervantes [1,†,‡,*], Juan Adrián Ramírez-Belman [2,‡] and Juan Bory-Reyes [3,‡]

1 Departamento de Matemáticas, ESFM-Instituto Politécnico Nacional. 07338, México
2 SEPI, ESFM-Instituto Politécnico Nacional. 07338, México
3 SEPI, ESIME-Zacatenco-Instituto Politécnico Nacional. 07338, México
* Correspondence: jogc200678@gmail.com; Tel.: 52 55 4318017
† Current address: Departamento de Matemáticas, ESFM-Instituto Politécnico Nacional. 07338, México: IPN.
‡ These authors contributed equally to this work.

Abstract: This paper introduce a fractional-fractal ψ-Fueter operator in the quaternionic context inspired in the concepts of proportional fractional derivative and Hausdorff derivative of a function with respect to a fractal measure. Moreover, we establish the corresponding Stokes and Borel-Pompeiu formulas associated to this generalized fractional-fractal ψ-Fueter operator.

Keywords: quaternionic analysis; fractal-fractional derivatives; Borel-Pompeiu type formula; cauchy type formula

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0. Introduction

The fractal derivative or Hausdorff derivative, is a relatively new concept of differentiation that extends Leibniz's derivative for discontinuous fractal media. In the literature, there are various definitions of this new concept. For instance, in 2006 Chen introduced the concept of the Hausdorff derivative of a function with respect to a fractal measure t^η, where η is the order of the fractal derivative. A treatment of a more general case goes back to the work of Jeffery in 1958.

Fractal calculus is extremely effective in branches such as fluid mechanics where hierarchical or porous media, turbulence or aquifers present fractal properties, which do not necessarily follow a Euclidean geometry.

Fractional calculus deals with the generalization of the concepts of differentiation and integration of non-integer orders. This generalization is not merely a purely mathematical curiosity, but it has demonstrated its application in various disciplines such as physics, biology, engineering, and economics.

Unlike fractional calculus, fractal calculus maintains the chain rule in a very direct way, which relates the fractal derivative to the classical derivative.

The fractal-fractional derivative (a new class of fractional derivative, which has many applications in real world problems) is a mathematical concept that combines two different ideas: fractals and fractional derivatives. Fractals are complex geometric patterns that repeat at different scales, while fractional derivatives are a generalization of ordinary derivatives that allow for non-integer orders. The combination of fractal theory and fractional calculus gave rise to new concepts of differentiation and integration.

A considerable literature has grown up around new fractal, fractional and fractal-fractional derivatives. For references connected with the subject being considered in this work we refer the reader to [1–12].

Quaternionic analysis (the most natural and close generalization of complex analysis) concerns the connection between analysis (even topology / geometry) in $\mathbb{R}^4$ and the algebraic structure of quaternions $\mathbb{H}$. At the heart of this function theory lies the notion of $\psi-$hyperholomorphic functions

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defined on domains in $\mathbb{R}^4$ with values in $\mathbb{H}$, i.e., null solutions of the so-called $\psi-$Fueter operator (to be defined later) in which the standard basis of $\mathbb{R}^4$ is replaced by a structural set $\psi = \{1, \psi_1, \psi_2, \psi_3\} \in \mathbb{H}^4$.

In the last years, there is an increasing interest in finding a framework for a fractal or fractional counterpart of quaternionic analysis, see [13–19] and the references given there.

This paper introduce a fractional-fractal ψ-Fueter operator in the quaternionic context inspired in the concepts of proportional fractional derivative and Hausdorff derivative of a function with respect to a fractal measure. Moreover, we establish the corresponding Stokes and Borel-Pompeiu formulas associated to this generalized fractional-fractal ψ-Fueter operator.

The outline of this paper is summarized as follows. In Section 2 we give a brief exposition of the generalized fractal-fractional derivative considered. Section 3 presents some preliminaries on quaternionic analysis. In Section 4 we develop the rudiments of a function theory induced by a quaternionic β-proportional fractal Fueter operator and finally in Section 5 we will be concerned with a quaternionic β-proportional fractal Fueter operator with truncated exponential functions as fractals measure.

1. Generalized Fractal-Fractional Derivative

Definition 1. *The fractal derivative of a function f, defined on an interval I, with respect to a fractal measure $\nu(\eta, t)$ is given by*

$$\frac{d_\nu f(t)}{dt^\eta} := \lim_{\tau \to t} \frac{f(t) - f(\tau)}{\nu(\eta, t) - \nu(\eta, \tau)}, \quad \eta > 0.$$

If $\frac{d_\nu f(t)}{dt^\eta}$ exists for all $t \in I$ then f is real fractal differentiable on I with order η.

Some well-known cases. If $\nu(h, t) = t$ for all $t \in I$ then $\frac{d_\nu}{dt^\eta} = \frac{d}{dt}$ is the derivative operator. In addition, if $\nu(h, t) = h(t)$ for all $t \in I$ where $h'(t) > 0$ for all $t \in I$ then $\frac{d_\nu f(t)}{dt^\eta} = \frac{f'(t)}{h'(t)}$ for all $f \in C^1(I)$

On the other hand, if $\nu(\eta, t) = t^\eta$ for all $t \in I$ then $\frac{d_\nu f(t)}{dt^\eta}$ reduces to the Hausdorff derivative. Another useful fractal measure is $\nu(\eta, t) = e^{t^\alpha}$ for all $t \in I$ and $\alpha \in (0, 1]$.

We consider an well-known extension of the previous fractal derivative.

Definition 2. *Given $\beta \in [0, 1]$ we present the β-fractal derivative of a function f, defined on an interval I, with respect to a fractal measure $\nu(\eta, t)$:*

$$\frac{d^\beta_\nu f(t)}{dt^\eta} := \lim_{\tau \to t} \frac{(f(t))^\beta - (f(\tau))^\beta}{\nu(\eta, t) - \nu(\eta, \tau)}, \quad \eta > 0.$$

In order to make our description of the concept of fractal-fractional derivatives to be used precise, we introduce the notion of fractional proportional derivative, following [20].

Definition 3. *Let $\chi_0, \chi_1 : [0, 1] \times I$ be continuous functions such that*

$$\lim_{\sigma \to 0^+} \chi_1(\sigma, t) = 1, \lim_{\sigma \to 0^+} \chi_0(\sigma, t) = 0, \lim_{\sigma \to 1^-} \chi_1(\sigma, t) = 0, \lim_{\sigma \to 1^-} \chi_0(\sigma, t) = 1.$$

The proportional derivative of $f \in C^1(I)$ of order $\sigma \in [0, 1]$ is given by

$$D^\sigma f(t) = \chi_1(\sigma, t) f(t) + \chi_0(\sigma, t) f'(t), \quad \forall t \in I.$$

A combination of Definitions 2 and 3 yields.

 doi:10.20944/preprints202502.2042.v1

Definition 4. *Let $\beta \in [0,1]$, the proportional β-fractal derivative of $f : I \to \mathbb{R}$ with respect to $\nu(\eta, t)$ and σ is defined to be*

$$\frac{d_\nu^{\sigma,\beta} f(t)}{dt^\eta}(t) := \chi_1(\sigma,t) f(t) + \chi_0(\sigma,t) \frac{d_\nu^\beta f(t)}{dt^\eta},$$

if it exists for all $t \in I$.

Remark 1. *Given $\alpha \in (0,1]$ and $k \in \mathbb{N}$ we will consider the k-truncated exponential function defined as follows*

$$e(t^\alpha)_k := \sum_{i=0}^{k} \frac{(t^\alpha)^i}{i!}$$

for all $t \in \mathbb{R}$. For $k = 1$ we have $e(t^\alpha)_1 = 1 + t^\alpha$ and for $k = \infty$ we have $e(t^\alpha)_\infty = e^{t^\alpha}$.

Remark 2. *Important particular case, when the proportional and fractal measure in Definition 4 are given by*

$$\chi_1(\sigma,t) = 1 - \sigma, \quad \chi_0(\sigma,t) = \sigma, \quad \nu(k,t) = e(t^\alpha)_k$$

for all $\sigma \in [0,1]$ and $t \in I$ allows to introduce some cases of generalized fractal-fractional derivative to consider. For $f \in C^1(I)$ we have

$$\frac{d^{\sigma,\beta}}{dt_{\alpha,k}} f(t) := (1-\sigma) f(t) + \sigma \frac{(f^\beta)'(t)}{e(t^\alpha)_k'},$$

for all $t \in I$. The particular cases $k = 1, \infty$ reduces to

$$\frac{d^{\sigma,\beta}}{dt_{\alpha,1}} f(t) = (1-\sigma) f(t) + \sigma \frac{(f^\beta)'(t)}{\alpha t^{\alpha-1}},$$
$$\frac{d^{\sigma,\beta}}{dt_{\alpha,\infty}} f(t) = (1-\sigma) f(t) + \sigma \frac{(f^\beta)'(t)}{\alpha t^{\alpha-1} e^{t^\alpha}},$$

where clearly the conditions $\alpha \in (0,1]$ and $t > 0$ are necessary.

Addressing the issue $\sigma = \alpha$ requires that the case $\alpha = 0$ should be omitted.

$$\frac{d^{\alpha,\beta}}{dt_{\alpha,1}} f(t) = (1-\alpha) f(t) + \frac{(f^\beta)'(t)}{t^{\alpha-1}},$$
$$\frac{d^{\alpha,\beta}}{dt_{\alpha,\infty}} f(t) = (1-\alpha) f(t) + \frac{(f^\beta)'(t)}{t^{\alpha-1} e^{t^\alpha}}.$$

The k-truncated exponential function as fractal measure provides the generalized fractal-fractional drivative in much generality

$$\frac{d^{\alpha,\beta}}{dt_{\alpha,k}} f(t) := (1-\alpha) f(t) + \frac{(f^\beta)'(t)}{t^{\alpha-1} e(t^\alpha)_{k-1}}.$$

2. Preliminaries on Quaternionic Analysis

We begin by recalling some background and fixing notation that will be used throughout the entire document. For more details, we refer the interested reader to [21–23].

A real quaternion is an element of the form $x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the imaginary units e_1, e_2, e_3 satisfy:

$$e_1^2 = e_2^2 = e_3^2 = -1, e_1 e_2 = -e_2 e_1 = e_3, e_2 e_3 = -e_3 e_2 = e_1, e_3 e_1 = -e_1 e_3 = e_2.$$

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Quaternions form a skew-field denoted by $\mathbb{H}$. The set $\{1, e_1, e_2, e_3\}$ is the standard basis of $\mathbb{H}$.

The vector part of $x \in \mathbb{H}$ is by definition, $\mathbf{x} := x_1e_1 + x_2e_2 + x_3e_3$ while its real part is $x_0 := x_0$. The quaternionic conjugation of x, denoted by $\bar{x}$ is defined by $\bar{x} =: x_0 - \mathbf{x}$ and norm the $x \in \mathbb{H}$ is given by

$$\|x\| := \sqrt{x_0^2 + x_1^2 + x_2^3 + x_3^2} = \sqrt{x\bar{x}} = \sqrt{\bar{x}x}.$$

The quaternionic scalar product of $x, y \in \mathbb{H}$ is given by

$$\langle x, y\rangle := \frac{1}{2}(\bar{x}y + \bar{y}x) = \frac{1}{2}(x\bar{y} + y\bar{x}).$$

A set of quaternions $\psi = \{\psi_0, \psi_1, \psi_2, \psi_2\}$ is called structural set if $\langle \psi_k, \psi_s\rangle = \delta_{k,s}$, for $k, s = 0, 1, 2, 3$ and any quaternion x can be rewritten as $x_\psi := \sum_{k=0}^3 x_k\psi_k$, where $x_k \in \mathbb{R}$ for all k. Notion of structural sets is due to Nôno [24,25].

Given $q, x \in \mathbb{H}$ we follow the notation used in [21] to write

$$\langle q, x\rangle_\psi = \sum_{k=0}^{3} q_k x_k,$$

where $q_k, x_k \in \mathbb{R}$ for all k.

Let ψ an structural set. From now on, we will use the mapping

$$\sum_{k=0}^{3} x_k\psi_k \to (x_0, x_1, x_2, x_3). \tag{1}$$

in essential way.

Given a domain $\Omega \subset \mathbb{H} \cong \mathbb{R}^4$ and a function $f : \Omega \to \mathbb{H}$. Then f is written as: $f = \sum_{k=0}^3 f_k\psi_k$, where $f_k, k = 0, 1, 2, 3$, are $\mathbb{R}$-valued functions. Properties of f are due to properties of all components f_k such as continuity, differentiability, integrability and so on. For example, $C^1(\Omega, \mathbb{H})$ denotes the set of continuously differentiable $\mathbb{H}$-valued functions defined in Ω.

The left- and the right-ψ-Fueter operators are given by ${}^\psi\mathcal{D}[f] := \sum_{k=0}^3 \psi_k\partial_k f$ and ${}^\psi\mathcal{D}_r[f] := \sum_{k=0}^3 \partial_k f\psi_k$, for all $f \in C^1(\Omega, \mathbb{H})$, respectively, where $\partial_k f = \dfrac{\partial f}{\partial x_k}$ for all k.

Let $\partial\Omega$ be a 3−dimensional smooth surface. Then recall the Borel-Pompieu and differential and integral versions of Stokes' formulas

$$\begin{aligned}
&\int_{\partial\Omega} (K_\psi(\tau - x)\sigma_\tau^\psi f(\tau) + g(\tau)\sigma_\tau^\psi K_\psi(\tau - x)) \\
&- \int_\Omega (K_\psi(y - x)\,{}^\psi\mathcal{D}[f](y) + {}^\psi\mathcal{D}_r[g](y)K_\psi(y - x))dy \\
&= \begin{cases} f(x) + g(x), & x \in \Omega, \\ 0, & x \in \mathbb{H} \setminus \overline{\Omega}, \end{cases}
\end{aligned} \tag{2}$$

for all $f, g \in C^1(\Omega, \mathbb{H})$.

$$d(g\sigma_x^\psi f) = (g\,{}^\psi\mathcal{D}[f] + {}^\psi\mathcal{D}_r[g]f)dx, \tag{3}$$

$$\int_{\partial\Omega} g\sigma_x^\psi f = \int_\Omega (g\,{}^\psi\mathcal{D}[f] + {}^\psi\mathcal{D}_r[g]f)dx, \tag{4}$$

for all $f, g \in C^1(\overline{\Omega}, \mathbb{H})$. Here d represents the exterior differentiation operator, dx is the differential form of the 4-dimensional volume in $\mathbb{R}^4$ and

$$\sigma_x^\psi := -sgn\psi\left(\sum_{k=0}^{3}(-1)^k\psi_k d\hat{x}_k\right)$$

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is the quaternionic differential form of the 3-dimensional volume in $\mathbb{R}^4$ according to ψ, where $d\hat{x}_k = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ omitting factor dx_k. In addition, $sgn\psi$ is 1, or -1, if ψ and $\psi_{std} := \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ have the same orientation, or not, respectively. Note that, $|\sigma_x^\psi| = dS_3$ is the differential form of the 3-dimensional volume in $\mathbb{R}^4$ and write $\sigma_x = \sigma_x^{\psi_{std}}$. Let us recall that the ψ-Cauchy Kernel is given by

$$K_\psi(\tau - x) = \frac{1}{2\pi^2} \frac{\overline{\tau_\psi - x_\psi}}{|\tau_\psi - x_\psi|^4}.$$

3. A Function Theory Generated by a β-Proportional Fractal Fueter Operator

Let us extend Definition 4 to a quaternionic differential operator associate to an arbitrary structural set ψ.

Definition 5. *Let $\Omega \subset \mathbb{H}$ a domain. Fix $\beta = (\beta_0, \beta_1, \beta_2, \beta_3) \in [0,1]^4$ and $\nu = (\nu_0, \nu_1, \nu_2, \nu_3)$ where $\nu_k(\eta_k, x_k)$ is a fractal measure for $k = 0,1,2,3$ according to Definition 1. Denote $\chi_1 = (\chi_{0,1}, \chi_{1,1}, \chi_{2,1}, \chi_{3,1})$, $\chi_0 = (\chi_{0,0}, \chi_{1,0}, \chi_{2,0}, \chi_{3,0})$ and $\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \in [0,1]^4$ where $\chi_{k,1}(\sigma_k, x_k)$ and $\chi_{k,0}(\sigma_k, x_k)$ and are given by Definition 3 on coordinate x_k for $k = 0,1,2,3$.*

Let $f : \Omega \to \mathbb{H}$ such that $\dfrac{\partial_{\nu_n}^{\beta_n} f(x)}{\partial (x_n)^{\eta_n}}$ exists for all $x \in \Omega$ and all $n = 0,1,2,3$. Then, the quaternionic ψ-proportional β-fractal derivative of f with respect to ν and σ, is given by

$$\begin{aligned}
{}^\psi\mathcal{D}_\nu^{\sigma,\beta}[f](x) &:= \sum_{n=0}^{3} \psi_n \frac{\partial_{\nu_n}^{\sigma_n,\beta_n} f(x)}{\partial (x_n)^{\eta_n}} \\
&= \sum_{n=0}^{3} \psi_n \left(\chi_{n,1}(\sigma_n, x_n) f(x) + \chi_{n,0}(\sigma_n, x_n) \frac{\partial_{\nu_n}^{\beta_n} f(x)}{\partial (x_n)^{\eta_n}} \right) \\
&= \sum_{n=0=m}^{3} \psi_n \psi_m \left(\chi_{n,1}(\sigma_n, x_n) f_m(x) + \chi_{n,0}(\sigma_n, x_n) \frac{\partial_{\nu_n}^{\beta_n} f_m(x)}{\partial (x_n)^{\eta_n}} \right).
\end{aligned}$$

Proposition 1. *Given $f \in C^1(\Omega, \mathbb{H})$ as above let us assume that*

$$\lambda_{\nu_n}^{\beta_n}(f_m)(x) := \int_0^{x_n} \frac{\partial_{\nu_n}^{\beta_n} f_m(x)}{\partial (t_n)^{\eta_n}} dt, \quad \lambda_{\nu_n}^{\beta_n}(f)(x) := \sum_{m=0}^{3} \psi_m \int_0^{x_n} \frac{\partial_{\nu_n}^{\beta_n} f_m(x)}{\partial (t_n)^{\eta_n}} dt$$

exist for $n, m = 0,1,2,3$. Under conditions $\chi_{n,0}(\sigma_n, x_n) \neq 0$ and $\lambda_{\nu_n}^{\beta_n}(f_m)(x) \neq 0$ for all $x = (x_0, x_1, x_2, x_3) \in \Omega$ and all $m, n = 0,1,2,3$ we have

$${}^\psi\mathcal{D} \circ \mathfrak{L}_\nu^{\sigma,\beta}[f](x) = {}^\psi\mathcal{D}_\nu^{\sigma,\beta}[f](x) + \mathcal{E}_\nu^{\sigma,\beta}[f](x) + \sum_{n=0=m}^{3} \psi_n \psi_m L_{n,m}[f](x) \lambda_{\nu_n}^{\beta_n}(f_m)(x),$$

for all $x \in \Omega$, where

$$\begin{aligned}
\mathcal{E}_\nu^{\sigma,\beta}[f](x) &:= \sum_{\substack{n=0=k \\ n \neq k}}^{3} \psi_n \frac{\partial}{\partial x_n} \left[(\chi_{k,0}(\sigma_k, x_k)) \lambda_{\nu_k}^{\beta_k}(f)(x) \right], \\
\mathfrak{L}_\nu^{\sigma,\beta}(f)(x) &= \sum_{k=0}^{3} (\chi_{k,0}(\sigma_k, x_k)) \lambda_{\nu_k}^{\beta_k}(f)(x), \\
L_{n,m}[f](x) &:= \frac{\partial}{\partial x_n} \left(\frac{\chi_{n,0}(\sigma_n, x_n)}{e^{h_{n,m}(x)}} \right) e^{h_{n,m}(x)}, \\
h_{n,m}(x) &= \int_0^{x_n} \frac{\chi_{n,1}(\sigma_n, t_n)}{\chi_{n,0}(\sigma_n, t_n)} \frac{f_m}{\lambda_{\nu_n}^{\beta_n}(f_m)} dt,
\end{aligned}$$

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for $n, m \in \{0, 1, 2, 3\}$.

Proof. To simplify notation consider $\lambda_n = \lambda_{\nu_n}^{\beta_n}$ for all $n = 0, 1, 2, 3$. From direct computations we have that

$$\begin{aligned}
\frac{\partial}{\partial x_n}(e^{h_{n,m}(x)}\lambda_n(f_m)(x)) =& e^{h_{n,m}(x)}\left[\frac{\chi_{n,1}(\sigma_n, x_n)}{\chi_{n,0}(\sigma_n, x_n)}\frac{f_m}{\lambda_n(f_m)}\lambda_n(f_m)(x) + \frac{\partial_{\nu_n}^{\beta_n} f_m(x)}{\partial(x_n)^{\eta_n}}\right] \\
=& \frac{e^{h_{n,m}(x)}}{\chi_{n,0}(\sigma_n, x_n)}\left[\chi_{n,1}(\sigma_n, t_n) f_m + \chi_{n,0}(\sigma_n, x_n)\frac{\partial_{\nu_n}^{\beta_n} f_m(x)}{\partial(x_n)^{\eta_n}}\right] \\
=& \frac{e^{h_{n,m}(x)}}{\chi_{n,0}(\sigma_n, x_n)}\frac{\partial_{\nu_n}^{\sigma_n,\beta_n} f_m(x)}{\partial(x_n)^{\eta_n}}, \\
\frac{\partial_{\nu_n}^{\sigma_n,\beta_n} f_m(x)}{\partial(x_n)^{\eta_n}} =& \frac{\chi_{n,0}(\sigma_n, x_n)}{e^{h_{n,m}(x)}}\frac{\partial}{\partial x_n}\left(e^{h_{n,m}(x)}\lambda_n(f_m)(x)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
{}^{\psi}\mathcal{D}_{\nu}^{\sigma,\beta}[f](x) =& \sum_{n=0=m}^{3}\psi_n\psi_m\frac{\chi_{n,0}(\sigma_n, x_n)}{e^{h_{n,m}(x)}}\frac{\partial}{\partial x_n}\left(e^{h_{n,m}(x)}\lambda_n(f_m)(x)\right) \\
=& \sum_{n=0=m}^{3}\psi_n\psi_m\frac{\partial}{\partial x_n}(\chi_{n,0}(\sigma_n, x_n)\lambda_n(f_m)(x))) - \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](x)\lambda_n(f_m)(x) \\
=& \sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}(\chi_{n,0}(\sigma_n, x_n)\sum_{m=0}^{3}\psi_m\lambda_n(f_m)(x))) - \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](x)\lambda_n(f_m)(x) \\
=& \sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}[\chi_{n,0}(\sigma_n, x_n)\lambda_n(f)(x)] - \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](x)\lambda_n(f_m)(x) \\
=& \sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}[\varsigma(x)\lambda_n(f)(x)] - \sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}[\kappa_n\lambda_n(f)(x)] - \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](x)\lambda_n(f_m)(x),
\end{aligned}$$

where $\varsigma(x) = \sum_{\ell=0}^{3}\chi_{\ell,0}(\sigma_\ell, x_\ell)$ and $\kappa_n = \sum_{\substack{\ell=0 \\ \ell\neq n}}^{3}\chi_{\ell,0}(\sigma_\ell, x_\ell)$.

Then

$$\begin{aligned}
{}^{\psi}\mathcal{D}_{\nu}^{\sigma,\beta}[f](x) :=& \sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}\left[\varsigma(x)\left(\sum_{k=0}^{3}\lambda_k(f)(x)\right)\right] - \sum_{\substack{n=0=k \\ n\neq k}}^{3}\psi_n\frac{\partial}{\partial x_n}[\varsigma(x)\lambda_k(f)(x)] \\
&- \sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}\left[\sum_{k=0}^{3}\kappa_k\lambda_k(f)(x)\right] + \sum_{\substack{n=0=k \\ n\neq k}}^{3}\psi_n\frac{\partial}{\partial x_n}[\kappa_k\lambda_k(f)(x)] \\
&- \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](x)\lambda_n(f_m)(x).
\end{aligned}$$

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As a consequence we have that

$$
{}^{\psi}\mathcal{D}_{\nu}^{\sigma,\beta}[f](x) := \sum_{n=0}^{3} \psi_n \frac{\partial}{\partial x_n}\left[\varsigma(x)\left(\sum_{k=0}^{3}\lambda_k(f)(x)\right) - \sum_{k=0}^{3}\kappa_k\lambda_k(f)(x)\right]
$$

$$
+ \sum_{\substack{n=0=k \\ n\neq k}}^{3} \psi_n \frac{\partial}{\partial x_n}[\kappa_k\lambda_k(f)(x) - \varsigma(x)\lambda_k(f)(x)] - \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](x)\lambda_n(f_m)(x),
$$

i.e.,

$$
\begin{aligned}
{}^{\psi}\mathcal{D}_{\nu}^{\sigma,\beta}[f](x) := &\sum_{n=0}^{3} \psi_n \frac{\partial}{\partial x_n}\left[\sum_{k=0}^{3}(\chi_{k,0}(\sigma_k,x_k))\lambda_k(f)(x)\right] \\
&- \sum_{\substack{n=0=k \\ n\neq k}}^{3} \psi_n \frac{\partial}{\partial x_n}[(\chi_{k,0}(\sigma_k,x_k))\lambda_k(f)(x)] \\
&- \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](x)\lambda_n(f_m)(x)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
{}^{\psi}\mathcal{D}_{\nu}^{\sigma,\beta}[f](x) :={}^{\psi}\mathcal{D}\circ\mathfrak{L}_{\nu}^{\sigma,\beta}[f](x) &- \sum_{\substack{n=0=k \\ n\neq k}}^{3} \psi_n \frac{\partial}{\partial x_n}[(\chi_{k,0}(\sigma_k,x_k))\lambda_k(f)(x)] \\
&- \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](x)\lambda_n(f_m)(x).
\end{aligned}
$$

□

Notation $L_{n,m}$ and h_n for all $n,m = 0,1,2,3$, can be improved but we have decided to keep it at this level to make easier to write and read the following computations.

Definition 6. *For $\delta = (\delta_0,\delta_1,\delta_2,\delta_3) \in [0,1]^4$, and $\mu = (\mu_0,\mu_1,\mu_2,\mu_3)$ where $\mu_k(\zeta_k,x_k)$ is a fractal measure for $k = 0,1,2,3$ according to Definition 1. Denote $\varkappa_1 = (\varkappa_{0,1},\varkappa_{1,1},\varkappa_{2,1},\varkappa_{3,1})$, $\varkappa_0 = (\varkappa_{0,0},\varkappa_{1,0},\varkappa_{2,0},\varkappa_{3,0})$ and $\rho = (\rho_0,\rho_1,\rho_2,\rho_3) \in [0,1]^4$ where $\varkappa_{k,1}(\rho_k,x_k)$ and $\varkappa_{k,0}(\rho_k,x_k)$ and are given by Definition 3 for $k = 0,1,2,3$.*

Given $g:\Omega\to\mathbb{H}$ such that $\dfrac{\partial_{\mu_n}^{\delta_n}g(x)}{\partial(x_n)^{\zeta_n}}$ there exists for all $x\in\Omega$ and all $n = 0,1,2,3$. The quaternionic right ψ-proportional δ-fractal derivative of g with respect to μ and ρ, is given by

$$
\begin{aligned}
{}^{\psi}\mathcal{D}_{r,\mu}^{\rho,\delta}[g](x) := &\sum_{n=0}^{3}\frac{\partial_{\mu_n}^{\rho_n,\delta_n}g(x)}{\partial(x_n)^{\zeta_n}}\psi_n \\
= &\sum_{n=0=m}^{3}\psi_m\psi_n\left(\varkappa_{n,1}(\rho_n,x_n)g_m(x) + \varkappa_{n,0}(\rho_n,x_n)\frac{\partial_{\mu_n}^{\delta_n}g_m(x)}{\partial(x_n)^{\zeta_n}}\right).
\end{aligned}
$$

Remark 3. *Consider $g:\Omega\to\mathbb{H}$ such that*

$$
\lambda_{\mu_n}^{\delta_n}(g_m)(x) = \int_0^{x_n}\frac{\partial_{\mu_n}^{\delta_n}g_m(x)}{\partial(t_n)^{\zeta_n}}dt, \quad \lambda_{\mu_n}^{\delta_n}(g)(x) = \sum_{m=0}^{3}\psi_m\int_0^{x_n}\frac{\partial_{\mu_n}^{\delta_n}g_m(x)}{\partial(t_n)^{\zeta_n}}dt
$$

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there exist for $n, m = 0, 1, 2, 3$. If $\varkappa_{n,0}(\rho_n, x_n) \neq 0$ and $\lambda^{\delta_n}_{\mu_n}(g_m)(x) \neq 0$ for all $x = (x_0, x_1, x_2, x_3) \in \Omega$ and all $n = 0, 1, 2, 3$, then repeating several computations of the previous proof we can see that

$$ {}^{\psi}\mathcal{D}_r \circ \mathfrak{L}^{\rho,\delta}_{\mu}[g](x) := {}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\mu}[g](x) + \mathcal{E}^{\rho,\delta}_{r,\mu}[g](x) + \sum_{n=0=m}^{3} \psi_m \psi_n T_{n,m}[g](x) \lambda^{\delta_n}_{\mu_n}(g_m)(x), \tag{5} $$

for all $x \in \Omega$, where

$$ \begin{aligned} \mathcal{E}^{\rho,\delta}_{r,\mu}[g](x) &:= \sum_{\substack{n=0=k \\ n\neq k}}^{3} \frac{\partial}{\partial x_n}\Big[(\varkappa_{k,0}(\rho_k, x_k))\lambda^{\delta_k}_{\mu_k}(g)(x)\Big]\psi_n \\ \mathfrak{L}^{\rho,\delta}_{\mu}(g)(x) &= \sum_{k=0}^{3}(\varkappa_{k,0}(\rho_k, x_k))\lambda^{\delta_k}_{\nu_k}(g)(x), \\ T_{n,m}[g](x) &:= \frac{\partial}{\partial x_n}\Big(\frac{\varkappa_{n,0}(\rho_n, x_n)}{e^{l_{n,m}(x)}}\Big)e^{l_{n,m}(x)}, \\ l_{n,m}(x) &= \int_0^{x_n} \frac{\varkappa_{n,1}(\rho_n, t_n)}{\varkappa_{n,0}(\rho_n, t_n)} \frac{g_m}{\lambda^{\delta_n}_{\mu_n}(g_m)} dt, \end{aligned} $$

for $n, m \in \{0, 1, 2, 3\}$.

Assuming hypothesis and notations of Proposition 1 and Remark 3 let us present some consequences of quatertionic Borel-Pompeiu and Stokes formulas.

Proposition 2. *Let $\Omega \subset \mathbb{H}$ be a domain such that $\partial\Omega$ is a 3-dimensional smooth surface. If $\mathfrak{L}^{\sigma,\beta}_{\nu}[f], \mathfrak{L}^{\rho,\delta}_{\mu}[g] \in C^1(\Omega, \mathbb{H})$ then*

$$ \begin{aligned} &\int_{\partial\Omega}(K_{\psi}(\tau - x)\sigma^{\psi}_{\tau}\mathfrak{L}^{\sigma,\beta}_{\nu}[f](\tau) + \mathfrak{L}^{\rho,\delta}_{\mu}[g](\tau)\sigma^{\psi}_{\tau}K_{\psi}(\tau - x)) \\ &- \int_{\Omega}\Big(K_{\psi}(y - x)\,{}^{\psi}\mathcal{D}^{\sigma,\beta}_{\nu}[f](y) - {}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\mu}[g](y)K_{\psi}(y - x)\Big)dy \\ &- \int_{\Omega}K_{\psi}(y - x)\Bigg(\mathcal{E}^{\sigma,\beta}_{\nu}[f](y) + \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](y)\lambda^{\beta_n}_{\nu_n}(f_m)(y)\Bigg)dy \\ &- \int_{\Omega}\Bigg(\mathcal{E}^{\rho,\delta}_{r,\mu}[g](y) + \sum_{n=0=m}^{3}\psi_m\psi_n T_{n,m}[g](y)\lambda^{\delta_n}_{\mu_n}(g_m)(y)\Bigg)K_{\psi}(y - x)dy \\ =&\begin{cases} \mathfrak{L}^{\sigma,\beta}_{\nu}[f](x) + \mathfrak{L}^{\rho,\delta}_{\mu}[g](x), & x \in \Omega, \\ 0, & x \in \mathbb{H}\setminus\overline{\Omega}, \end{cases} \end{aligned} \tag{6} $$

and

$$ \begin{aligned} &\int_{\partial\Omega}\mathfrak{L}^{\rho,\delta}_{\mu}[g]\sigma^{\psi}_{x}\mathfrak{L}^{\sigma,\beta}_{\nu}[f] = \int_{\Omega}\Big(\mathfrak{L}^{\rho,\delta}_{\mu}[g]\,{}^{\psi}\mathcal{D}^{\sigma,\beta}_{\nu}[f] + {}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\mu}[g]\mathfrak{L}^{\sigma,\beta}_{\nu}[f]\Big)dx + \\ &+ \int_{\Omega}\mathfrak{L}^{\rho,\delta}_{\mu}[g]\Bigg(\mathcal{E}^{\sigma,\beta}_{\nu}[f] + \sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f]\lambda^{\beta_n}_{\nu_n}(f_m)\Bigg)dx \\ &+ \int_{\Omega}\Bigg(\mathcal{E}^{\rho,\delta}_{r,\mu}[g] + \sum_{n=0=m}^{3}\psi_m\psi_n T_{n,m}[g]\lambda^{\delta_n}_{\mu_n}(g_m)\Bigg)\mathfrak{L}^{\sigma,\beta}_{\nu}[f]dx. \end{aligned} \tag{7} $$

Proof. The formulas follow by application of quaternionic Borel-Pompieu and Stokes formula, functions $\mathfrak{L}^{\sigma,\beta}_{\nu}[f]$, $\mathfrak{L}^{\rho,\delta}_{\mu}[g]$ and the usage of identities given in Proposition 1 and Remark 3. □

Remark 4. *In case in which $\mathfrak{L}^{\sigma,\beta}_{\nu}$ and $\mathfrak{L}^{\rho,\delta}_{\mu}$ are invertible operators we can improve formula (6) to obtain the quaternionc values of f and g. In addition, if $f \in Ker({}^{\psi}\mathcal{D}^{\sigma,\beta}_{\nu})$ and $g \in Ker({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\mu})$ then*

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$$\begin{aligned}&\int_{\partial\Omega}(K_\psi(\tau-x)\sigma_\tau^\psi\mathfrak{L}_\nu^{\sigma,\beta}[f](\tau)+\mathfrak{L}_\mu^{\rho,\delta}[g](\tau)\sigma_\tau^\psi K_\psi(\tau-x))\\&-\int_\Omega K_\psi(y-x)\left(\mathcal{E}_\nu^{\sigma,\beta}[f](y)+\sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f](y)\lambda_{\nu_n}^{\beta_n}(f_m)(y)\right)dy\\&-\int_\Omega\left(\mathcal{E}_{r,\mu}^{\rho,\delta}[g](y)+\sum_{n=0=m}^{3}\psi_m\psi_n T_{n,m}[g](y)\lambda_{\mu_n}^{\delta_n}(g_m)(y)\right)K_\psi(y-x)dy\\=&\begin{cases}\mathfrak{L}_\nu^{\sigma,\beta}[f](x)+\mathfrak{L}_\mu^{\rho,\delta}[g](x), & x\in\Omega,\\0, & x\in\mathbb{H}\setminus\overline{\Omega},\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\int_{\partial\Omega}\mathfrak{L}_\mu^{\rho,\delta}[g]\sigma_x^\psi\mathfrak{L}_\nu^{\sigma,\beta}[f]=\int_\Omega\mathfrak{L}_\mu^{\rho,\delta}[g]\left(\mathcal{E}_\nu^{\sigma,\beta}[f]+\sum_{n=0=m}^{3}\psi_n\psi_m L_{n,m}[f]\lambda_{\nu_n}^{\beta_n}(f_m)\right)dx\\&+\int_\Omega\left(\mathcal{E}_{r,\mu}^{\rho,\delta}[g]+\sum_{n=0=m}^{3}\psi_m\psi_n T_{n,m}[g]\lambda_{\mu_n}^{\delta_n}(g_m)\right)\mathfrak{L}_\nu^{\sigma,\beta}[f]dx.\end{aligned}$$

4. Quaternionic β-Proportional Fractal Fueter Operator with Truncated Exponential Fractal Measure

From now on, partial differential operators given by Remarks 1 and 2 are considered, and let $k:=(k_0,k_1,k_2,k_3)\in\mathbb{N}^4$, $\sigma=(\sigma_0,\sigma_1,\sigma_2,\sigma_3)$, $\beta=(\beta_0,\beta_1,\beta_2,\beta_3)\in[0,1]^4$, $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in(0,1]$ and for $n=0,1,2,3$.

Let $\Omega\subset\mathbb{H}$ be a domain and $f\in C^1(\Omega,\mathbb{R})$. We will use the proportional β_n-fractal partial derivatives

$$\frac{\partial^{\sigma_n,\beta_n}f}{\partial x_{\alpha_n,k_n}}(x):=(1-\sigma_n)f(x)+\sigma_n\frac{\frac{\partial f^{\beta_n}}{\partial x_n}(x)}{\frac{\partial e(x_n^{\alpha_n})_{k_n}}{\partial x_n}},$$

for all $x=\sum_{n=0}^3\psi x_n\in\Omega$.

Definition 7. *Let $\Omega\subset\mathbb{H}$ be a domain. Given $f=\sum_{\ell=0}^3\psi_\ell f_\ell\in C^1(\Omega,\mathbb{H})$, where f_0,f_1,f_2,f_3 are real valued functions. Define*

$$({}^\psi\mathcal{D}_{\alpha,k}^{\sigma,\beta}f)(x):=\sum_{n=0=\ell}^{3}\psi_n\psi_\ell\frac{\partial^{\sigma_n,\beta_n}f_\ell}{\partial x_{\alpha_n,k_n}}(x) \tag{8}$$

$$=\sum_{n=0=\ell}^{3}\psi_n\psi_\ell\left((1-\sigma_n)f_\ell(x)+\sigma_n\frac{\frac{\partial f_\ell^{\beta_n}}{\partial x_n}(x)}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\right), \tag{9}$$

$$H_{\alpha_n,k_n}^{\sigma_n,\beta_n}[f_\ell](x):=\int_0^{x_n}\frac{\sigma_n-1}{\sigma_n}\left(\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}\right)f_\ell(x)^{1-\beta_n}dx_n \tag{10}$$

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and to simplify the notation in the proof of the next statement use $h_{n,\ell}(x) = H^{\sigma_n,\beta_n}_{\alpha_n,k_n}(f_\ell)(x)$ *for all* $n,\ell = 0,1,2,3$. *In addition,*

$$T^{\sigma_n,\beta_n}_{\alpha_n,k_n}[f_\ell](x) := \frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{e^{h_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\right)e^{h_{n,\ell}(x)}f_\ell(x)^{\beta_n},$$

$$^{\psi}W^{\sigma,\beta}_{\alpha,k}[f](x) = \sum_{\substack{\ell=0=n \\ \ell\neq n}}^{3} \frac{\sigma_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\alpha_\ell})_{k_\ell}}\psi_n\frac{\partial}{\partial x_n}I^{\beta_\ell}[f](x),$$

where $I^{\beta_n}[f](x) = \sum_{\ell=0}^{3}\psi_\ell f_\ell(x)^{\beta_n}$ *for all* $x\in\Omega$ *and* $n,\ell = 0,1,2,3.$

Proposition 3. *Given* $f = \sum_{\ell=0}^{3}\psi_\ell f_\ell \in C^1(\Omega,\mathbb{H})$. *Then*

$$\begin{aligned}&{}^{\psi}\mathcal{D}\left[\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\alpha_\ell})_{k_\ell}}I^{\beta_\ell}[f]\right](x)\\ =&({}^{\psi}\mathcal{D}^{\rho,\beta}_{\alpha,k}f)(x) + \sum_{n=0=\ell}^{3}\psi_n\psi_\ell T^{\sigma_n,\beta_n}_{\alpha_n,k_n}[f_\ell](x) + {}^{\psi}W^{\sigma,\beta}_{\alpha,k}[f](x),\end{aligned} \tag{11}$$

for all $x\in\Omega$.

Proof.

$$\begin{aligned}\frac{\partial}{\partial x_n}\left(e^{h_{n,\ell}(x_n)}f_\ell(x)^{\beta_n}\right) &= \left[\frac{1-\sigma_n}{\sigma_n}\left(\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}\right)(f_\ell(x))^{1-\beta_n}f_\ell^{\beta_n}(x) + \frac{\partial f_\ell^{\beta_n}}{\partial x_n}(x)\right]e^{h_{n,\ell}(x)}\\ &= \left[(1-\sigma_n)f_\ell(x) + \sigma_n\frac{\frac{\partial f_\ell^{\beta_n}}{\partial x_n}(x)}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\right]\frac{1}{\sigma_n}e^{h_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n},\\ \frac{\partial^{\sigma_n,\beta_n}}{\partial x_{\alpha_n,k_n}}f_\ell(x) &= \frac{\sigma_n}{e^{h_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\frac{\partial}{\partial x_n}\left(e^{h_{n,\ell}(x)}f_\ell(x)^{\beta_n}\right)\end{aligned}$$

and

$$\begin{aligned}({}^{\psi}\mathcal{D}^{\sigma,\beta}_{\alpha,k}f)(x) &= \sum_{n=0}^{3}\psi_n\frac{\partial^{\sigma_n,\beta_n}}{\partial x_{\alpha_n,k_n}}f(x)\\ &= \sum_{n=0=\ell}^{3}\psi_n\psi_\ell\frac{\sigma_n}{e^{h_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\frac{\partial}{\partial x_n}\left(e^{h_{n,\ell}(x)}f_\ell(x)^{\beta_n}\right).\end{aligned}$$

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The identities

$$\frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}f_\ell(x)^{\beta_n}\right)=\frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{e^{h_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}e^{h_{n,\ell}(x)}f_\ell(x)^{\beta_n}\right)$$
$$=\frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{e^{h_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\right)e^{h_{n,\ell}(x)}f_\ell(x)^{\beta_n}+\frac{\sigma_n}{e^{h_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\frac{\partial}{\partial x_n}\left(e^{h_{n,\ell}(x)}f_\ell(x)^{\beta_n}\right)$$

and

$$\frac{\sigma_n}{e^{h_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\frac{\partial}{\partial x_n}\left(e^{h_{n,\ell}(x)}f_\ell(x)^{\beta_n}\right)$$
$$=\frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}f_\ell(x)^{\beta_n}\right)-\frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{e^{h_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\right)e^{h_{n,\ell}(x)}f_\ell(x)^{\beta_n}$$
$$=\frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}f_\ell(x)^{\beta_n}\right)-T^{\sigma_n,\beta_n}_{\alpha_n,k_n}[f_\ell](x)(x)$$

imply that

$$({}^{\psi}\mathcal{D}^{\sigma,\beta}_{\alpha,k}f)(x)=\sum_{n=0=\ell}^{3}\psi_n\psi_\ell\left[\frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}f_\ell(x)^{\beta_n}\right)-T^{\sigma_n,\beta_n}_{\alpha_n,k_n}[f_\ell](x)\right]$$
$$=\sum_{n=0=\ell}^{3}\psi_n\psi_\ell\frac{\partial}{\partial x_n}\left[\frac{\sigma_n}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}f_\ell(x)^{\beta_n}\right]-\sum_{n=0=\ell}^{3}\psi_n\psi_\ell T^{\sigma_n,\beta_n}_{\alpha_n,k_n}[f_\ell](x)$$
$$=\sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}\left[\frac{\sigma_n}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}\sum_{\ell=0}^{3}\psi_\ell f_\ell(x)^{\beta_n}\right]-\sum_{n=0=\ell}^{3}\psi_n\psi_\ell T^{\sigma_n,\beta_n}_{\alpha_n,k_n}[f_\ell](x)$$
$$=\sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}\left[\frac{\sigma_n}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}I^{\beta_n}[f](x)\right]-\sum_{n=0=\ell}^{3}\psi_n\psi_\ell T^{\sigma_n,\beta_n}_{\alpha_n,k_n}[f_\ell](x).$$

For each $n=0,1,2,3$ we see that

$$\frac{\sigma_n}{\frac{d}{dx_n}e(x_n^{\alpha_n})_{k_n}}I^{\beta_n}[f](x)=\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\alpha_\ell})_{k_\ell}}I^{\beta_\ell}[f](x)-\sum_{\substack{\ell=0\\ \ell\neq n}}^{3}\frac{\sigma_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\alpha_\ell})_{k_\ell}}I^{\beta_\ell}[f](x).$$

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Therefore,

$$
\begin{aligned}
&{}^{\psi}\mathcal{D}\left[\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\alpha_\ell})_{k_\ell}}I^{\beta_\ell}[f]\right](x)\\
=&({}^{\psi}\mathcal{D}^{\rho,\beta}_{\alpha,k}f)(x)+\sum_{n=0=\ell}^{3}\psi_n\psi_\ell T^{\sigma_n,\beta_n}_{\alpha_n,k_n}[f_\ell](x)+\sum_{\substack{\ell=0=n\\ \ell\neq n}}^{3}\frac{\sigma_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\alpha_\ell})_{k_\ell}}\psi_n\frac{\partial}{\partial x_n}I^{\beta_\ell}[f](x).
\end{aligned}
$$

□

Remark 5. *Denote* $v=(v_0,v_1,v_2,v_3)\in\mathbb{N}^4$, $\rho=(\rho_0,\rho_1,\rho_2,\rho_3)$, $\delta=(\delta_0,\delta_1,\delta_2,\delta_3)\in(0,1]^4$, $\gamma=(\gamma_0,\gamma_1,\gamma_2,\gamma_3)\in(0,1]^4$ *and for* $n=0,1,2,3$. *We will use the the proportional* δ_n*-fractal partial derivative* $\frac{\partial^{\rho_n,\delta_n}}{\partial x_{\gamma_n,k_n}}$. *Recall that if* $g\in C^1(\Omega,\mathbb{R})$ *then*

$$
\frac{\partial^{\rho_n,\delta_n}g}{\partial x_{\gamma_n,k_n}}(x):=(1-\rho_n)g(x)+\rho_n\frac{\frac{\partial g^{\delta_n}}{\partial x_n}(x)}{\frac{\partial e(x_n^{\gamma_n})_{k_n}}{\partial x_n}},
$$

for all $x=\sum_{n=0}^{3}\psi x_n\in\Omega$.

If $g=\sum_{\ell=0}^{3}\psi_\ell g_\ell\in C^1(\Omega,\mathbb{H})$, *where* g_0,g_1,g_2,g_3 *are real valued functions. Define the right version of the operator given by* (8) *as follows:*

$$
\begin{aligned}
({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x)&:=\sum_{n=0=\ell}^{3}\psi_\ell\frac{\partial^{\rho_n,\delta_n}g_\ell}{\partial x_{\gamma_n,m_n}}(x)\psi_n,\\
H^{\rho_n,\delta_n}_{\gamma_n,k_n}[g_\ell](x)&:=\int_0^{x_n}\frac{\rho_n-1}{\rho_n}\left(\frac{d}{dx_n}e(x_n^{\gamma_n})_{m_n}\right)g_\ell(x)^{1-\delta_n}dx_n.
\end{aligned}
$$

and use $j_{n,\ell}(x)=H^{\sigma_n,\beta_n}_{\alpha_n,k_n}(g_\ell)(x)$ *for all* $n,\ell=0,1,2,3$. *Denote*

$$
\begin{aligned}
S^{\rho_n,\delta_n}_{\gamma_n,m_n}[g_\ell](x)&:=\frac{\partial}{\partial x_n}\left(\frac{\rho_n}{e^{j_{n,\ell}(x)}\frac{d}{dx_n}e(x_n^{\gamma_n})_{m_n}}\right)e^{j_{n,\ell}(x)}g_\ell(x)^{\delta_n},\\
{}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x)&:=\sum_{\substack{\ell=0=n\\ \ell\neq n}}^{3}\frac{\rho_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\gamma_\ell})_{m_\ell}}\frac{\partial}{\partial x_n}I^{\delta_\ell}[g](x)\psi_n.
\end{aligned}
$$

From similar computations to presented in the previous proof we can obtain the right version of (11)*:*

$$
\begin{aligned}
&{}^{\psi}\mathcal{D}_r\left[\sum_{\ell=0}^{3}\frac{\rho_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\gamma_\ell})_{m_\ell}}I^{\delta_\ell}[g]\right](x)\\
=&({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x)+\sum_{n=0=\ell}^{3}\psi_\ell S^{\rho_n,\delta_n}_{\gamma_n,m_n}[g_\ell](x)\psi_n+{}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x),
\end{aligned}
\tag{12}
$$

for all $x\in\Omega$

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Corollary 1. *Let $\Omega \subset \mathbb{H}$ be a domain such that $\partial\Omega$ is a 3-dimensional smooth surface. In agreement with notation in Definition 7 and Remark 5 we have:*

1. *If $\beta = (1,1,1,1)$ then operators given in Definition 7 are represented as follows:*

$$
\begin{aligned}
({}^{\psi}\mathcal{D}^{\sigma,\beta}_{\alpha,k} f)(x) &= \sum_{n=0=\ell}^{3} \psi_n \psi_\ell \left((1-\sigma_n) f_\ell(x) + \sigma_n \frac{\frac{\partial f_\ell}{\partial x_n}(x)}{\frac{d}{dx_n} e(x_n^{\alpha_n})_{k_n}} \right),\\
h_n(x) = H^{\sigma_n,1}_{\alpha_n,k_n}[f_\ell](x) &= \frac{\sigma_n - 1}{\sigma_n} [e(x_n^{\alpha_n})_{k_n} - 1],\\
T^{\sigma_n,1}_{\alpha_n,k_n}[f_\ell](x) &= \frac{\partial}{\partial x_n} \left(\frac{\sigma_n}{e^{h_n(x)} \frac{d}{dx_n} e(x_n^{\alpha_n})_{k_n}} \right) e^{h_n(x)} f_\ell(x),\\
I^1[f] &= f\\
{}^{\psi}W^{\sigma,\beta}_{\alpha,k}[f](x) &= \sum_{\substack{\ell=0=n\\ \ell\neq n}}^{3} \frac{\sigma_\ell}{\frac{d}{dx_\ell} e(x_\ell^{\alpha_\ell})_{k_\ell}} \psi_n \frac{\partial}{\partial x_n} f(x),
\end{aligned}
$$

for all $x \in \Omega$ and (11) becomes at

$$
{}^{\psi}\mathcal{D}\left[\sum_{\ell=0}^{3} \frac{\sigma_\ell}{\frac{d}{dx_\ell} e(x_\ell^{\alpha_\ell})_{k_\ell}} f\right](x) = ({}^{\psi}\mathcal{D}^{\rho,\beta}_{\alpha,k} f)(x) + A(x) f(x) + {}^{\psi}W^{\sigma,\beta}_{\alpha,k}[f](x),
$$

where

$$
A(x) := \sum_{n=0}^{3} \psi_n \frac{\partial}{\partial x_n} \left(\frac{\sigma_n}{e^{h_n(x)} \frac{d}{dx_n} e(x_n^{\alpha_n})_{k_n}} \right) e^{h_n(x)},
$$

for all $x \in \Omega$.
Another important cases are the following:

(a) *If $\beta = (1,1,1,1)$ and $k = (1,1,1,1)$ then*

$$
\begin{aligned}
({}^{\psi}\mathcal{D}^{\sigma,\beta}_{\alpha,k} f)(x) &= \sum_{n=0=\ell}^{3} \psi_n \psi_\ell \left((1-\sigma_n) f_\ell(x) + \sigma_n \frac{\frac{\partial f_\ell}{\partial x_n}(x)}{\alpha_n x_n^{\alpha_n - 1}} \right),\\
h_n(x) = H^{\sigma_n,1}_{\alpha_n,1}[f_\ell](x) &= \frac{\sigma_n - 1}{\sigma_n} x_n^{\alpha_n},\\
T^{\sigma_n,1}_{\alpha_n,1}[f_\ell](x) &= \frac{\partial}{\partial x_n} \left(\frac{\sigma_n}{e^{h_n(x)} \alpha_n x_n^{\alpha_n - 1}} \right) e^{h_n(x)} f_\ell(x),\\
I^1[f] &= f\\
{}^{\psi}W^{\sigma,\beta}_{\alpha,k}[f](x) &= \sum_{\substack{\ell=0=n\\ \ell\neq n}}^{3} \frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell - 1}} \psi_n \frac{\partial}{\partial x_n} f(x),
\end{aligned}
$$

for all $x \in \Omega$ and (11) becomes at

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$$
{}^{\psi}\mathcal{D}\left[\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell-1}}f\right](x) = ({}^{\psi}\mathcal{D}^{\rho,\beta}_{\alpha,k}f)(x) + A(x)f(x) + {}^{\psi}W^{\sigma,\beta}_{\alpha,k}[f](x),
$$

where

$$
A(x) := \sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{e^{h_n(x)}\alpha_n x_n^{\alpha_n-1}}\right)e^{h_n(x)},
$$

for all $x\in\Omega$.

(b) *If* $\beta=(1,1,1,1)$ *and* $k=(\infty,\infty,\infty,\infty)$ *then*

$$
(\mathcal{D}^{\sigma,\beta}_{\alpha,k}f)(x) = \sum_{n=0=\ell}^{3}\psi_n\psi_\ell\left((1-\sigma_n)f_\ell(x) + \sigma_n\frac{\frac{\partial f_\ell}{\partial x_n}(x)}{\alpha_n x_n^{\alpha_n-1}e^{x_n^{\alpha_n}}}\right),
$$

$$
\begin{aligned}
h_n(x) = H^{\sigma_n,1}_{\alpha_n,\infty}[f_\ell](x) &= \frac{\sigma_n-1}{\sigma_n}\left[e^{x_n^{\alpha_n}}-1\right],\\
T^{\sigma_n,1}_{\alpha_n,\infty}[f_\ell](x) &= \frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{e^{h_n(x)}\alpha_n x_n^{\alpha_n-1}e^{x_n^{\alpha_n}}}\right)e^{h_n(x)}f_\ell(x),\\
I^1[f] &= f\\
{}^{\psi}W^{\sigma,\beta}_{\alpha,k}[f](x) &= \sum_{\substack{\ell=0=n\\ \ell\neq n}}^{3}\frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell-1}e^{x_\ell^{\alpha_\ell}}}\psi_n\frac{\partial}{\partial x_n}f(x),
\end{aligned}
$$

for all $x\in\Omega$ *and* (11) *becomes at*

$$
{}^{\psi}\mathcal{D}\left[\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell-1}e^{x_\ell^{\alpha_\ell}}}f\right](x) = ({}^{\psi}\mathcal{D}^{\rho,\beta}_{\alpha,k}f)(x) + A(x)f(x) + {}^{\psi}W^{\sigma,\beta}_{\alpha,k}[f](x),
$$

where

$$
A(x) := \sum_{n=0}^{3}\psi_n\frac{\partial}{\partial x_n}\left(\frac{\sigma_n}{e^{h_n(x)}\alpha_n x_n^{\alpha_n-1}e^{x_n^{\alpha_n}}}\right)e^{h_n(x)},
$$

for all $x\in\Omega$.

2. *If* $\delta=(1,1,1,1)$ *then the operators given in Remark* 5 *are represented by*

$$
\begin{aligned}
({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x) &= \sum_{n=0=\ell}^{3}\psi_\ell\frac{\partial^{\rho_n,1}g_\ell}{\partial x_{\gamma_n,m_n}}(x)\psi_n,\\
j_n(x) = H^{\rho_n,1}_{\gamma_n,k_n}[g_\ell](x) &= \frac{\rho_n-1}{\rho_n}\left[e(x_n^{\gamma_n})_{m_n}-1\right],\\
S^{\rho_n,1}_{\gamma_n,m_n}[g_\ell](x) &= \frac{\partial}{\partial x_n}\left(\frac{\rho_n}{e^{j_n(x)}\frac{d}{dx_n}e(x_n^{\gamma_n})_{m_n}}\right)e^{j_n(x)}g_\ell(x),\\
{}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x) &= \sum_{\substack{\ell=0=n\\ \ell\neq n}}^{3}\frac{\rho_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\gamma_\ell})_{m_\ell}}\frac{\partial}{\partial x_n}g(x)\psi_n
\end{aligned}
$$

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and identity (12) is

$$^{\psi}\mathcal{D}_r\left[\sum_{\ell=0}^{3}\frac{\rho_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\gamma_\ell})_{m_\ell}}g\right](x)=({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x)+g(x)B(x)+{}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x),$$

where

$$B(x)=\sum_{n=0}^{3}\frac{\partial}{\partial x_n}\left(\frac{\rho_n}{e^{j_n(x)}\frac{d}{dx_n}e(x_n^{\gamma_n})_{m_n}}\right)e^{j_n(x)}\psi_n,$$

for all $x\in\Omega$.

(a) *If* $\delta=(1,1,1,1)$ *then the operators given in Remark 5 are represented by*

$$({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x)=\sum_{n=0=\ell}^{3}\psi_\ell\frac{\partial^{\rho_n,1}g_\ell}{\partial x_{\gamma_n,m_n}}(x)\psi_n,$$

$$j_n(x)=H^{\rho_n,1}_{\gamma_n,k_n}[g_\ell](x)=\frac{\rho_n-1}{\rho_n}[e(x_n^{\gamma_n})_{m_n}-1],$$

$$S^{\rho_n,1}_{\gamma_n,m_n}[g_\ell](x)=\frac{\partial}{\partial x_n}\left(\frac{\rho_n}{e^{j_n(x)}\frac{d}{dx_n}e(x_n^{\gamma_n})_{m_n}}\right)e^{j_n(x)}g_\ell(x),$$

$${}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x)=\sum_{\substack{\ell=0=n\\ \ell\neq n}}^{3}\frac{\rho_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\gamma_\ell})_{m_\ell}}\frac{\partial}{\partial x_n}g(x)\psi_n$$

and identity (12) is

$$^{\psi}\mathcal{D}_r\left[\sum_{\ell=0}^{3}\frac{\rho_\ell}{\frac{d}{dx_\ell}e(x_\ell^{\gamma_\ell})_{m_\ell}}g\right](x)=({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x)+g(x)B(x)+{}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x),$$

where

$$B(x)=\sum_{n=0}^{3}\frac{\partial}{\partial x_n}\left(\frac{\rho_n}{e^{j_n(x)}\frac{d}{dx_n}e(x_n^{\gamma_n})_{m_n}}\right)e^{j_n(x)}\psi_n,$$

for all $x\in\Omega$.

(b) *If* $\delta=(1,1,1,1)$ *and* $m=(1,1,1,1)$ *then the operators given in Remark 5 are represented by*

$$({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x)=\sum_{n=0=\ell}^{3}\psi_\ell\frac{\partial^{\rho_n,1}g_\ell}{\partial x_{\gamma_n,1}}(x)\psi_n,$$

$$j_n(x)=H^{\rho_n,1}_{\gamma_n,k_n}[g_\ell](x)=\frac{\rho_n-1}{\rho_n}x_n^{\gamma_n},$$

$$S^{\rho_n,1}_{\gamma_n,1}[g_\ell](x)=\frac{\partial}{\partial x_n}\left(\frac{\rho_n}{e^{j_n(x)}\gamma_n x_n^{\gamma_n-1}}\right)e^{j_n(x)}g_\ell(x),$$

$${}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x)=\sum_{\substack{\ell=0=n\\ \ell\neq n}}^{3}\frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}}\frac{\partial}{\partial x_n}g(x)\psi_n$$

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and identity (12) *is*

$$ {}^{\psi}\mathcal{D}_r\left[\sum_{\ell=0}^{3}\frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}}g\right](x) = ({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x) + g(x)B(x) + {}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x), $$

where

$$ B(x) = \sum_{n=0}^{3}\frac{\partial}{\partial x_n}\left(\frac{\rho_n}{e^{j_n(x)}\gamma_n x_n^{\gamma_n-1}}\right)e^{j_n(x)}\psi_n, $$

for all $x\in\Omega$.

(c) *If* $\delta=(1,1,1,1)$ *and* $m=(\infty,\infty,\infty,\infty)$ *then the operators given in Remark* 5 *are represented by*

$$ \begin{aligned} ({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x) &= \sum_{n=0=\ell}^{3}\psi_\ell\frac{\partial^{\rho_n,1}g_\ell}{\partial x_{\gamma_n,\infty}}(x)\psi_n, \\ j_n(x) = H^{\rho_n,1}_{\gamma_n,k_n}[g_\ell](x) &= \frac{\rho_n-1}{\rho_n}\left[e^{x_n^{\gamma_n}}-1\right], \\ S^{\rho_n,1}_{\gamma_n,\infty}[g_\ell](x) &= \frac{\partial}{\partial x_n}\left(\frac{\rho_n}{e^{j_n(x)}\gamma_n x_n^{\gamma_n-1}e^{x_n^{\gamma_n}}}\right)e^{j_n(x)}g_\ell(x), \\ {}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x) &= \sum_{\substack{\ell=0=n\\ \ell\neq n}}^{3}\frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}e^{x_\ell^{\gamma_\ell}}}\frac{\partial}{\partial x_n}g(x)\psi_n \end{aligned} $$

and identity (12) *is*

$$ {}^{\psi}\mathcal{D}_r\left[\sum_{\ell=0}^{3}\frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}e^{x_\ell^{\gamma_\ell}}}g\right](x) = ({}^{\psi}\mathcal{D}^{\rho,\delta}_{r,\gamma,m}g)(x) + g(x)B(x) + {}^{\psi}V^{\rho,\delta}_{\gamma,m}[g](x), $$

where

$$ B(x) = \sum_{n=0}^{3}\frac{\partial}{\partial x_n}\left(\frac{\rho_n}{e^{j_n(x)}\gamma_n x_n^{\gamma_n-1}e^{x_n^{\gamma_n}}}\right)e^{j_n(x)}\psi_n, $$

for all $x\in\Omega$.

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Proposition 4. *Let $\Omega \subset \mathbb{H}$ be a domain such that $\partial\Omega$ is a 3-dimensional smooth surface. In agreement with notation in Definition 7 and Remark 5 let $f = \sum_{\ell=0}^{3} \psi_\ell f_\ell, \quad g = \sum_{\ell=0}^{3} \psi_\ell g_\ell \in C^1(\Omega, \mathbb{H})$, where f_ℓ, g_ℓ are real valued functions. Then*

$$\begin{aligned}
&\int_{\partial\Omega} K_\psi(\tau - x)\sigma_\tau^\psi \left(\sum_{\ell=0}^{3} \frac{\sigma_\ell}{\frac{d}{d\tau_\ell} e(\tau_\ell^{\alpha_\ell})_{k_\ell}} I^{\beta_\ell}[f](\tau) \right) \\
&+ \int_{\partial\Omega} \left(\sum_{\ell=0}^{3} \frac{\rho_\ell}{\frac{d}{d\tau_\ell} e(\tau_\ell^{\gamma_\ell})_{m_\ell}} I^{\delta_\ell}[g](\tau) \right) \sigma_\tau^\psi K_\psi(\tau - x) \\
&- \int_\Omega \left[K_\psi(y-x)({}^\psi\mathcal{D}_{\alpha,k}^{\rho,\beta} f)(y) + ({}^\psi\mathcal{D}_{r,\gamma,m}^{\rho,\delta} g)(y) K_\psi(y-x) \right] dy \\
&- \int_\Omega K_\psi(y-x) \left[\sum_{n=0=\ell}^{3} \psi_n \psi_\ell T_{\alpha_n,k_n}^{\sigma_n,\beta_n}[f_\ell](y) + {}^\psi W_{\alpha,k}^{\sigma,\beta}[f](y) \right] dy \\
&- \int_\Omega \left[\sum_{n=0=\ell}^{3} \psi_\ell S_{\gamma_n,m_n}^{\rho_n,\delta_n}[g_\ell](y)\psi_n + {}^\psi V_{\gamma,m}^{\rho,\delta}[g](y) \right] K_\psi(y-x) dy \\
&= \begin{cases} \displaystyle\sum_{\ell=0}^{3} \frac{\sigma_\ell}{\frac{d}{dx_\ell} e(x_\ell^{\alpha_\ell})_{k_\ell}} I^{\beta_\ell}[f](x) + \sum_{\ell=0}^{3} \frac{\rho_\ell}{\frac{d}{dx_\ell} e(x_\ell^{\gamma_\ell})_{m_\ell}} I^{\delta_\ell}[g](x), & x \in \Omega, \\ 0, & x \in \mathbb{H} \setminus \overline{\Omega}. \end{cases}
\end{aligned} \tag{13}$$

In addition,

$$\begin{aligned}
&\int_{\partial\Omega} \left(\sum_{\ell=0}^{3} \frac{\rho_\ell}{\dfrac{d}{dx_\ell} e(x_\ell^{\gamma_\ell})_{m_\ell}} I^{\delta_\ell}[g] \right) \sigma_x^\psi \left(\sum_{\ell=0}^{3} \frac{\sigma_\ell}{\dfrac{d}{dx_\ell} e(x_\ell^{\alpha_\ell})_{k_\ell}} I^{\beta_\ell}[f](x) \right) \\
&= \int_\Omega \Big(g({}^\psi\mathcal{D}_{\alpha,k}^{\rho,\beta} f)(x) + ({}^\psi\mathcal{D}_{r,\gamma,m}^{\rho,\delta} g)(x) f(x \Big) dx \\
&+ \int_\Omega g(x) \left[\sum_{n=0=\ell}^{3} \psi_n \psi_\ell T_{\alpha_n,k_n}^{\sigma_n,\beta_n}[f_\ell](x) + {}^\psi W_{\alpha,k}^{\sigma,\beta}[f](x) \right] dx \\
&+ \int_\Omega \left[\sum_{n=0=\ell}^{3} \psi_\ell S_{\gamma_n,m_n}^{\rho_n,\delta_n}[g_\ell](x)\psi_n + {}^\psi V_{\gamma,m}^{\rho,\delta}[g](x) \right] f(x) dx
\end{aligned} \tag{14}$$

Proof. It is a direct consequence of Definition 7 and Remark 5 using functions $\displaystyle\sum_{\ell=0}^{3} \frac{\sigma_\ell}{\dfrac{d}{dx_\ell} e(x_\ell^{\alpha_\ell})_{k_\ell}} I^{\beta_\ell}[f](x)$ and $\displaystyle\sum_{\ell=0}^{3} \frac{\rho_\ell}{\dfrac{d}{dx_\ell} e(x_\ell^{\gamma_\ell})_{m_\ell}} I^{\delta_\ell}[g](x)$ and identities (11) and (12) in formulas (2) and (3). □

Remark 6. *In formulas (13) and (14), the operators ${}^\psi\mathcal{D}_{\alpha,k}^{\rho,\beta}$ and ${}^\psi\mathcal{D}_{r,\gamma,m}^{\rho,\delta}$ reflect the phenomenon of duality in quaternionic analysis due to the non-commutativity of quaterinonic algebra.*

Corollary 2. *Let $\Omega \subset \mathbb{H}$ be a domain such that $\partial\Omega$ is a 3-dimensional smooth surface. In agreement with notation in Definition 7 and Remark 5 let $f = \sum_{\ell=0}^{3} \psi_\ell f_\ell, \quad g = \sum_{\ell=0}^{3} \psi_\ell g_\ell \in C^1(\Omega, \mathbb{H})$, where f_ℓ, g_ℓ are real valued functions. Suppose that $f \in Ker({}^\psi\mathcal{D}_{\alpha,k}^{\rho,\beta})$ and $g \in Ker({}^\psi\mathcal{D}_{r,\gamma,m}^{\rho,\delta})$. Then*

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$$
\begin{aligned}
&\int_{\partial\Omega} K_\psi(\tau-x)\sigma_\tau^\psi \left(\sum_{\ell=0}^{3} \frac{\sigma_\ell}{\frac{d}{d\tau_\ell} e(\tau_\ell^{\alpha_\ell})_{k_\ell}} I^{\beta_\ell}[f](\tau) \right) \\
&+ \int_{\partial\Omega} \left(\sum_{\ell=0}^{3} \frac{\rho_\ell}{\frac{d}{d\tau_\ell} e(\tau_\ell^{\gamma_\ell})_{m_\ell}} I^{\delta_\ell}[g](\tau) \right) \sigma_\tau^\psi K_\psi(\tau-x) \\
&- \int_\Omega K_\psi(y-x) \left[\sum_{n=0=\ell}^{3} \psi_n \psi_\ell T_{\alpha_n,k_n}^{\sigma_n,\beta_n}[f_\ell](y) + {}^\psi W_{\alpha,k}^{\sigma,\beta}[f](y) \right] dy \\
&- \int_\Omega \left[\sum_{n=0=\ell}^{3} \psi_\ell S_{\gamma_n,m_n}^{\rho_n,\delta_n}[g_\ell](y)\psi_n + {}^\psi V_{\gamma,m}^{\rho,\delta}[g](y) \right] K_\psi(y-x)dy \\
&= \begin{cases} \displaystyle\sum_{\ell=0}^{3} \frac{\sigma_\ell}{\frac{d}{dx_\ell} e(x_\ell^{\alpha_\ell})_{k_\ell}} I^{\beta_\ell}[f](x) + \sum_{\ell=0}^{3} \frac{\rho_\ell}{\frac{d}{dx_\ell} e(x_\ell^{\gamma_\ell})_{m_\ell}} I^{\delta_\ell}[g](x), & x \in \Omega, \\ 0, & x \in \mathbb{H} \setminus \overline{\Omega} \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\partial\Omega} \left(\sum_{\ell=0}^{3} \frac{\rho_\ell}{\frac{d}{dx_\ell} e(x_\ell^{\gamma_\ell})_{m_\ell}} I^{\delta_\ell}[g] \right) \sigma_x^\psi \left(\sum_{\ell=0}^{3} \frac{\sigma_\ell}{\frac{d}{dx_\ell} e(x_\ell^{\alpha_\ell})_{k_\ell}} I^{\beta_\ell}[f](x) \right) \\
&= \int_\Omega g(x) \left[\sum_{n=0=\ell}^{3} \psi_n \psi_\ell T_{\alpha_n,k_n}^{\sigma_n,\beta_n}[f_\ell](x) + {}^\psi W_{\alpha,k}^{\sigma,\beta}[f](x) \right] dx \\
&+ \int_\Omega \left[\sum_{n=0=\ell}^{3} \psi_\ell S_{\gamma_n,m_n}^{\rho_n,\delta_n}[g_\ell](x)\psi_n + {}^\psi V_{\gamma,m}^{\rho,\delta}[g](x) \right] f(x)dx
\end{aligned}
$$

Corollary 3. *Let $\Omega \subset \mathbb{H}$ be a domain such that $\partial\Omega$ is a 3-dimensional smooth surface. In agreement with notation in Definition 7 and Remark 5 let $f = \sum_{\ell=0}^{3} \psi_\ell f_\ell$, $g = \sum_{\ell=0}^{3} \psi_\ell g_\ell \in C^1(\Omega, \mathbb{H})$, where f_ℓ, g_ℓ are real valued functions. Suppose that $f \in Ker({}^\psi\mathcal{D}_{\alpha,k}^{\rho,\beta})$ and $g \in Ker({}^\psi\mathcal{D}_{r,\gamma,m}^{\rho,\delta})$. For fix $\beta = (1,1,1,1)$ and $\delta = (1,1,1,1)$ we have:*

1. *If $k = (1,1,1,1)$ and $m = (1,1,1,1)$, then*

$$
\begin{aligned}
&\int_{\partial\Omega} K_\psi(\tau-x)\sigma_\tau^\psi \left(\sum_{\ell=0}^{3} \frac{\sigma_\ell}{\alpha_\ell \tau_\ell^{\alpha_\ell-1}} \right) f(\tau) + \int_{\partial\Omega} g(\tau) \left(\sum_{\ell=0}^{3} \frac{\rho_\ell}{\gamma_\ell \tau_\ell^{\gamma_\ell-1}} \right) \sigma_\tau^\psi K_\psi(\tau-x) \\
&- \int_\Omega K_\psi(y-x) \left[\sum_{n=0=\ell}^{3} \psi_n \psi_\ell T_{\alpha_n,1}^{\sigma_n,1}[f_\ell](y) + {}^\psi W_{\alpha,k}^{\sigma,\beta}[f](y) \right] dy \\
&- \int_\Omega \left[\sum_{n=0=\ell}^{3} \psi_\ell S_{\gamma_n,1}^{\rho_n,1}[g_\ell](y)\psi_n + {}^\psi V_{\gamma,m}^{\rho,\delta}[g](y) \right] K_\psi(y-x)dy \\
&= \begin{cases} \displaystyle f(x) \sum_{\ell=0}^{3} \frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell-1}} + g(x) \sum_{\ell=0}^{3} \frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}}, & x \in \Omega, \\ 0, & x \in \mathbb{H} \setminus \overline{\Omega} \end{cases}
\end{aligned}
$$

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and

$$\begin{aligned}
&\int_{\partial\Omega} g(x)\left(\sum_{\ell=0}^{3}\frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}}\right)\sigma_x^{\psi}\left(\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell-1}}\right)f(x)\\
&=\int_{\Omega} g(x)\left[\sum_{n=0=\ell}^{3}\psi_n\psi_\ell T_{\alpha_n,1}^{\sigma_n,1}[f_\ell](x)+{}^{\psi}W_{\alpha,k}^{\sigma,\beta}[f](x)\right]dx\\
&+\int_{\Omega}\left[\sum_{n=0=\ell}^{3}\psi_\ell S_{\gamma_n,1}^{\rho_n,1}[g_\ell](x)\psi_n+{}^{\psi}V_{\gamma,m}^{\rho,\delta}[g](x)\right]f(x)dx,
\end{aligned}$$

where operators $T_{\alpha_n,1}^{\sigma_n,1}$, ${}^{\psi}W_{\alpha,k}^{\sigma,\beta}$, $S_{\gamma_n,1}^{\rho_n,1}$ *and* ${}^{\psi}V_{\gamma,m}^{\rho,\delta}$ *are represented in Corollary 1.*

2. *If* $k=(\infty,\infty,\infty,\infty)$ *and* $m=(\infty,\infty,\infty,\infty)$, *then*

$$\begin{aligned}
&\int_{\partial\Omega} K_\psi(\tau-x)\sigma_\tau^{\psi}\left(\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\alpha_\ell\tau_\ell^{\alpha_\ell-1}e^{\tau_\ell^{\alpha_\ell}}}\right)f(\tau)+\int_{\partial\Omega} g(\tau)\left(\sum_{\ell=0}^{3}\frac{\rho_\ell}{\gamma_\ell\tau_\ell^{\gamma_\ell-1}e^{\tau_\ell^{\gamma_\ell}}}\right)\sigma_\tau^{\psi}K_\psi(\tau-x)\\
&-\int_{\Omega}K_\psi(y-x)\left[\sum_{n=0=\ell}^{3}\psi_n\psi_\ell T_{\alpha_n,\infty}^{\sigma_n,1}[f_\ell](y)+{}^{\psi}W_{\alpha,k}^{\sigma,\beta}[f](y)\right]dy\\
&-\int_{\Omega}\left[\sum_{n=0=\ell}^{3}\psi_\ell S_{\gamma_n,\infty}^{\rho_n,1}[g_\ell](y)\psi_n+{}^{\psi}V_{\gamma,m}^{\rho,\delta}[g](y)\right]K_\psi(y-x)dy\\
&=\begin{cases} f(x)\displaystyle\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell-1}e^{x_\ell^{\alpha_\ell}}}+g(x)\sum_{\ell=0}^{3}\frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}e^{x_\ell^{\gamma_\ell}}}, & x\in\Omega,\\ 0, & x\in\mathbb{H}\setminus\overline{\Omega}\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\partial\Omega} g(x)\left(\sum_{\ell=0}^{3}\frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}e^{x_\ell^{\gamma_\ell}}}\right)\sigma_x^{\psi}\left(\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell-1}e^{x_\ell^{\alpha_\ell}}}\right)f(x)\\
&=\int_{\Omega} g(x)\left[\sum_{n=0=\ell}^{3}\psi_n\psi_\ell T_{\alpha_n,\infty}^{\sigma_n,1}[f_\ell](x)+{}^{\psi}W_{\alpha,k}^{\sigma,\beta}[f](x)\right]dx\\
&+\int_{\Omega}\left[\sum_{n=0=\ell}^{3}\psi_\ell S_{\gamma_n,\infty}^{\rho_n,1}[g_\ell](x)\psi_n+{}^{\psi}V_{\gamma,m}^{\rho,\delta}[g](x)\right]f(x)dx,
\end{aligned}$$

where $T_{\alpha_n,\infty}^{\sigma_n,1}$, ${}^{\psi}W_{\alpha,k}^{\sigma,\beta}$, $S_{\gamma_n,\infty}^{\rho_n,1}$ *and* ${}^{\psi}V_{\gamma,m}^{\rho,\delta}$ *are given in Corollary 1.*

3. *If* $k=(1,1,1,1)$ *and* $m=(\infty,\infty,\infty,\infty)$, *then*

$$\begin{aligned}
&\int_{\partial\Omega} K_\psi(\tau-x)\sigma_\tau^{\psi}\left(\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\alpha_\ell\tau_\ell^{\alpha_\ell-1}}\right)f(\tau)+\int_{\partial\Omega} g(\tau)\left(\sum_{\ell=0}^{3}\frac{\rho_\ell}{\gamma_\ell\tau^{\gamma_\ell-1}e^{\tau_\ell^{\gamma_\ell}}}\right)\sigma_\tau^{\psi}K_\psi(\tau-x)\\
&-\int_{\Omega}K_\psi(y-x)\left[\sum_{n=0=\ell}^{3}\psi_n\psi_\ell T_{\alpha_n,1}^{\sigma_n,1}[f_\ell](y)+{}^{\psi}W_{\alpha,k}^{\sigma,\beta}[f](y)\right]dy\\
&-\int_{\Omega}\left[\sum_{n=0=\ell}^{3}\psi_\ell S_{\gamma_n,\infty}^{\rho_n,1}[g_\ell](y)\psi_n+{}^{\psi}V_{\gamma,m}^{\rho,\delta}[g](y)\right]K_\psi(y-x)dy\\
&=\begin{cases} f(x)\displaystyle\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell-1}}+g(x)\sum_{\ell=0}^{3}\frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}e^{x_\ell^{\gamma_\ell}}}, & x\in\Omega,\\ 0, & x\in\mathbb{H}\setminus\overline{\Omega}\end{cases}
\end{aligned}$$

and

$$\int_{\partial\Omega} g(x)\left(\sum_{\ell=0}^{3}\frac{\rho_\ell}{\gamma_\ell x_\ell^{\gamma_\ell-1}e^{x_\ell^{\gamma_\ell}}}\right)\sigma_x^{\psi}\left(\sum_{\ell=0}^{3}\frac{\sigma_\ell}{\alpha_\ell x_\ell^{\alpha_\ell-1}}\right)f(x)$$
$$=\int_{\Omega} g(x)\left[\sum_{n=0=\ell}^{3}\psi_n\psi_\ell T_{\alpha_n,1}^{\sigma_n,1}[f_\ell](x)+{}^{\psi}W_{\alpha,k}^{\sigma,\beta}[f](x)\right]dx$$
$$+\int_{\Omega}\left[\sum_{n=0=\ell}^{3}\psi_\ell S_{\gamma_n,\infty}^{\rho_n,1}[g_\ell](x)\psi_n+{}^{\psi}V_{\gamma,m}^{\rho,\delta}[g](x)\right]f(x)dx,$$

where operators $T_{\alpha_n,1}^{\sigma_n,1}$, ${}^{\psi}W_{\alpha,k}^{\sigma,\beta}$, $S_{\gamma_n,\infty}^{\rho_n,1}$ *and* ${}^{\psi}V_{\gamma,m}^{\rho,\delta}$ *are given in Corollary 1.*

4. *For* $k = (\infty,\infty,\infty,\infty)$ *and* $m = (1,1,1,1)$ *a similar result is in fact true.*

5. Discussion

This paper establishes the foundations of a quaternionic function theory associated to a proportional and fractional-fractal ψ-Fueter operator associated to a fractal measure. Also this work extends the quaternionic hiperholomorphic function theory. So what other results can be extended to this recent function theory?

Acknowledgments: This work was partially supported by Instituto Politécnico Nacional (grant numbers SIP20241638, SIP20241237) and CONAHCYT (grant number 1077475).

Conflicts of Interest: " The authors declare that they have no competing interests regarding the publication of this paper."

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