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Coherence and Uniqueness in Risk Capital Allocation: Linking Exposure Curves, Expected Shortfall, and Aumann–Shapley Values

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Abstract

This paper develops a unified framework for risk capital allocation, centered on Expected Shortfall (ES). We establish the structural equivalence between Complementary Risk Exposure (CRE) and ES, showing that CRE is not an independent risk measure but a distributional representation of ES. Under continuous differentiability, ES ensures analytical uniqueness: capital allocation is uniquely determined via Euler decomposition and strictly coincides with the Aumann–Shapley allocation. Incorporating Denault's (2001) axioms further secures axiomatic uniqueness, demonstrating that ES-based allocation is stable both mathematically and institutionally. For non-smooth or atomic distributions, Rockafellar–Uryasev optimization and subgradient methods are applied, with portfolio-based CRE proposed as a canonical selection, preserving uniqueness in practice. The paper's contribution is twofold: it is the first to integrate coherent risk measures with the dual uniqueness of capital allocation (analytical and axiomatic) within a single framework, and it positions the CRE–ES equivalence as the key nexus. This result provides both mathematical rigor and practical applicability, offering a robust foundation for theory, regulation, and practice in risk capital allocation.

Keywords: Expected Shortfall (ES); complementary risk exposure (CRE); Aumann–Shapley allocation; Euler decomposition; coherent risk measures; uniqueness; Denault axioms

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1. Introduction

In modern financial and insurance risk management, the allocation of capital across multiple risk units has long been a central issue in both theoretical research and practical operations. Capital allocation is not only relevant to internal risk assessment and incentive mechanisms within firms but also directly tied to regulatory frameworks, such as capital adequacy ratios and solvency evaluations. Consequently, identifying a capital allocation method that is both mathematically rigorous and institutionally compliant has become a common concern for both academia and industry.

Since Artzner, Delbaen, Eber, and Heath (1999)[1] introduced the four axioms of coherent risk measures, axiomatic research in risk measurement has developed rapidly. Among these, Expected Shortfall (ES) has gradually become the preferred tool of both academia and regulators, as it satisfies coherence, convexity, and tail sensitivity. It has been formally adopted in the Basel capital accords (Basel III/IV) and Solvency II in the European Union. Nevertheless, how to allocate capital within the ES framework remains an open question, with multiple parallel research paths yet to converge into a unified theoretical system.

Current research can be broadly divided into two approaches. First, the analytical path: Tasche (1999)[2] and Aumann–Shapley (1974)[3] demonstrated that when the risk measure satisfies homogeneity of degree one and differentiability, capital allocation can be uniquely determined via Euler decomposition and is strictly equivalent to the Aumann–Shapley allocation, thus establishing analytical uniqueness. Second, the axiomatic path: Denault (2001)[4], building upon the coherent risk measure framework of Artzner et al., proposed that capital allocation must satisfy five axioms (full

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allocation, no undercut, risk neutrality, symmetry, and consistency), and proved that under this framework, the only admissible allocation rule is also the Euler allocation, thereby establishing axiomatic uniqueness.

Despite these important contributions, three notable gaps remain:

1. Separation: Analytical uniqueness and axiomatic uniqueness are often discussed independently, without a unified logical chain.
2. The unclear role of CRE: Although the Complementary Risk Exposure (CRE) function has been introduced in actuarial literature, its strict correspondence with ES has yet to be systematically formalized.
3. Insufficient treatment of non-smooth cases: When risk distributions contain atoms, ES becomes non-differentiable, analytical uniqueness degenerates into multiplicity, yet no widely accepted normative resolution exists.

Against this background, this paper proposes several innovative contributions. By formally defining CRE and proving its structural equivalence to ES, this study connects the analytical and axiomatic paths and introduces the concept of capital allocation uniqueness:

Capital allocation uniqueness refers to the property that, under the ES framework, capital allocation simultaneously satisfies analytical uniqueness (ensured by functional properties) and axiomatic uniqueness (ensured by institutional norms), thereby establishing its irreplaceability at both mathematical and institutional levels.

Furthermore, for non-smooth cases, this study introduces Rockafellar–Uryasev's optimization representation and subgradient tools, and proposes portfolio-based CRE as a normative choice to ensure that capital allocation continues to maintain institutional uniqueness in practice. Through this set of derivations and integrations, the paper not only fills theoretical gaps in the existing literature but also provides a unified framework for regulatory and industry practice.

1.1. Research Background

Since the rise of insurance and financial risk management, the issue of risk capital allocation has always been central in both theory and practice. With increasing complexity in financial markets and tightening regulatory regimes, how to allocate total risk capital across multiple risk units in a reasonable, transparent, and consistent manner has become a shared concern for regulators, institutions, and researchers. Traditional allocation methods (such as proportional allocation or marginal contribution approaches), although simple, often ignore inter-risk correlations and tail distribution features, leading to capital allocations that lack coherence and fairness.

Within this context, the axiomatization of risk measures became a key breakthrough. Artzner et al. (1999) proposed the four axioms of coherent risk measures, providing a unified theoretical framework for risk management. Subsequently, Kusuoka (2001)[5] developed the representation theorem, Föllmer and Schied (2002, 2004)[6] extended convex risk measures, and Acerbi (2002)[7] proposed the theory of spectral risk measures (SRM), further deepening this framework. Particularly, Expected Shortfall (ES), because it satisfies the coherence axioms while maintaining computability and stability, has gradually become the standard risk measure in regulatory and market practice, and has been adopted in both Basel and Solvency II frameworks.

1.2. Research Motivation

Although ES has become a widely accepted risk measure, several deficiencies remain in the literature concerning the uniqueness of capital allocation:

- Mathematically: Tasche (1999) and Aumann–Shapley (1974) demonstrated that under homogeneity and differentiability conditions, capital allocation enjoys analytical uniqueness, i.e., Euler decomposition yields a unique solution. However, this result depends heavily on differentiability assumptions and lacks integration with the axiomatic framework.

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- Axiomatically: Denault (2001) and Kalkbrener (2005)[8], from an axiomatic perspective, proposed a set of normative requirements that capital allocation must satisfy and proved that the unique admissible solution is again the Euler allocation, i.e., axiomatic uniqueness. However, this strand of research rarely links directly to ES, leaving the connection underexplored.
- Functionally: Bernegger (1997)[9] introduced the risk exposure function (RE) and the complementary risk exposure function (CRE), which have been applied in actuarial science. Yet, their mathematical relationship with ES has not been systematically clarified, leaving CRE's role in the capital allocation framework ambiguous.

Thus, while prior research has revealed uniqueness in different dimensions, a unified logical framework that integrates ES's coherence, CRE's tail-functional representation, Euler/A–S analytical uniqueness, and Denault's axiomatic uniqueness has not yet been established. This gap forms the entry point of this paper.

1.3. Research Questions

This study seeks to answer the following core questions:

1. How can the relationship between CRE and ES be formalized? Can CRE be regarded as a standardized representation of ES, thereby establishing their structural correspondence?
2. Within the ES framework, is capital allocation unique? Can uniqueness be demonstrated simultaneously from functional properties (analytical path) and axiomatic principles (institutional path)?
3. How should uniqueness be addressed under non-smooth distributions? Can Rockafellar–Uryasev optimization and subgradient methods, combined with extensions of CRE, provide a unified explanation?

1.4. Research Contributions

The contributions of this paper are threefold:

- Theoretical Unification: For the first time, this study integrates CRE–ES structural correspondence, Euler/A–S analytical uniqueness, and Denault's axiomatic uniqueness into a single logical chain, establishing a unified framework for consistent capital allocation.
- Dual Uniqueness: It demonstrates that under ES, capital allocation simultaneously satisfies both analytical and axiomatic uniqueness, ensuring robustness at both mathematical and institutional levels.
- Boundary Case Treatment: In non-smooth distributions, by employing Rockafellar–Uryasev subgradient methods and normative CRE choices, the scope of the uniqueness framework is expanded, enhancing its practical applicability.

1.5. Structure of the Paper

This paper is organized into five chapters:

- Chapter 2 systematically reviews the development of coherent risk measures, Expected Shortfall (ES), capital allocation methods, and risk exposure functions (RE/CRE), and identifies the research gaps addressed in this study.
- Chapter 3 develops the theoretical framework, establishes the structural correspondence between CRE and ES, and proposes the concept of "dual uniqueness" in capital allocation.
- Chapter 4 presents the derivations, proving both analytical and axiomatic uniqueness under ES, and discusses the closure of the framework under non-smooth cases.
- Chapter 5 concludes by summarizing contributions, addressing the identified research gaps, highlighting institutional and practical implications, and suggesting directions for future research.

1.6. Summary

The research logic of this paper follows a progressive sequence of literature review → theoretical framework → result derivation → discussion and conclusion. By formalizing the structural correspondence between CRE and ES, it unifies risk measurement and capital allocation into a single logical chain, ultimately establishing the dual uniqueness of capital allocation under ES. This not only fills existing gaps in the literature but also provides regulators and practitioners with a systematic, robust, and explanatory theoretical foundation.

2. Literature Review and Research Context

This chapter systematically reviews the theoretical development of risk capital allocation, covering coherent risk measures, Expected Shortfall (ES), both analytical and axiomatic approaches to capital allocation, and the introduction and applications of risk exposure functions (RE/CRE). In particular, this study distinguishes between functional CRE (used for tail characterization) and portfolio-based CRE (used for capital allocation), and rigorously demonstrates their equivalence and derivative relationship with ES. This lays the foundation for later establishing the CRE–ES correspondence and the unified framework of capital allocation uniqueness.

2.1. The Proposal and Extension of Coherent Risk Measures

The "axiomatic turn" in risk measurement began with Artzner, Delbaen, Eber, and Heath (1999), who proposed that risk measures should satisfy four coherence axioms: monotonicity, subadditivity, translation invariance, and positive homogeneity. This breakthrough established the theoretical framework of coherent risk measures, provided a unified language for risk management, and revealed the deficiency of Value-at-Risk (VaR) in terms of subadditivity.

On this basis, Kusuoka (2001) proved that law-invariant coherent risk measures can be represented as weighted averages of conditional tail risk measures, laying the mathematical foundation for spectral risk measures (SRM). Föllmer and Schied (2002, 2004) further extended coherent risk measures to more general optimization and hedging scenarios by introducing convex risk measures and their dual representations, thereby strengthening the applicability of the theory.

Acerbi (2002) introduced the SRM framework, integrating VaR through a weighting function ϕ. He showed that when the weighting function is uniform on the tail, the resulting risk measure is exactly Expected Shortfall (ES). This result confirmed ES as a special case within the spectrum of coherent risk measures, gradually elevating it to a common benchmark in both theory and practice. The SRM framework demonstrated the existence of infinitely many risk measures satisfying coherence, introducing a non-uniqueness problem at the level of risk measure selection. This paper, however, focuses on ES as the chosen coherent risk measure and investigates the uniqueness problem at the level of capital allocation.

2.2. The Role and Computational Tractability of ES

Expected Shortfall (ES) not only satisfies coherence axioms but also exhibits desirable mathematical properties. Acerbi and Tasche (2002) demonstrated that ES is stable and operationally feasible under both continuous and discrete distributions, which led to its gradual adoption as a regulatory standard. Both Basel III/IV and Solvency II have formally enshrined ES as the benchmark for risk capital requirements, underscoring its institutional role.

From a computational perspective, Rockafellar and Uryasev (2000, 2002)[10] proposed an optimization-based representation of ES:

$$ES_{\alpha}(L) = \min_{s \in R} \left\{ s + \frac{1}{1-\alpha} E[(L-s)^{+}] \right\}$$

This representation not only solved the computational challenge of ES but, through the convexity and subdifferentiability of the objective function in ss and portfolio weights, also provided the mathematical foundation for the (sub)gradient structure of Euler capital allocation. It thus serves as a crucial bridge connecting ES and analytical uniqueness.

2.3. Analytical and Axiomatic Approaches to Capital Allocation

The uniqueness of risk capital allocation has long been a focal concern in academia, and existing research can be grouped into two main approaches:

- **Analytical Approach**: Tasche (1999), based on the Euler principle, proposed that if the risk measure is homogeneous of degree one and differentiable, then capital allocation can be uniquely decomposed by marginal derivatives:

$$\rho\big(L(x)\big) = \sum_i x_i \, \partial_{x_i}\rho\big(L(x)\big).$$

  Aumann and Shapley (1974), using continuous game theory, further proved that Euler decomposition coincides with the Aumann–Shapley allocation derived from path integrals, thereby guaranteeing analytical uniqueness.
- **Axiomatic Approach**: Shapley (1953)[11], in cooperative game theory, introduced four axioms and proved that the unique solution is the Shapley value. Drawing on this, Denault (2001) proposed five axioms for capital allocation (full allocation, no undercut, risk neutrality, symmetry, and consistency) and proved that the only allocation satisfying them is the Euler allocation. Kalkbrener (2005) further reinforced this conclusion, emphasizing the importance of coherence and uniqueness from an axiomatic perspective.

Although the two approaches appear distinct—one being mathematical, the other institutional—the works of Denault (2001) and Kalkbrener (2005) have demonstrated that within the framework of coherent risk measures, the only reasonable allocation rule reduces to the Euler allocation. Since ES is the most representative and regulatorily adopted coherent risk measure, this unification is particularly significant. One of this paper's contributions is to make this logical chain explicit and to embed both paths within the same framework through the structural correspondence of CRE and ES.

2.4. Functional RE/CRE

Beyond risk measurement and allocation methods, another important research direction is the use of exposure curves to functionally characterize the tail behavior of loss distributions. The Swiss Re curve proposed by Bernegger (1997) is a classic example.

- Risk Exposure (RE):

$$RE_i(t) = \frac{1}{\mu_i}\int_0^t S_i(y)\,dy, \quad S_i(y) = P(L_i > y), \quad \mu_i = E[L_i].$$

- Complementary Risk Exposure (CRE):

$$CRE_i(t) = 1 - RE_i(t) = \frac{1}{\mu_i}\int_t^\infty S_i(y)\,dy = \frac{E[(L_i - t)^+]}{E[L_i]}.$$

It is worth noting that ES can be viewed as the unnormalized tail expectation, while CRE is its normalized counterpart. Under continuous distributions, when $t = VaR_\alpha(L)$, they are linearly related:

$$ES_\alpha(L) = t + \frac{\mu}{1-\alpha}\,CRE(t).$$

This demonstrates that functional CRE and ES share the same mathematical origin, providing intuitive evidence for their structural correspondence.

Actuarial science has long studied exposure curves. Bernegger (1997), in his work on Swiss Re's MBBEFD distribution class, first formalized exposure curves to describe excess tail risk in claim distributions. This line of research has been expanded further: Drees (2011)[12] investigated tail risk estimation and model validation in extreme value statistics and actuarial contexts, strengthening the statistical foundation of tail integrals (the core of CRE). Hirz, Schmock, and Shevchenko (2015)[13] discussed the application and sensitivity of tail risk measures (VaR and ES) in CreditRisk+ aggregation, offering practical computational perspectives. Hillairet, Jiao, and Réveillac (2017)[14] applied Malliavin calculus to derive pricing formulas for insurance derivatives, structurally representing tail loss expectations (the essence of ES), thereby mathematically reinforcing the CRE–ES link.

Building on these studies, this paper formally defines CRE in Chapter 3 and demonstrates its structural correspondence with ES. CRE is thus not an isolated innovation but one deeply rooted in actuarial and extreme value traditions, providing a solid academic lineage for the unified capital allocation framework developed here.

2.5 Portfolio-Based CRE

In the context of capital allocation, beyond functional CRE, one can also define portfolio-based CRE as:

$$CRE_{i,\alpha}(x) = ES_{\alpha}!\left(\sum_{j=1}^{n} x_j L_j\right) - ES_{\alpha}!\left(\sum_{j\neq i} x_j L_j\right).$$

This definition measures the incremental contribution of the ith risk unit to the portfolio risk measure at confidence level α—that is, the difference in ES before and after excluding unit ii. Its significance lies in providing an incremental contribution interpretation, making the economic role of each risk unit transparent.

Portfolio-based CRE is closely related to Euler allocation:

- Under smooth distributions, the optimization representation of ES by Rockafellar and Uryasev (2000, 2002) guarantees differentiability. In this case,

$$CRE_{i,\alpha}(x) = x_i\ \partial_{x_i} ES_{\alpha}\left(L_{\text{port}}(x)\right),$$

which is exactly equivalent to Euler decomposition (Tasche, 1999; Acerbi & Tasche, 2002).

- Under non-smooth or atomic distributions, the gradient of ES with respect to portfolio weights degenerates into a subgradient set, so Euler allocation ceases to be unique. Pflug (2000)[15] formally established the subgradient representation of risk measures such as ES under non-differentiable settings, providing a mathematical foundation for this degeneration. Meanwhile, Emmer, Tasche, and Kratz (2015)[16] emphasized from a backtesting and interpretability perspective that maintaining stability of capital allocation under irregular distributions is critical—thus supporting the adoption of CRE as a consistent and interpretable normative choice.

Therefore, portfolio-based CRE is essentially a capital allocation rule under ES, not a new risk measure. It coincides with Euler allocation under smooth conditions, while under non-smooth conditions it serves as a normative implementation. In other words, portfolio-based CRE is a functional realization of Euler allocation under ES, with a clear economic interpretation as the incremental contribution of each risk unit to portfolio ES.

2.6 Research Gaps

In summary, while significant progress has been made in coherent risk measures, ES optimization and computation, analytical and axiomatic approaches to capital allocation, and the introduction of functional RE/CRE, several gaps remain:

- Gap A (Unlinked Chain): Coherent risk measures, ES optimization, Euler/A–S analytical uniqueness, and Denault's axiomatic uniqueness have not yet been systematically unified.
- Gap B (Unclear Role of CRE): Although functional CRE and ES are mathematically equivalent (or linearly related), the literature lacks formal statements and proofs of this relationship; portfolio-based CRE is used in allocation but has not been systematically positioned as a derivative rule of ES.
- Gap C (Terminological Confusion): The coherence of risk measures and the consistency/uniqueness of capital allocation are often conflated, leading to conceptual ambiguity.

- Gap D (Non-smooth Scenarios): While RU subgradient methods exist, their integration into the CRE–ES–uniqueness framework under non-smooth distributions remains insufficient.

2.7. Summary

This chapter reviewed the theoretical foundations of risk capital allocation, tracing the research path from the coherence of risk measures to the uniqueness of capital allocation. Although existing contributions are rich, the components—including ES theory and optimization, analytical and axiomatic approaches to allocation, and functional and portfolio-based CRE—remain relatively fragmented and lack a unified framework.

Building on this, subsequent chapters will use the CRE–ES structural correspondence as a central nexus to integrate ES's coherence, Euler/Aumann–Shapley analytical uniqueness, and Denault's axiomatic uniqueness into a unified theoretical framework. Ultimately, this study will demonstrate that capital allocation under ES enjoys both mathematical rigor and axiomatic legitimacy, thereby establishing its dual uniqueness.

3 Theoretical Framework and Methods

This chapter establishes a coherent theoretical framework by taking the structural correspondence between CRE and ES as the point of departure. First, we formally define functional CRE and portfolio-based CRE and clarify their common origin with ES in tail-integration structures. Second, building on this correspondence, we develop two complementary routes:

1. an analytical uniqueness route grounded in functional properties, showing that under continuous differentiability the ES-based capital allocation is equivalent to the Euler/A–S allocation and is therefore unique; and
2. an axiomatic uniqueness route grounded in institutional norms, showing that under Denault's five axioms the ES-based capital allocation is likewise unique. Finally, we discuss extensions to non-smooth settings, demonstrating that even in the presence of atoms or empirical distributions, one can preserve consistency of capital allocation via subgradients and normative selections.

Through these three layers of derivation, this chapter establishes a unified result: under ES, capital allocation is unique in both the analytical and axiomatic senses.

3.1. Functional Risk Exposure and Complementary Risk Exposure

In insurance and risk management practice, exposure curves provide an intuitive tool for depicting the tail characteristics of loss distributions. Bernegger (1997) proposed the Swiss Re exposure curve and formally defined the risk exposure function via an integral form:

- Risk Exposure (RE)

$$RE_i(t) = \frac{1}{\mu_i}\int_0^t S_i(y)\,dy,$$

where $S_i(y) = P(L_i > y)$ is the survival function of the loss unit L_i, and $\mu_i = E[L_i]$ is its expected loss. The quantity $RE_i(t)$ represents the proportion of cumulative exposure below threshold t.

- Complementary Risk Exposure (CRE)

$$CRE_i(t) = 1 - RE_i(t) = \frac{1}{\mu_i}\int_t^\infty S_i(y)\,dy.$$

This function precisely characterizes tail risk above the threshold t. Its economic meaning can be understood in two ways:

- Relative tail-exposure ratio. It represents the share of losses exceeding threshold t (tail risk) in the total expected loss μ_i.
- Standardized mean excess loss. It is equivalent to the mean excess function at threshold t, $e(t) = E[L_i - t \mid L_i > t]$, multiplied by the exceedance probability $S_i(t)$, and then standardized by μ_i:

$$CRE_i(t) = \frac{E[(L_i - t)^+]}{\mu_i} = \frac{S_i(t)\cdot e(t)}{\mu_i}.$$

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This representation makes explicit the intrinsic link between CRE and conditional tail expectations and provides an intuitive bridge to the equivalence with ES developed in the next section.

From a functional standpoint, $CRE_i(t)$ is a smooth, monotonically decreasing curve from (0,1) to $(\infty, 0)$. Its boundary behavior is:

- $As \to 0^+, E[(L_i - t)^+] \to E[L_i] = \mu_i, hence\ CRE_i(t) \to 1$
- $Ast \to \infty, P(L_i > t) \to 0, thus\ E[(L_i - t)^+] \to 0\ and\ CRE_i(t) \to 0.$

These properties guarantee that CRE has well-behaved mathematical features as a standardized proportion function.

3.2. The Structural Correspondence Between CRE and Expected Shortfall (ES)

The previous chapter surveyed the exposure-curve tradition, including the seminal work of Bernegger (1997) on the MBBEFD distribution class and subsequent contributions by Drees (2011), Hirz–Schmock–Shevchenko (2015), Hillairet–Jiao–Réveillac (2017), Frees–Valdez (1998)[17], and Embrechts–Klüppelberg–Mikosch (1997)[18], which together broaden the theoretical background of Complementary Risk Exposure (CRE). This literature shows that tail integrals are not only central tools in insurance and reinsurance but also provide a solid theoretical underpinning for unifying risk measurement and capital allocation.

Against this background, this section derives the structural relationship between CRE and Expected Shortfall (ES) and proves their strict correspondence in the sense of tail integration.

3.2.1. Definition and Optimization Representation of ES

At confidence level α \alpha, Expected Shortfall (ES) is defined (Acerbi & Tasche, 2002; Rockafellar & Uryasev, 2000) as

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)\, du = VaR_\alpha(L) + \frac{1}{1-\alpha} E\left[\left(L - VaR_\alpha(L)\right)^+\right].$$

This definition shows that ES equals the quantile $VaR_\alpha(L)$ plus a weighted expectation of tail excess losses above that quantile; it is a convex, coherent tail-risk measure.

3.2.2. Definition and Expression of CRE

In actuarial science, the exposure-curve approach introduced by Bernegger (1997) quantifies cumulative exposure across loss thresholds. The associated CRE is defined as

$$CRE(t) = \frac{1}{\mu}\int_t^\infty S(y)\, dy,$$

where $S(y) = P(L > y)$ is the survival function and $\mu = E[L]$ is the expected loss. CRE thus represents "the proportion of expected excess loss above threshold tt relative to the total expected loss," i.e., a standardized tail-exposure function.

3.2.3. Correspondence Between CRE and ES

Setting the threshold $t = VaR_\alpha(L)$ and comparing the expressions for ES and CRE yields:

$$ES_\alpha(L) = t + \frac{\mu}{1-\alpha} CRE(t),$$
$$CRE(t) = \frac{1-\alpha}{\mu}(ES_\alpha(L) - t).$$

Hence:

- Functional CRE and ES are in one-to-one correspondence (under continuous distributions) within the tail-integration structure.
- The difference lies in parameterization and scale: CRE is the normalized tail-expectation curve, while ES is the tail-conditional expectation operator.

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When the distribution has atoms or is otherwise non-smooth, the mapping $\alpha \leftrightarrow t$ becomes set-valued (interval correspondence). The relationship still holds but must be interpreted via subgradient sets.

3.2.4. Literature Lineage and Theoretical Support

CRE is not an isolated innovation; it is rooted in several traditions of actuarial science and extreme-value theory:

- Bernegger (1997) first proposed a functional exposure-curve representation, revealing a structured depiction of tail excess in claim distributions;
- Panjer (2006) and McNeil, Frey & Embrechts (2015) systematized the links among exposure functions, tail distributions, and risk measures; in particular, McNeil et al.'s integration with EVT provides rigorous background for the CRE–ES equivalence;
- Frees & Valdez (1998) employed copulas to capture tail dependence across multiple risk units, enabling portfolio-based extensions of CRE;
- Embrechts, Klüppelberg & Mikosch (1997) brought EVT to tail distributions, supporting the stability and asymptotics of CRE under tail-integral conditions.

3.2.5. Summary

CRE and ES exhibit a strict correspondence within tail-integration structures. Under continuous differentiability, CRE can be viewed as a distribution-level, normalized representation of ES; in non-smooth settings, the correspondence persists but becomes set-valued. The literature demonstrates that CRE is anchored in exposure-curve and tail-distribution theory, providing both mathematical rigor and practical lineage in insurance and financial risk management. This correspondence lays a solid foundation for the CRE–ES–A–S main line of argument and for proving the dual uniqueness of capital allocation.

3.3. A Dual Framework of Uniqueness

With the structural correspondence between CRE and ES established, we proceed to analyze the uniqueness of capital allocation. As the literature indicates, uniqueness rests on both functional properties and axiomatic norms—a dual framework:

- Analytical Uniqueness.
  If the risk measure $\rho_\alpha(L)$ is positively homogeneous of degree one and differentiable, then by the Euler principle (Tasche, 1999),
  $$\rho_\alpha\big(L(x)\big) = \sum_i x_i\, \partial_{x_i}\rho_\alpha\big(L(x)\big).$$
  Under ES, marginal risk contributions coincide with the Aumann–Shapley allocation (Aumann & Shapley, 1974), ensuring uniqueness.
- Axiomatic Uniqueness.
  According to Denault (2001), if an allocation satisfies full allocation, no undercut, risk neutrality, symmetry, and consistency, then the unique allocation rule meeting these axioms is the Euler allocation. Since ES is a coherent risk measure, capital allocation under ES is unique in the axiomatic sense as well.

Thus, ES occupies a special position in capital allocation not only because of its structural correspondence with CRE but also because of its dual uniqueness (analytical and axiomatic).

3.4. Portfolio-Based CRE as a Derived Rule

To keep the main line of derivation clear, we treat portfolio-based CRE as a derived extension of ES. It is defined as

$$CRE_{i,\alpha}(x) = ES_\alpha\left(\sum_{j=1}^{n} x_j L_j\right) - ES_\alpha\left(\sum_{j\neq i} x_j L_j\right),$$

which represents the incremental contribution of risk unit ii to the portfolio risk at confidence level $\alpha\alpha$.

- Under continuous distributions, one can show that this is identical to the Euler/A–S allocation.
- In non-smooth settings, portfolio-based CRE provides a normative selection compatible with the RU subgradient set.

Therefore, portfolio-based CRE is an ES-derived allocation rule—an extension rather than a peer risk measure.

3.5. Summary

Starting from functional RE/CRE, this chapter has established the structural correspondence between CRE and ES and, on this basis, proposed a dual-uniqueness framework for capital allocation. The main line of argument remains:

$$ES_\alpha \implies \text{Euler/A-S (analytical uniqueness)} \implies \text{Denault (axiomatic uniqueness).}$$

Functional CRE provides an intuitive representation of ES's tail features at the distributional level, while portfolio-based CRE serves as an ES-derived allocation rule that facilitates extensions to non-smooth settings and practical allocation problems.

4. Results and Derivations

Building on the theoretical framework developed earlier, this chapter states the core result of the paper: under Expected Shortfall (ES), risk capital allocation is unique in both the analytical and axiomatic senses. To that end, we first restate the correspondence between CRE and ES, then develop the derivation chain CRE ⇒ ES ⇒ A–S, discuss analytical uniqueness and axiomatic uniqueness in turn, and finally address treatments and extensions in non-smooth settings.

4.1. The Correspondence Between CRE and ES

Chapter 3 shows that Complementary Risk Exposure (CRE) and Expected Shortfall (ES) admit a parameterized correspondence within a tail-integration structure:

$$ES_\alpha(L) = t + \frac{\mu}{1-\alpha} CRE(t), \qquad CRE(t) = \frac{1-\alpha}{\mu}(ES_\alpha(L) - t),$$

$$where\ t = VaR_\alpha(L)\ and\ \mu = E[L].$$

This relationship implies:

- CRE is not a new risk measure, but a standardized tail representation of ES;
- As a coherent risk measure, ES shares a common tail-integral origin with CRE;
- Consequently, the subsequent uniqueness results for capital allocation can be developed entirely within the ES framework, with CRE providing intuitive interpretation and structural support.

4.2. The Derivation Chain: CRE ⇒ ES ⇒ A–S

Under continuous differentiability, CRE, ES, and A–S allocation form a closed logical chain:

$$\text{CRE} \Rightarrow \text{ES} \Rightarrow \text{Euler/A-S Allocation.}$$

- Step 1 (CRE ⇒ ES).
  By the definition of coherent risk measures, Expected Shortfall (ES) can be viewed as a special case of tail-weighted functionals represented by CRE. Concretely, at confidence level α, ES can be expressed as the sum of the quantile VaR_α and a weighted, standardized tail exposure. By Kusuoka's (2001) representation theorem, under law-invariance, CRE-type tail functionals can be subsumed as mixtures of ES.

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- Step 2 (ES ⇒ Euler/A–S).
  Let the portfolio loss be

$$L(x) = \sum_i x_i L_i\,, \quad f(x) = ES_\alpha\big(L(x)\big).$$

  If f is continuously differentiable in the weight vector x, then:
- Since ff is positively homogeneous of degree one, Euler's theorem applies:

$$f(x) \;=\; \sum_i x_i\; \partial_{x_i} f(x).$$

- The Aumann–Shapley (A–S) allocation is defined by

$$\varphi_i^{AS}(x) \;=\; \int_0^1 \partial_{x_i} f(\theta x) \cdot x_i\, d\theta.$$

- Because the gradient of a 1-homogeneous function is 0-homogeneous, the integral reduces to its point value:

$$\varphi_i^{AS}(x) \;=\; x_i\; \partial_{x_i} f(x).$$

- The portfolio-based CRE coincides with the same expression:

$$CRE_{i,\alpha}(x) \;=\; ES_\alpha\big(L(x)\big) \;-\; ES_\alpha\left(L(x^{-i})\right) \;=\; x_i\; \partial_{x_i} ES_\alpha\big(L(x)\big).$$

Conclusion:

Under continuous differentiability, the three allocation rules are identical:

$$\textbf{CRE Allocation} \;=\; \textbf{Euler Risk Contribution} \;=\; \textbf{A-S Allocation}$$

Boundary remark:

The above CRE–ES–A–S logic is built on the four axioms of coherent risk measures, where positive homogeneity is crucial for the normative uniqueness of A–S allocations. If positive homogeneity is relaxed while retaining monotonicity, translation invariance, and convexity, one arrives at convex risk measures (Föllmer & Schied, 2002, 2004). In that framework, dual representations introduce penalty functions to model uncertainty and heterogeneous risk aversion; however, capital allocation then loses A–S–style uniqueness and can only be characterized by subgradient sets, which is less normative and interpretable than the CRE-based setting.

For these reasons, we focus on the CRE–ES–A–S chain: it is mathematically tighter and satisfies Denault's (2001) axioms, thus providing a more normative basis for capital allocation in theory and practice.

A further note: Rockafellar & Uryasev (2000, 2002) proved the convexity of ES (a.k.a. CVaR) by showing it is the supremum of affine functions, guaranteeing numerical stability and tractability in portfolio optimization. Meanwhile, Föllmer & Schied (2002, 2004) elevate convexity to an axiom for risk measures (replacing positive homogeneity), deliver dual representations with penalty terms, and define a broader class that includes ES as a typical member but also non-homogeneous measures (e.g., entropic risk). Thus, RU's convexity result is a functional guarantee within the F&S setting, while F&S provide the wider theoretical umbrella.

4.3. Analytical Uniqueness

Theorem 1 (Analytical Uniqueness).

Let the portfolio loss be $L_{port}(x) = \sum_{i=1}^{n} x_i L_i$ *and take* $\rho_\alpha = ES_\alpha$ *as the risk measure. If* $ES_\alpha\left(L_{port}(x)\right)$ *is 1-homogeneous and differentiable in* $x = (x_1, \dots, x_n)$*, then*

$$ES_\alpha\left(L_{\text{port}}(x)\right) \;=\; \sum_{i=1}^{n} x_i\; \partial_{x_i} ES_\alpha\left(L_{\text{port}}(x)\right).$$

Proof sketch (see Appendix B).

- RU's representation ensures convexity and 1-homogeneity of ES in the relevant setting;
- Euler's theorem applies to 1-homogeneous functions;

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- Tasche (1999) and Aumann–Shapley (1974) establish the coincidence of marginal decomposition and path-integral allocation, ensuring uniqueness.

Corollary 1.
Under continuous distributions, ES-based capital allocation is analytically unique and coincides with both Euler decomposition and Aumann–Shapley allocation.

Remark.
Analytical uniqueness matters because it provides a purely mathematical guarantee: allocation flows from functional properties (coherence, 1-homogeneity, differentiability) rather than ad hoc assumptions, thereby underpinning stability and consistency and providing a solid foundation for numerical algorithms in practice.

4.4. Axiomatic Uniqueness

Theorem 2 (Axiomatic Uniqueness).
Let ρ be a coherent risk measure and suppose the allocation rule satisfies Denault's (2001) five axioms: full allocation, no undercut, riskless allocation, symmetry, and consistency. Then the allocation rule is uniquely the Euler allocation.

Corollary 2.
Since ES satisfies the four coherence axioms of Artzner et al. (1999) (monotonicity, subadditivity, translation invariance, positive homogeneity), it is a coherent risk measure and thus fits directly into Denault's axiomatic framework. Within this framework, the only admissible allocation is Euler's. Together with Kalkbrener (2005) and practical arguments in Acerbi & Tasche (2002), we conclude: under ES, capital allocation is unique both analytically and axiomatically.

Remark (on symmetry).
The strength of the symmetry axiom varies across literatures:

- In Shapley (1953), strong symmetry (label irrelevance) is required: swapping any two players strictly swaps allocations—crucial for uniqueness of the Shapley value.
- In Denault (2001), a weak symmetry suffices: if two risk units are identical in weights and distributions, they receive the same allocation—closer to regulatory practice and "equal treatment of homogeneous risks."

Thus both paths use "symmetry" as a core axiom, but with different strengths: Shapley's strong symmetry guarantees uniqueness in cooperative games; Denault's weak symmetry emphasizes fairness and practicality for capital allocation. This distinction explains differences in logical strength and application contexts.

4.5. Non-Smooth Settings

In applications, loss distributions frequently have atoms (e.g., clustered defaults in credit portfolios), rendering ES non-differentiable in the weight space. Analytical uniqueness then degenerates into a set of solutions, yet theoretical and practical consistency can be preserved as follows:

- Analytical view. Using the RU optimization representation, derivatives of ES are replaced by subgradients:
$$\partial_{x_i} ES_\alpha(L(x)) \in \mathrm{SubGrad}_{x_i} ES_\alpha(L(x)).$$
Different subgradient selections yield different allocation schemes.
- Axiomatic view. Within Denault's framework, one can select from the subgradient set the allocation consistent with Euler's rule, preserving institutional uniqueness.

- Normative selection. In practice, adopt portfolio-based CRE as the normative choice within the RU subgradient set, thereby enhancing transparency and operability of the outcome.

Note: institutional uniqueness and the role of CRE.

With non-smooth distributions, ES's analytical uniqueness degenerates into a family of subgradient-based solutions, inviting ambiguity that can undermine comparability and normativity. Adopting portfolio-based CRE as a normative selection resolves this by providing a clear, interpretable, and single allocation even when differentiability fails. Thus, CRE is not merely a technical patch; it is an institutional support for uniqueness and stability of capital allocation.

International practice.

- Basel III/IV. ES-based capital attribution must be additive, interpretable, and allocable. In non-smooth settings, banks commonly use Incremental ES (iES)—essentially the difference definition that is equivalent to portfolio-based CRE.
- Solvency II. EIOPA guidance requires allocation to be consistent, operationalizable, and fair. For non-smooth distributions, regulators accept incremental allocation methods, fully aligned with CRE logic.
- Academia and practice. Tasche (1999) and Kalkbrener (2005) note that Euler decomposition degenerates to a multi-solution family when differentiability fails, necessitating normative selection. The industry-standard iES allocation is precisely a standardized embodiment of the CRE idea.

4.6. Summary of Results

In sum, this chapter establishes the central claim: under ES, capital allocation is unique both analytically and axiomatically.

- Analytical level. If the loss distribution is continuous and differentiable, the 1-homogeneity and convex optimization structure of ES guarantee that capital allocation is uniquely determined via Euler decomposition and is strictly equivalent to Aumann–Shapley allocation.
- Axiomatic level. Denault's five axioms imply that for coherent risk measures, the unique admissible allocation is Euler. Since ES is coherent, axiomatic uniqueness holds fully in this framework.
- Non-smooth settings. Although analytical uniqueness degenerates into a subgradient family, CRE as a normative selection within Denault's framework preserves institutional uniqueness.

To display these relations and conditions more transparently, Tables 4.1 and 4.2 summarize the chain and the associated assumptions.

Table 4.1. Conditions and Exceptions in the Chain CRE ⇒ ES ⇒ A–S

Link	Conditions (Supported)	When It Fails (Not Supported)	Treatment
CRE ⇒ ES	Holds by definition; unique VaR_α under continuity	Atoms ⇒ VaR_α is an interval	RU optimization; CRE–ES becomes many-to-one
ES ⇒ Euler	ES is 1-homogeneous; differentiable in x	ES non-differentiable (atoms)	RU + subgradient set ⇒ family of solutions
Euler ⇒ A–S	Gradient exists; 0-homogeneous; path independence	ES non-differentiable, subgradient non-unique	A–S reduces to subgradient integral; **normative selection** needed
Whole chain	ES continuously differentiable; CRE–ES consistent param'zn	Non-smooth ⇒ multi-solutions	Axioms preserve uniqueness; **use CRE** in practice

Table 4.2. Conditions for Coherence and Uniqueness

Layer	Holds When	Fails When	Remedy
Risk-measure coherence	ES / SRM satisfy Artzner's axioms	VaR and other non-coherent measures	Switch to ES
Allocation consistency (Denault)	Compatible under aggregation/scaling/sub-portfolios	Allocation violates consistency	Under ES, use Euler
Analytical uniqueness	ES is smooth; no atoms	Non-smooth; ES non-diff.	RU + subgradients; **CRE** as normative choice
Axiomatic uniqueness	Denault's five axioms + ES	Non-coherent measure or rule breaks axioms	Switch to ES + Euler
Dual uniqueness	ES + smooth + Denault axioms	Non-smooth ⇒ analytical part fails	Axioms still ensure uniqueness; practice uses CRE

*Interpretation.

- Table 4.1 shows when each link in the CRE → ES → Euler/A–S chain holds and how to respond when it does not.
- Table 4.2 separates coherence/consistency from uniqueness (analytical/axiomatic), noting that analytical uniqueness depends on smoothness, whereas axiomatic uniqueness under ES continues to hold.

Taken together, these results clarify the dual-uniqueness conclusion:

- Analytical uniqueness stresses functional properties (1-homogeneity, differentiability);
- Axiomatic uniqueness stresses institutional norms (Denault's axioms);
- Under smooth conditions, the two coincide;
- Under non-smooth conditions, analytical uniqueness becomes a family, but CRE and the axiomatic framework preserve institutional uniqueness.

Accordingly, the conclusions here not only consolidate ES's theoretical status but also highlight, at regulatory and practical levels, its robustness as the benchmark for capital allocation. Through the joint analytical-and-axiomatic uniqueness result, the chapter provides a consistent and solid theoretical base for further research and real-world implementation.

5. Discussion and Conclusion

This paper develops a unified theoretical framework for risk capital allocation grounded in Expected Shortfall (ES). We first prove a strict correspondence between CRE and ES within a tail-integration structure, clarifying that CRE is not an independent risk measure but a normalized, distribution-level representation of ES. This result provides intuitive and structural support for the subsequent uniqueness derivations.

Building on this, the paper advances along two main tracks:

- Analytical path. Under continuous differentiability, the 1-homogeneity and convexity of ES guarantee that capital allocation is uniquely determined via Euler decomposition and is strictly equivalent to the Aumann–Shapley (A–S) allocation, thereby establishing analytical uniqueness.
- Axiomatic path. Within Denault's (2001) five-axiom framework, and using ES's coherence, the only allocation rule satisfying the axioms is the Euler allocation, thereby establishing axiomatic uniqueness.

Together these yield the paper's core proposition: capital allocation uniqueness. For the first time, this concept systematically unifies analytical and axiomatic uniqueness, showing that ES-based capital allocation possesses robust normativity at both the mathematical and institutional levels.

Furthermore, for the non-smooth settings commonly encountered in financial and insurance portfolios, we show that although ES may lose analytical uniqueness, Rockafellar–Uryasev (RU) subgradient sets combined with a portfolio-based CRE normative selection can still preserve institutional uniqueness. This not only secures the closure of the theoretical framework but also ensures comparability and operability in practice.

5.1. Summary of Contributions

Aiming at a "unified theoretical framework for consistent capital allocation," and focusing on ES, this paper proposes and proves dual uniqueness—analytical and axiomatic—for capital allocation. The main contributions are:

- Establishing the CRE–ES structural correspondence. By introducing the functional definition of CRE and comparing it with ES's tail-integration structure, we show that CRE is, in essence, a normalized, distribution-level representation of ES. This provides an intuitive distributional interpretation of ES and a necessary bridge to the subsequent allocation results.
- Proving both analytical and axiomatic uniqueness under ES. Under continuous differentiability, ES-based allocation is equivalent to Euler/Aumann–Shapley allocation, yielding analytical uniqueness; since ES is coherent, it also yields axiomatic uniqueness under Denault's framework. Hence two previously parallel lines of research are unified under ES, revealing ES's dual robustness for capital allocation.
- Ensuring closure in non-smooth settings. When the loss distribution has atoms, analytical uniqueness degenerates to a subgradient family, yet institutional uniqueness can still be preserved through normative selection within the axiomatic framework (e.g., portfolio-based CRE). This guarantees adaptability to real-world financial and insurance contexts and bridges theory and practice.

5.2. Addressing the Research Gaps

Revisiting the gaps raised in Chapter 2, this paper responds on several fronts:

- Gap A (Unlinked chain). By using CRE–ES as the bridge, we unify coherent risk measures, RU optimization structure, Euler/A–S decomposition, and Denault's axioms within a single derivational chain.
- Gap B (Unclear role of CRE). We formalize the correspondence between CRE and ES, clarifying CRE's role as a distribution-level representation of ES.
- Gap C (Terminological confusion). We distinguish coherence of the risk measure from consistency/uniqueness of allocation, clarifying their respective scopes.
- Gap D (Insufficient treatment of non-smooth settings). Combining RU subgradients with Denault's axioms, we propose a normative allocation scheme for non-smooth distributions, filling a gap in the literature.

Thus, beyond integrating and clarifying prior work, the paper contributes new theoretical and methodological insights.

5.3. Practical and Institutional Implications

The results have direct implications for financial risk management and insurance capital regulation. Since ES is widely adopted in Basel III/IV and Solvency II, our dual-uniqueness conclusion ensures that:

- Capital allocation does not depend on the computational path or the choice among competing allocation rules, avoiding arbitrariness in practice;
- Allocation results are unique not only mathematically but also institutionally, enhancing transparency and stability of regulatory frameworks;
- Even with non-smooth or empirical distributions, one can maintain consistent allocations via normative selection, thereby reducing institutional disputes.

In other words, the paper provides a unified theoretical foundation for regulators and market participants, aligning theory and practice of capital allocation within the ES framework.

5.4. Limitations and Avenues for Future Research

Despite the progress, several limitations invite further exploration:

- Distributional assumptions. The proof of analytical uniqueness relies on continuity and differentiability. More complex non-continuous settings (e.g., compound distributions or jump processes) warrant systematic extensions.
- Computation and numerics. Although RU provides convex optimization tools for ES, efficiently implementing Euler/A–S allocation in high-dimensional portfolios and large samples remains a computational challenge.
- Broader coherent measures. This paper focuses on ES; uniqueness under spectral risk measures (SRM) or other coherent measures remains an open question.
- Institutional implementations. Extending the CRE–ES–uniqueness framework to different regulatory regimes (e.g., U.S. RBC, China's C-RBC) calls for deeper institutional research.

5.5. Conclusion

Overall, by establishing the correspondence between CRE and ES, the paper completes the following logical chain:

$$ES \;\Rightarrow\; Euler/A-S\; decomposition\; (analytical\; uniqueness)$$
$$\Rightarrow\; Denault's\; axioms\; (axiomatic\; uniqueness).$$

This framework shows that under ES, capital allocation enjoys analytical uniqueness at the mathematical level and axiomatic uniqueness at the institutional level—i.e., dual uniqueness. The result fills the gap in the CRE–ES–allocation chain theoretically and provides a unified benchmark for regulators and markets in practice.

Future research can extend along broader risk measures, more complex distributional settings, and institutional implementations, further advancing capital allocation theory in both academia and practice.

Appendix A: Core Axioms and Extensions in Risk-Measure Theory

This appendix compiles the main theoretical foundations of coherent risk measures and their extensions, spanning from the foundational definition of Artzner et al. (1999), to Denault (2001)'s axioms for capital allocation, and to the representations and generalizations due to Kusuoka (2001), Rockafellar & Uryasev (2000, 2002), and Föllmer & Schied (2002, 2004).

A.1 Coherent Risk Measures (Artzner et al., 1999)

Let $\rho: L^{\infty} \to R$ be a risk measure. If ϱ\rho satisfies the following four properties, it is called coherent:

- Monotonicity. If $X \le Y$ almost surely, then $\rho(X) \ge \rho(Y)$.
- Subadditivity. $\rho(X+Y) \le \rho(X) + \rho(Y)\; (diversification\, ffect)$.
- Positive Homogeneity. $\rho(\lambda X) = \lambda\, \rho(X), for\; ll\, \lambda \ge 0$.
- Translation Invariance. $\rho(X+m) = \rho(X) - m, for\; All\, m \in R$

A.2 Consistent Capital Allocation (Denault, 2001)

Let ρ be a coherent risk measure. A capital allocation $\{\varphi_i\}$ for risk units ii is called consistent if it satisfies:

- No Undercut. For any subset S, $\sum_{i \in S} \varphi_i \;\ge\; \rho(L_S)$.
- Efficiency (Full Allocation). $\sum_{i=1}^{n} \varphi_i = \rho(L)$.
- Symmetry. Statistically indistinguishable risk units receive the same allocation.
- Consistency. The allocation is compatible with the risk measure (aggregation/scaling coherence).

A.3 Kusuoka Representation (Kusuoka, 2001)

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If ρ is law-invariant and coherent, then there exists a family of probability measures $\mathcal{M}$ on [0,1]such that

$$\rho(X) \; = \; \sup_{\mu \in \mathcal{M}} \int_0^1 ES_\alpha(X) \; \mu(d\alpha).$$

That is, every law-invariant coherent risk measure can be represented as a mixture of Expected Shortfall (ES).

A.4 Optimization Representation and Convexity of ES (Rockafellar & Uryasev, 2000, 2002)

Expected Shortfall admits the optimization representation

$$ES_\alpha(X) \; = \; \min_{s \in R} \left\{ s + \frac{1}{1-\alpha} \, E[(X-s)_+] \right\}.$$

This ensures that ES is convex as an optimization functional, enabling efficient solution by convex optimization and providing theoretical underpinnings for portfolio optimization and algorithmic implementation.

A.5 Convex Risk Measures (Föllmer & Schied, 2002, 2004)

Relaxing positive homogeneity while retaining monotonicity, translation invariance, and convexity yields the class of convex risk measures:

$$\rho(\lambda X + (1-\lambda)Y) \; \le \; \lambda\rho(X) + (1-\lambda)\rho(Y).$$

They admit the dual representation

$$\rho(X) \; = \; \sup_{Q \in Q} \big(E_Q[-X] - \alpha(Q) \big)$$

where $\alpha(Q)$ is a penalty function capturing model uncertainty and risk aversion. This generalization widens the scope beyond homogeneity. However, for capital allocation, the uniqueness of the Aumann–Shapley allocation generally fails and allocations must be described via subgradient sets.

Appendix B: Proof Details

B.0 Notation and Definitions

- Random variables and distributions.
  L_i: loss r.v. of risk unit i;
  x_i: portfolio weight;
  $L_{\text{port}}(x) = \sum_i x_i \, L_i : portfolio\ loss.$
- Exposure functions.

$$RE_i(t) = \frac{1}{\mu_i} \int_0^t S_i(y) \; dy,$$

$$CRE_i(t) = \frac{1}{\mu_i} \int_t^\infty S_i(y) \; dy,$$

  $with \mu_i = E[L_i] and S_i(y) = P(L_i > y).$
- Expected Shortfall (ES).

$$ES_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L) \; du \; = \; \min_{s \in R} \left\{ s + \frac{1}{1-\alpha} E[(L-s)^+] \right\}.$$

- Portfolio-based CRE.

$$CRE_{i,\alpha}(x) \; = \; ES_\alpha Big\left(\sum_j x_j \, L_j \right) - ES_\alpha \left(\sum_{j \ne i} x_j \, L_j \right)$$

- Probability space.

$$(\Omega, \mathcal{F}, \mathcal{P}); L_i \in L^1; x = (x_1, \dots, x_n) \in R_+^n.$$

- Distribution and quantiles.

$$F_L: cdf; left-continuous\ \ quantile q_L(u) = \inf\{t: F_L(t) \geq u\}.$$
$$If F_L\ is\ continuous\ and\ strictly\ increasing, then\ q_L(u) = VaR_u(L).$$

- ES (continuous case).

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 q_L(u)\ du = q_L(\alpha) + \frac{1}{1-\alpha} E\left[\left(L - q_L(\alpha)\right)^+\right].$$

- RU representation. For any $L \in L^1$, $\alpha \in (0,1)$, define

$$\Phi_\alpha(s) = s + \frac{1}{1-\alpha} E[(L-s)^+], \quad s \in R.$$

Then $ES_\alpha(L) = \min_s \Phi_\alpha(s) and \arg\min \Phi_\alpha \subset [q_L^-(\alpha), q_L^+(\alpha)]$ (see Appendix C).

- Functional CRE (unit level).

$$CRE_i(t) = \frac{1}{\mu_i}\int_t^\infty oP(L_i > y)\ dy = \frac{E[(L_i - t)^+]}{E[L_i]}, \quad \mu_i = E[L_i] > 0.$$

B.1 CRE–ES Correspondence

Proposition B.1 (Parametric correspondence in the continuous case).
Let L have a continuous distribution, set $t = q_L(\alpha) = VaR_\alpha(L), \mu = E[L]$. *Then*

$$ES_\alpha(L) = t + \frac{\mu}{1-\alpha} CRE(t), \qquad CRE(t) = \frac{1-\alpha}{\mu}\backslash big(ES_\alpha(L) - t\backslash big).$$

Proof. Using the "threshold–excess" representation,

$$ES_\alpha(L) = t + \frac{1}{1-\alpha} E[(L-t)^+], \quad t = q_L(\alpha).$$

Since $CRE(t) = \frac{1}{\mu} E[(L-t)^+]$, substitution yields the first identity; rearrange for the second. □

Note 1 (Set-valued correspondence for general distributions).
If F_L is discontinuous, the optimal threshold $t^\star \in \arg\min \Phi_\alpha$ lies in an interval of $\alpha - quantiles$:

$$ES_\alpha(L) = t^\star + \frac{1}{1-\alpha} E[(L - t^\star)^+], \quad t^\star \in [q_L^-(\alpha), q_L^+(\alpha)].$$

Then $CRE(t^\star) = E[(L - t^\star)^+]$ still satisfies the same affine relation with $ES_\alpha(L)$, but the mapping $(\alpha, t^\star)$ is many-to-one (see Appendix C).

B.2 Euler Decomposition and Aumann–Shapley Equivalence

Let

$$f(x) = ES_\alpha\left(\sum_{i=1}^n x_i L_i\right), \qquad x \in R_+^n.$$

Under continuous/differentiable conditions we show:
(i) Euler identity: $f(x) = \sum_i x_i\ \partial_{x_i} f(x)$;
(ii) A–S equivalence: $\varphi_i^{AS}(x) = x_i\ \partial_{x_i} f(x)$.

B.2.1 Euler Identity

Lemma B.2 (1-homogeneity). For any \lambda\geq0, f\left(\lambda x\right)=\lambda f\left(x\right).
Proof. ES satisfies $ES_\alpha(cZ) = c\, ES_\alpha(Z) for c \geq 0 c \geq 0$. Hence

$$f(\lambda x) = ES_\alpha\left(\sum_i \lambda x_i L_i\right) = ES_\alpha\left(\lambda \sum_i x_i L_i\right) = \lambda ES_\alpha\left(\sum_i x_i L_i\right) = \lambda f(x)$$

□

Proposition B.2 (Euler identity).
If f is differentiable at x, then

$$f(x) = \sum_{i=1}^{n} x_i \, \partial_{x_i} f(x).$$

Proof. By Euler's theorem for differentiable 1-homogeneous functions on R^n,

$$\frac{d}{d\lambda} f(\lambda x)|_{\lambda=1} = \sum_{i=1}^{n} \partial_{x_i} f(x)\, x_i = f(x).$$

□

Note 2: (Differentiability).
If the distribution of $L_{\text{port}}(x)$ varies continuously with x and ES is Gâteaux differentiable in x, the above holds; for non-smooth cases use subgradients (Appendix C).

B.2.2 Aumann–Shapley Equivalence

Define the Aumann–Shapley (A–S) allocation (in the differentiable case):

$$\varphi_i^{AS}(x) = \int_0^1 \partial_{x_i} f(\theta x)\, x_i \, d\theta.$$

Lemma B.3 (0-homogeneity of the gradient).
If f is differentiable and 1-homogeneous, then for any $\theta > 0, \nabla f(\theta x) = \nabla f(x)$*; equivalently,* $\partial_{x_i} f(\theta x) = \partial_{x_i} f(x)$ *for all i.*

Proof. From $f(\theta x) = \theta f(x)$, differentiate w.r.t. x_i and use the chain rule:

$$\partial_{x_i} f(\theta x) \cdot \theta = \theta \, \partial_{x_i} f(x) \Rightarrow \partial_{x_i} f(\theta x) = \partial_{x_i} f(x).$$

□

Proposition B.3 (A–S = Euler).
If f *is differentiable and 1-homogeneous,*

$$\varphi_i^{AS}(x) = \int_0^1 \partial_{x_i} f(\theta x)\, x_i \, d\theta = x_i \, \partial_{x_i} f(x).$$

Proof. By Lemma B.3, $\partial_{x_i} f(\theta x)$ is θ-independent; integrate over [0,1]. □

Conclusion.
Under ES and differentiability, A - S allocation ≡ Euler risk contribution, and by Proposition B.2, $\sum_i \varphi_i^{AS}(x) = f(x)$.

B.3 Uniqueness Under Denault's Axioms

We reduce Denault (2001)'s axiomatic capital-allocation framework to a unique cost-sharing problem for the convex, 1-homogeneous cost $f(x) = ES_\alpha\left(L_{\text{port}}(x)\right)$, and prove that the unique allocation satisfying the axioms is A–S/Euler.

B.3.1 Problem Setup and Axioms

Define $f: R_+^n \to R_+, f(x) = ES_\alpha(\sum_{i=1}^{n} x_i L_i)$,
which is convex, 1-homogeneous, monotone, and subadditive (by coherence of ES).
Let an allocation rule $A: R_+^n \to R_+^n, A(x) = \left(A_1(x), \dots, A_n(x)\right)$, satisfy Denault's axioms (phrased in cost-sharing terms):

- Efficiency (full allocation): $\sum_i A_i(x) = f(x)$.
- Riskless/dummy: If $x_i = 0$ or $L_i \equiv 0$, then $A_i(x) = 0$.
- Symmetry: If (x_i, L_i) and $\left(x_j, L_j\right)$ are indistinguishable in weight and law, then $A_i(x) = A_j(x)$.
- Consistency: Compatibility under aggregation and scaling (including $A(\lambda x) = \lambda A(x)$).
- No Undercut: For any $S \subset \{1, \dots, n\}, \sum_{i \in S} A_i(x) \le f(x^S)$, where x^S retains only coordinates in S.

B.3.2 A–S Satisfies Denault's Axioms

With

$$\varphi_i^{AS}(x) = \int_0^1 \partial_{x_i} f(\theta x)\, x_i\, d\theta,$$

and using convexity, 1-homogeneity, and differentiability (or Gâteaux differentiability), one verifies:

- Efficiency:
$$\sum_i \varphi_i^{AS}(x) = \int_0^1 \sum_i x_i\, \partial_{x_i} f(\theta x)\, d\theta = \int_0^1 f(\theta x)\ d\theta = f(x).$$
- Riskless: If $x_i = 0$, then $\varphi_i^{AS}(x) = 0$.
- Symmetry: Symmetric players have identical partials/path integrals.
- Consistency (incl. homogeneity): $\varphi^{AS}(\lambda x) = \lambda \varphi^{AS}(x)$ (0-homogeneous gradient and change of variables).
- No Undercut: For any S, $\sum_{i\in S} \varphi_i^{AS}(x) \le f(x^S)$ follows from convexity/subadditivity and comparison of path integrals (standard argument; see B.3.4 notes).

B.3.3 Uniqueness: Denault's Axioms ⇒ A–S (hence Euler)

For convex, 1-homogeneous f with f\left(0\right)=0, any allocation satisfying efficiency, symmetry, riskless, homogeneity, and consistency corresponds to integrating marginal prices along some "fair path" $\gamma: [0,1] \to R_+^n$ *from* 0 *to* x. *Path-independence (via symmetry/consistency) forces* $\gamma(\theta) = \theta x$ *(the ray). No undercut plus efficiency pins the marginal prices down to the true partial derivatives, hence A–S; with differentiability, A–S = Euler.*

- Step 1 (Marginal-pricing representation).
By efficiency and consistency (additivity/scalability), there exists a path γ with
$$A_i(x) = \int_0^1 x_i\ w_{i\backslash} big(\gamma(\theta)\backslash big)\, d\theta,$$
where $w(\cdot)$ is a "marginal price" vector; riskless and symmetry require ww to be determined by local properties of ff and invariant under symmetric coordinates.
- Step 2 (Symmetry & consistency ⇒ ray path).
Different paths would violate consistency/symmetry (same x but different allocations). Hence $\gamma(\theta) = \theta x$.
- Step 3 (No undercut ⇒ true partials).
For convex f, no undercut forces $w_i(\theta x)$ not to understate any directional marginal $\partial_{x_i} f(\theta x)$, while efficiency prevents overstatement; thus $w_i(\theta x) = \partial_{x_i} f(\theta x)$ a.e. $\theta \in [0,1]$, and therefore $A_i(x) = \varphi_i^{AS}(x)$.
- Step 4 (Equivalence with Euler).
By §B.2.2, if f is 1-homogeneous and differentiable, $\varphi_i^{AS}(x) = x_i\, \partial_{x_i} f(x)$, i.e., the unique allocation is Euler. □

Notes:

- Path independence. Under differentiability and standard symmetry of the Hessian, integrating along exchangeable coordinate orders yields identical allocations, aligning with consistency/symmetry.
- No undercut. For any S, convexity gives $f(x) \le f(x^S) + f\left(x^{S^c}\right)$; the sum of A–S contributions along the ray for S does not exceed the cost of moving S alone from 0 to x^S, hence $\sum_{i\in S} \varphi_i^{AS}(x) \le f(x^S)$.
- General (non-smooth) case. If f is only Gâteaux differentiable or merely subdifferentiable, define A–S using measurable selections of subgradients; uniqueness becomes "select the subgradient family consistent with the axioms" (see Appendix C).

B.3.5 Levels of Symmetry: Strong vs. Weak

"Symmetry" is central but differs across literatures:

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- Weak symmetry. If two units are identical in inputs (equal weights and homogeneous distributions), then
$$x_i = x_j, \quad L_i \overset{\text{def}}{=} L_j \implies A_i(x) = A_j(x).$$
This ensures equal treatment of homogeneous risks. Denault (2001) adopts this level.
- Strong symmetry. Full label-invariance: swapping indices i and j leaves the allocation vector invariant. This stricter condition underpins uniqueness in Shapley (1953) cooperative games.
- Comparison. Mathematically, strong symmetry ⇒ weak symmetry (not conversely). Shapley's uniqueness relies on strong symmetry; Denault's regulatory relevance is captured by weak symmetry—appropriate for finance/insurance practice where "like-for-like" fairness is required without strict label-indifference.

Appendix B Summary

- B.1: In the continuous case, ES and functional CRE are affinely linked via $t = VaR_\alpha$; for general distributions the link becomes set-valued (Appendix C).
- B.2: For $f(x) = ES_\alpha\left(L_{\text{port}}(x)\right)$, 1-homogeneity + differentiability ⇒ Euler identity and A–S ≡ Euler.
- B.3: Under Denault's axioms, the unique allocation is A–S, hence (under differentiability) Euler.

Appendix C: Treatment of Non-Smooth Settings

In practical risk management, loss distributions often contain atoms (e.g., clustered defaults), so ES need not be differentiable everywhere. The conclusion of analytical uniqueness must then be adapted using subgradients from convex analysis.

C.1 Subgradient Sets

From the RU optimization representation,
$$ES_\alpha(L) = \min_{s \in R}\left\{s + \frac{1}{1-\alpha} E[(L-s)^+]\right\},$$
at the optimum, derivatives of ES with respect to portfolio weights may not be single-valued, but belong to a subgradient set:
$$\partial_{x_i} ES_\alpha(L) \in \text{SubGrad}_{x_i} ES_\alpha(L).$$
Thus capital allocation may form a family of marginal allocations, not a singleton.

C.2 Analytical Meaning

Under continuous differentiability, $\partial_{x_i} ES_\alpha(L)$ exists uniquely and Euler's principle yields a unique allocation. In non-smooth cases, derivatives become subgradients, and marginal allocations become set-valued.

C.3 Axiomatic Meaning

Within Denault's five-axiom framework (full allocation, no undercut, riskless/dummy, symmetry, consistency), one can select from the subgradient set the allocation consistent with Euler/A–S, thereby preserving institutional uniqueness. Hence even when analytical uniqueness fails, axiomatic (normative) uniqueness can be maintained.

C.4 Propositions and Remarks

Proposition 1 (Closure in non-smooth settings).
With atoms or empirical distributions, analytical uniqueness becomes set-valued, but within Denault's axioms a normative selection ensures consistent, institutionally usable allocations.

Proposition 2 (Subgradient characterization under non-smoothness).
Let $L_{\text{port}}(x) = \sum_{i=1}^{n} x_i L_i$ and $\rho_\alpha = ES$. If the loss distribution has atoms/discontinuities, then ρ_α may be non-differentiable in x. Being convex, it possesses at each x a nonempty subgradient set $\partial\rho_\alpha(x)$; for any unit i,

$$\partial_{x_i} ES_\alpha(L(x)) \in \text{SubGrad}_{x_i} ES_\alpha(L(x)).$$

Interpretation:

- Mathematical: Smooth ⇒ unique derivative ⇒ unique Euler allocation; non-smooth ⇒ subgradient family ⇒ multiple analytical solutions.
- Allocation: Analytically, capital attribution depends on subgradient selection; axiomatically, Denault's framework picks a unique normative allocation (e.g., one consistent with Euler/A–S).
- Practice: Even with jumps/discreteness, ES provides a closed, feasible allocation set. Using the axiomatic framework with a normative choice—e.g., portfolio-based CRE (incremental ES)—avoids arbitrariness and preserves consistency and interpretability.

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