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Analysis of Hybrid Fractional Differential Inclusion with Impulses in Ordered Banach Algebras

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Abstract: In this article, we are devoted to investigating a class of fractional differential inclusions with impulses in a concrete ordered Banach algebra. The existence results of solutions for the considered problem are derived by applying related hybrid fixed point theorem of multi-valued maps. A simple example is provided to illustrate and validate our proposed results.

Keywords: impulsive fractional differential inclusion; ordered space; hybrid fixed point theorem; existence of solutions

MSC: 34A08; 34B15; 34A60

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1. Introduction

Nowadays, many phenomena and problems in real life are presented or portrayed through a large number of meaningful mathematical concepts. Differential equation, has been practically utilized in establishing mathematical models, which can be used as a powerful mathematical tool for a more scientific and comprehensive understanding of nature. See, for instance, [1–3] for applications tied to models from mathematical biology and physics. The theory of solvability is one of the most basic and popular research directions in practical application. For example, researchers focus on the existence and uniqueness of solutions to differential equations, we can refer the interested readers to [4–6] for the recent developments. However, differential inclusion system, as a natural extension of differential equations, serve as important mathematical tools in describing some uncertainties under complex systems, such as optimal control theory, dynamical systems and stochastic processes. In a mathematical sense, differential inclusions often involve more functions and their derivatives. Moreover, the uncertain features of differential inclusions system are represented by set-value maps. Over the past few decades, numerous techniques and conclusions concerning the solvability of differential inclusion systems with boundary value problems have already been obtained.

Fractional differential equations arise in many fields of engineering and applied science, such as medical, chemistry, aerodynamics, blood flow phenomena and environment etc. Here, we can refer to some monographs [7–10] as well as recent research papers [11–15]. Correspondingly, in recent years, a lot of work has been published on fractional differential inclusions, and some existence results and other properties have been proved by applying the different fixed point theorems. From what we have observed, it is worth our attention that most studies on fractional differentiation and integration problems use mostly Riemann-Liouville and Caputo types, see [16–18] for instance.

The impulsive differential equations were first proposed and studied by Milman and Myshkis. These differential systems with impulse effects often occur in shocks, heart throbs, seasonal changes or harvesting in environmental sciences, abrupt changes of prices in economics, and Catastrophic events, and so on. Recently, fractional differential equations with impulse effects have attracted widespread attention and heated discussions among several scholars, see for instance [19–22].

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As we all know, it is very convenient and important to rely on fixed point theorems to solve the existence of solutions to nonlinear systems. Additionally, fixed point theory is a good help for researchers to study the existence and uniqueness results for differential equations and inclusions, see [6,20]. Recently, many researchers have proposed several hybrid point theorems to broaden and enrich the theories. For more details are found in Dhage [25,26], Lakshmikantham [27], Joshi [28].

In this article, we are interested in hybrid differential inclusions. For the study of hybrid systems, we can refer to see [32–35] and the references contained therein. The main objective of this work is to obtain an existence result for a class of fractional hybrid differential inclusion with impulses in ordered Banach algebras:

$$\begin{cases} {}^cD^\alpha\left(\frac{\vartheta(t)-h(t,\vartheta(t))}{f(t,\vartheta(t))}\right) \in \mathcal{Q}(t,\vartheta(t)), \quad t\in J', \\ \Delta\vartheta(t_k) = I_k(\vartheta(t_k)), \quad k=1,...,m, \\ \vartheta(0)=\vartheta_0, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ denotes the Caputo-type fractional derivative of order α, $0<\alpha\le 1$, $h\in C(J\times\mathbb{R},\mathbb{R}^+)$, $f\in C(J\times\mathbb{R},\mathbb{R}^+\setminus\{0\})$, $\mathcal{Q}: J\times\mathbb{R}\to 2^{\mathbb{R}^+}\setminus\{\emptyset\}$ is a multi−valued map, $I_k\in C(\mathbb{R},\mathbb{R})$, $k=1,...,m$, $J=[0,1]$, $0=t_0<t_1<...<t_k<...<t_m<t_{m+1}=1$, $J'=J\setminus\{t_1,...,t_m\}$, $J_0=[0,t_1]$, $J_1=(t_1,t_2]$,..., $J_m=(t_m,1]$, $\Delta\vartheta(t_k)=\vartheta(t_k^+)-\vartheta(t_k^-)$, $\vartheta(t_k^-)=\vartheta(t_k)$, $k=1,...,m$. We consider the existence of solutions for our problem (1.1) by applying famous fixed point theory together some basic concepts on multi-valued maps. The innovation of this work are listed:

1. The fractional differential inclusion with impulses in ordered Banach algebras has been taken into account.

2. We make use of a fixed point theorem of multi-valued maps for three operators in ordered Banach algebras combined with some facts of fractional calculus, and set-valued maps in order to ensure the existence result.

3. An application, illustrating the obtained abstract theory, is also dedicated.

4. By comparing with previous works, we expand the results to fractional differential inclusion problems.

For the rest of this article, we arrange as follows. In Section 2, review some fundamental concepts, as well as definitions and lemmas that will be utilized throughout the latter part of this study. In Section 3, we develop the theoretical results of the initial value problem (1.1) via applying hybrid fixed point theorem of multi-valued map and a complete proof is given at the same time. In Section 4, an example is given to support our theoretical result. Finally, in Section 5, we conclude with a comprehensive description of the findings that are shown.

2. Preliminaries

This section serves to recall some necessary definitions and properties of fractional calculus, as well as some set-valued analysis theories. Meanwhile, we present a series of lemmas, and relative theories within the context of linear ordered spaces, with the aim of providing a foundation for our subsequent findings.

Let $\mathcal{X}$ denote a Banach space with norm $\|\cdot\|$, $\mathscr{P}(\mathcal{X})$ denote the collection of all subsets of $\mathcal{X}$ and the notation $\mathscr{P}_p$ will be used to represent the class of all non-empty subsets of $\mathcal{X}$ that possess property p. Thus, we have

♢ $\mathscr{P}_0(\mathcal{X})=\{U\in\mathscr{P}(\mathcal{X}) : U\neq\emptyset\}$,
♢ $\mathscr{P}_{cl}(\mathcal{X})=\{U\in\mathscr{P}_0(\mathcal{X}) : U \, is \, closed\}$,
♢ $\mathscr{P}_{bd}(\mathcal{X})=\{U\in\mathscr{P}_0(\mathcal{X}) : U \, is \, bouned\}$,
♢ $\mathscr{P}_{cp}(\mathcal{X})=\{U\in\mathscr{P}_0(\mathcal{X}) : U \, is \, compact\}$,
♢ $\mathscr{P}_{bd,cl}(\mathcal{X})=\{U\in\mathscr{P}_0(\mathcal{X}) : U \, is \, closed \, and \, bouned\}$,
♢ $\mathscr{P}_{cp,cv}(\mathcal{X})=\{U\in\mathscr{P}_0(\mathcal{X}) : U \, is \, compact \, and \, convex\}$.

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2.1. Basic Material for Fractional Calculus

Before starting our work, we present some definitions and auxiliary lemmas of fractional calculus which are necessary for the proof of our main results.

Definition 2.1 (See [29]). *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *for a function* $\hbar : [0, +\infty) \to \mathbb{R}$ *is expressed as*

$$I_{0^+}^{\alpha}\hbar(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\hbar(s)ds, \quad t > 0,$$

provided that such integral exists and $\Gamma(\alpha)$ *is the Gamma function* $\Gamma : (0, \infty) \to \mathbb{R}$*, defined by*

$$\Gamma(u) = \int_0^{\infty} (t-s)^{u-1}e^{-t}dt.$$

Definition 2.2 (See [29]). *The Caputo fractional derivative of order* $\alpha > 0$ *for a function* $\hbar \in C^n[0, 1]$ *is defined by*

$$^cD^{\alpha}\hbar(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1}\hbar^{(n)}(s)ds,$$

$n - 1 < \alpha < n$, $n = [\alpha] + 1$.

Lemma 2.1 (See [29]). *Let* $\alpha > 0$*. Then the solution of the fractional differential equation*

$$^cD^{\alpha}\hbar(t) = 0$$

is given by

$$\hbar(t) = c_0 + c_1t + c_2t^2 + ... + c_{n-1}t^{n-1},$$

$c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n$, $n = [\alpha] + 1$.

2.2. Basic Materials for Ordered Banach Spaces

Some preliminary materials about ordered Banach spaces are introduced. More knowledge about the cone and its properties appears in [23] and [24].

Let $\mathbb{R}$ denote a real line and $\mathcal{X}$ be real Banach space. Given that $\mathcal{P}$ is a non-empty convex closed set in $\mathcal{X}$, the conditions that are satisfied are as follows:

(i) $\mathcal{P} + \mathcal{P} \subseteq \mathcal{P}$,
(ii) $\lambda\mathcal{P} \subseteq \mathcal{P}$ for all $\lambda \in \mathbb{R}^+$,
(iii) $\{-\mathcal{P}\} \bigcap \mathcal{P} = \{\theta\}$, where θ is a zero element of $\mathcal{X}$.

It is then asserted that $\mathcal{P}$ is a cone in $\mathcal{X}$.

After giving the definition of a cone, we now review some important concepts related to cones.

- If the norm is semi-monotone on $\mathcal{X}$, i.e. if $\vartheta, \zeta \in \mathcal{X}$, and $\vartheta \leq \zeta$ yields $\|\vartheta\| \leq N\|\zeta\|$, where $N > 0$ is a constant. Then, we denote the cone $\mathcal{P}$ in $\mathcal{X}$ as normal.
- If every sequence in $\mathcal{X}$ that is monotone and order-bounded is convergent with respect to the norm. It then follows that the cone $\mathcal{P}$ is regular.
- if $\mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P}$. Then, the cone $\mathcal{P}$ in a Banach algebra $\mathcal{X}$ is known as a positive cone. (The symbol "$\circ$" corresponds to a multiplicative composition in the space $\mathcal{X}$).

Remark 2.1. *It is well-established that if the cone* $\mathcal{P}$ *is normal, then it follows that every order-bounded set is bounded in norm.*

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Next, in order to define an order relation in $\mathcal{X}$, we depend on the cone $\mathcal{P}$. The following is the order relation defined in $\mathcal{X}$. Let $\vartheta, \zeta \in \mathcal{X}$, then

$$\vartheta \preceq \zeta \Leftrightarrow \zeta - \vartheta \in \mathcal{P}.$$

The Banach space $\mathcal{X}$, when considered in combination with the order relation $\preceq$, is referred to as an ordered Banach space. This new space is expressed as $(\mathcal{X}, \preceq)$. Let $p, q \in (\mathcal{X}, \preceq)$ be such that $p \preceq q$, then the order interval $[p, q]$ is a set in $\mathcal{X}$ to be defined by

$$[p, q] = \{\vartheta \in \mathcal{X} | p \preceq \vartheta \preceq q\}.$$

In what follows, we define the order relations of the different categories in $\mathscr{P}_p(\mathcal{P})$. Let $M, N \in \mathscr{P}_p(\mathcal{P})$. Then

$$M \overset{i}{\preceq} N \Leftrightarrow for\, each\, m \in M, \exists\, one\, n \in N\, such\, that\, m \preceq n,$$

$$M \overset{d}{\preceq} N \Leftrightarrow for\, each\, n' \in N, \exists\, one\, m' \in M\, such\, that\, m' \preceq n',$$

$$M \overset{id}{\preceq} N \Leftrightarrow M \overset{i}{\preceq} N\, and\, M \overset{d}{\preceq} N,$$

$$M \preceq N \Leftrightarrow m \preceq n\, for\, all\, m \in M\, and\, n \in N.$$

2.3. Basic Material for Multi-Valued Maps

Here we outline some basic concepts of multi-valued analysis. For more information about multi-functions, see the book of Deimling [37].

For a normed space $(\mathcal{X}, \|\cdot\|)$, we give some basic definitions of the multi-valued map $Q : \mathcal{X} \to \mathscr{P}_p(\mathcal{X})$:

(i) Q is convex (closed) valued if $Q(u)$ is convex (closed) for all $u \in \mathcal{X}$;

(ii) Q is bounded on bounded sets if $Q(V) = \bigcup_{v \in V} Q(v)$ is bounded in $\mathcal{X}$ for all $V \in \mathscr{P}_b(\mathcal{X})$ (i.e. $\sup_{v \in V}\{\sup\{|v| : v \in Q(v)\}\} < \infty$));

(iii) Q is called to be compact if $Q(\mathfrak{B})$ is relatively compact for every $\mathfrak{B} \in \mathscr{P}_b(\mathcal{X})$;

(iv) Q is called to be completely continuous if $Q(\mathfrak{B})$ is relatively compact for every $\mathfrak{B} \in \mathscr{P}_b(\mathcal{X})$;

(v) Q is said to be measurable if for every $u \in \mathcal{X}$, the function

$$t \mapsto d(u, Q(t)) = \inf\{|u - v| : v \in Q(t)\}$$

is measurable;

(vi) Q has a fixed point if there is $\bar{\vartheta} \in \mathcal{X}$ such that $\bar{\vartheta} \in Q(\bar{\vartheta})$. The set of all fixed points for the multi-valued operator Q will be written as $FixQ$.

Definition 2.3 ([30]). *A multi−valued map $Q : \mathcal{X} \to \mathscr{P}_p(\mathcal{X})$ is called right monotone increasing if $\vartheta \preceq \zeta$, then $Q\vartheta \overset{i}{\preceq} Q\zeta$ for all $\vartheta, \zeta \in X$. Similarly, Q is called left monotone increasing if $\vartheta \preceq \zeta$, then $Q\vartheta \overset{d}{\preceq} Q\zeta$ for all $\vartheta, \zeta \in \mathcal{X}$. Finally, the multi−valued map Q is simply called monotone increasing if $\vartheta \preceq \zeta$, then $Q\vartheta \overset{id}{\preceq} Q\zeta$ for all $\vartheta, \zeta \in \mathcal{X}$.*

Let $\mathcal{X}$ be an ordered Banach algebra, then for any $M, N \in \mathscr{P}_p(\mathcal{X})$, we denote

$$MN = \{mn \in \mathcal{X} | m \in M\, and\, n \in N\}.$$

Lemma 2.2 ([30]). *Let $[p, q]$ be an order interval in a subset Y of an ordered Banach algebra $\mathcal{X}$. Let $A, C : [p, q] \to \mathcal{X}$ and $B : [p, q] \to \mathscr{P}_{cp}(\mathcal{X})$ be three maps satisfying*

(a) A is compact and monotone increasing,

(b) B is completely continuous,

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(c) *C is a contraction, where the contraction constant* $k < 1/2$, *and*

(d) $AxBy + Cz \subset \mathscr{P}_{cv}([p,q])$ *for all* $x, y, z \in [p,q]$.

Further, we observe that if the cone $\mathscr{P}$ *in* $\mathcal{X}$ *is normal, then the operator inclusion* $x \in AxBx + Cx$ *has a solution in the interval* $[p,q]$.

In this article, we define the space $PC(J,\mathbb{R})$ on J , which is expressed as

$$PC(J,\mathbb{R}) := \{\vartheta : J \to \mathbb{R} \mid \vartheta \in C(J_k), \vartheta(t_k^+), \vartheta(t_k^-) \, exist, \, \vartheta(t_k^-) = \vartheta(t_k), \, k = 0,1,...,m\},$$

and $(PC(J,\mathbb{R}), \|\cdot\|)$ is a Banach space endowed with the norm $\|\cdot\|$, defined as follows:

$$\|\vartheta\| = \sup_{t\in J} |\vartheta(t)|. \tag{2.1}$$

We define the order cone $\mathcal{P}$ in $PC(J,\mathbb{R})$ by

$$\mathcal{P} = \{\vartheta \in PC(J,\mathbb{R}) \mid \vartheta(t) \geq 0 \, for \, all \, t \in J\}. \tag{2.2}$$

It is evident that cone $\mathcal{P}$ is normal cone in $PC(J,\mathbb{R})$. Furthermore, we also define an order relation $\preceq$ in $PC(J,\mathbb{R})$ by

$$\vartheta \preceq \zeta \quad \Leftrightarrow \quad \vartheta(t) \leq \zeta(t), \quad t \in J. \tag{2.3}$$

Definition 2.4 (See [36]). *The set* ϖ *is defined as quasi-equicontinuous in* $[0,r]$ *if for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that if* $u \in \varpi$, $k \in \mathbb{Z}^+$, $t', t'' \in (t_{k-1}, t_k] \bigcap [0,r]$, *and* $|t' - t''| < \delta$, *then* $|u(t') - u(t'')| < \varepsilon$.

Lemma 2.3 (Compactness criterion, See [36]). *The set* $\varpi \subset PC([0,r],\mathbb{R}^n)(r < +\infty)$ *is relatively compact if and only if the following two conditions are satisfied:*

(i) ϖ *is bounded, and*

(ii) ϖ *is quasi-equicontinuous in* $[0,r]$.

3. Main Results

We try to find the existence of solutions for the inclusion problem (1.1).

Lemma 3.1 (See [31]). *Let us consider the following impulsive hybrid fractional differential equation:*

$$\begin{cases} {}^cD^\alpha\left(\frac{\vartheta(t)-h(t,\vartheta(t))}{f(t,\vartheta(t))}\right) = v(t), \quad t \in J', \\ \Delta\vartheta(t_k) = I_k(\vartheta(t_k)), \quad k = 1,...,m, \\ \vartheta(0) = \vartheta_0. \end{cases} \tag{3.1}$$

We will let the solution of this equation be denoted by the function $\vartheta \in C^n[0,1]$, *then it is shown to satisfy the following impulsive hybrid fractional integral equation:*

$$\begin{aligned} \vartheta(t) =& f(t,\vartheta(t))\left(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)ds \right. \\ &\left. + \sum_{i=1}^{k}\frac{I_i(\vartheta(t_i))}{\mathrm{f}(t_i,\vartheta(t_i))} + \frac{\vartheta_0 - \hat{\mathrm{h}}}{\hat{\mathrm{f}}}\right) + h(t,\vartheta(t)), \quad t \in J_k, \, k = 0,1,...,m. \end{aligned}$$

Remark 3.1. *For the convenience of presentation, we use notation* $h(0,0) := \hat{\mathrm{h}}$, $f(0,0) := \hat{\mathrm{f}}$.

Definition 3.1. *We define a solution of (1.1) as function* $\vartheta \in PC(J,\mathbb{R})$, *and the function* ϑ *satisfies the following:*

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(i) there exists a $v \in L^1(J,\mathbb{R})$ *with* $v(t) \in \mathcal{Q}(t,\vartheta(t))$ *a.e.t* $\in J$ *satisfying* ${}^cD^\alpha\left(\frac{\vartheta(t)-h(t,\vartheta(t))}{f(t,\vartheta(t))}\right) = v(t), \quad t \in J'$;

(ii) $\Delta\vartheta(t_k) = I_k(\vartheta(t_k)), k = 1,...,m,$ *and* $\vartheta(0) = \vartheta_0$.

Definition 3.2. *A function* $p \in PC(J,\mathbb{R})$ *is defined as a strictly lower solution of (1.1) if for all* $\nu \in S_{\mathcal{Q},p}$, *the following conditions are satisfied*

(i) ${}^cD^\alpha\left(\frac{p(t)-h(t,p(t))}{f(t,p(t))}\right) \le \nu(t)$ *for all* $t \in J'$;

(ii) $\Delta p(t_k) = I_k(p(t_k)), k = 1,...,m,$ *and* $p(0) = \vartheta_0$.

Similarly, a function $q \in PC(J,\mathbb{R})$ *is defined as a strictly upper solution of (1.1) if for all* $\mu \in S_{\mathcal{Q},q}$, *the following are satisfied*

(i) ${}^cD^\alpha\left(\frac{q(t)-h(t,q(t))}{f(t,q(t))}\right) \ge \mu(t)$ *for all* $t \in J'$;

(ii) $\Delta q(t_k) = I_k(q(t_k)), k = 1,...,m,$ *and* $q(0) = \vartheta_0$.

Remark 3.2. *For any one of multi-valued maps* $\mathcal{Q} : J \times \mathbb{R} \to \mathscr{P}_p(\mathbb{R})$, *we can define*

$$S_{\mathcal{Q},P} = \{v \in L^1(J,\mathbb{R});\, v(t) \in \mathcal{Q}(t,p(t))\; a.e.t \in J\}.$$

Here we give the following hypotheses $(H1)-(H6)$ to better present our major conclusions of the inclusion problem (1.1):

$(H1)$ $f \in C(J \times \mathbb{R}, \mathbb{R}^+ \setminus \{0\})$, and $f(t,x)$ is monotone increasing in x almost everywhere for $t \in J$. a.e.

$$f(t,x) \le f(t,y),$$

where $t \in J, x,y \in \mathbb{R}, x \preceq y$.

$(H2)$ $h \in C(J \times \mathbb{R}, \mathbb{R}^+)$, $h(t,x)$ is monotone increasing in x almost everywhere for $t \in J$ and there exists a bounded function $l : J \to \mathbb{R}$ with bound $\| l \|$ such that

$$|h(t,x) - h(t,y)| \le l(t)|x-y|,$$

where $t \in J, x,y \in \mathbb{R}$.

$(H3)$ $I_k(\vartheta)/f(t,\vartheta)$ is non-decreasing in ϑ for every $t \in J$, and there is a constant G satisfying

$$0 \le \frac{I_k(\vartheta)}{f(t,\vartheta)} \le G,$$

where $t \in J$, $x \in \mathbb{R}$, $k = 1,...,m$.

$(H4)$ The multi-valued map $\mathcal{Q} : J \times \mathbb{R} \to \mathscr{P}_{cp,cv}(\mathbb{R}^+)$, and $\mathcal{Q}(t,\vartheta)$ is right monotone increasing in ϑ almost everywhere for $t \in J$.

$(H5)$ The multi-valued function $\mathcal{Q}(t,\vartheta)$ is integrably bounded, that is, there exists $\phi \in L^1(\mathbb{R})$, for $v \in S_{\mathcal{Q},w}$, we have

$$|v(t)| \le \phi(t), \text{ for almost } t \in J,$$

where

$$S_{\mathcal{Q},w} = \{v \in L^1([0,1],\mathbb{R}); v(t) \in \mathcal{Q}(t,w(t)) \text{ for a.e } 0 \le t \le 1\}.$$

$(H6)$ (1.1) has a strictly lower solution p and a strictly upper solution q on J with $p \le q$.

Theorem 3.1. *Assume that the hypotheses* $(H1)-(H6)$ *hold. Further if* $\| l \| < \frac{1}{2}$, *then* (1.1) *has at least a solution in* $[p,q]$.

Proof. Now, let's start by defining $\mathcal{X} = PC(J,\mathbb{R})$, and defining a norm $\| \cdot \|$, a cone $\mathcal{P}$ and an order relation $\preceq$ in $\mathcal{X}$ by (2.1), (2.2) and (2.3), respectively. According to this multiplication "$\cdot$", where is defined as: $(x,y)(t) = x(t)y(t)$ for $t \in J$, at this point, $\mathcal{X}$ is an ordered Banach algebra.

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As a direct consequence of hypothesis $(H6)$, it can be demonstrated that there exists an order interval $[p,q]$ in $\mathcal{X}$. In subsequent proofs, we shall consider the order interval $[p,q]$ in $\mathcal{X}$. Define three operators $A:[p,q]\to\mathcal{X}$, $B:[p,q]\to\mathscr{P}_p(\mathcal{X})$ and $C:[p,q]\to\mathcal{X}$ by

$$A\vartheta(t)=\{f(t,\vartheta(t))\},\quad t\in J_k, k=0,1,...,m. \tag{3.1}$$

$$\begin{aligned}B\vartheta(t)=&\Bigg\{u(t):u(t)=\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)ds\\&+\sum_{i=1}^{k}\frac{I_i(\vartheta(t_i))}{\mathrm{f}(t_i,\vartheta(t_i))}+\frac{\vartheta_0-\hat{\mathrm{h}}}{\hat{\mathrm{f}}},\ v\in S_{\mathcal{Q},\vartheta}\Bigg\},\quad t\in J_k, k=0,1,...,m.\end{aligned} \tag{3.2}$$

$$C\vartheta(t)=\{h(t,\vartheta(t))\},\quad t\in J_k, k=0,1,...,m. \tag{3.3}$$

In the following, we shall prove that the maps A, B and C satisfy all the conditions of Lemma 2.2. on $[p,q]$. To make it easier to read, we break down the proof process into the following steps.

Step I: First we show that $AxBy+Cz$ is a convex subset of $[p,q]$ for each $x,y,z\in[p,q]$.

Let $u_1,u_2\in AxBy+Cz$. Then there are $v_1,v_2\in S_{\mathcal{Q},y}$ such that

$$\begin{aligned}u_1=&f(t,x(t))\Bigg(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_1(s)ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_1(s)ds\\&+\sum_{i=1}^{k}\frac{I_i(y(t_i))}{\mathrm{f}(t_i,y(t_i))}+\frac{y_0-\hat{\mathrm{h}}}{\hat{\mathrm{f}}}\Bigg)+h(t,z(t)),\quad t\in J_k,\ k=0,1,...,m,\end{aligned}$$

and

$$\begin{aligned}u_2=&f(t,x(t))\Bigg(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_2(s)ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_2(s)ds\\&+\sum_{i=1}^{k}\frac{I_i(y(t_i))}{\mathrm{f}(t_i,y(t_i))}+\frac{y_0-\hat{\mathrm{h}}}{\hat{\mathrm{f}}}\Bigg)+h(t,z(t)),\quad t\in J_k,\ k=0,1,...,m.\end{aligned}$$

Now for any $\rho\in[0,1]$,

$$\begin{aligned}&\rho u_1(t)+(1-\rho)u_2(t)\\=&f(t,x(t))\Bigg(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}(\rho v_1(s)+(1-\rho)v_2(s))ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(\rho v_1(s)+(1-\rho)v_2(s))ds\\&+\sum_{i=1}^{k}\frac{I_i(y(t_i))}{\mathrm{f}(t_i,y(t_i))}+\frac{y_0-\hat{\mathrm{h}}}{\hat{\mathrm{f}}}\Bigg)+h(t,z(t)).\quad t\in J_k,\ k=0,1,...,m.\end{aligned}$$

Since $\mathcal{Q}(t,y(t))$ is convex, $\rho v_1(t)+(1-\rho)v_2(t)\in\mathcal{Q}(t,y(t))$ for all J and so $\rho v_1+(1-\rho)v_2\in S_{\mathcal{Q},y}$. As a result $\rho u_1+(1-\rho)u_2\in AxBy+Cz$. Hence, $AxBy+Cz$ is a convex subset of $\mathcal{X}$.

Step II: Next we show that A, B and C are monotone increasing and $AxBy+Cz\subset[p,q]$ for all x, $y,z\in[p,q]$.

Let $x,y\in[p,q]$ be such that $x\preceq y$. Then by $(H1)$,

$$Ax(t)=f(t,x(t))\leq f(t,y(t))=Ay(t)$$

for all $t\in J$. Hence $Ax\preceq Ay$. Similarly, by $(H2)$ have

$$Cx(t)=h(t,x(t))\leq h(t,y(t))=Cy(t)$$

for all $t\in J$. So, $Cx\preceq Cy$.

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Let $\flat_1 \in Bx$. Then there is a $\kappa_1 \in S_{\mathcal{Q},x}$ such that

$$\begin{aligned}\flat_1(t) =&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\kappa_1(s)ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\kappa_1(s)ds\\&+\sum_{i=1}^{k}\frac{I_i(x(t_i))}{\mathfrak{f}(t_i,x(t_i))}+\frac{x_0-\hat{\mathfrak{h}}}{\hat{\mathfrak{f}}},\quad t\in J_k,\ k=0,1,...m.\end{aligned}$$

Since $\mathcal{Q}(t,\vartheta)$ is right monotone increasing in $\mathcal{X}$, if $x \overset{i}{\preceq} y$, we have $S_{\mathcal{Q}.x} \overset{i}{\preceq} S_{\mathcal{Q}.y}$. Thus, we can find a $\kappa_2 \in S_{\mathcal{Q},y}$ such that $\kappa_1 \preceq \kappa_2$ on J . Therefore, we have

$$\begin{aligned}\flat_1(t) =&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\kappa_1(s)ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\kappa_1(s)ds\\&+\sum_{i=1}^{k}\frac{I_i(x(t_i))}{\mathfrak{f}(t_i,x(t_i))}+\frac{x_0-\hat{\mathfrak{h}}}{\hat{\mathfrak{f}}}\\\preceq&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\kappa_2(s)ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\kappa_2(s)ds\\&+\sum_{i=1}^{k}\frac{I_i(y(t_i))}{\mathfrak{f}(t_i,y(t_i))}+\frac{x_0-\hat{\mathfrak{h}}}{\hat{\mathfrak{f}}}\\&=\flat_2(t).\end{aligned}$$

for all $t \in J_k$, $k = 0,1,...m$, where $\flat_2 \in By$. So, $Bx \preceq By$. Thus, A, B and C are right monotone increasing on $[p,q]$.

By $(H6)$, $p \overset{d}{\preceq} ApBp + Cp$ and $AqBq + Cq \overset{i}{\preceq} q$, and because the cone $\mathcal{P}$ in $\mathcal{X}$ is positive. Based on these, conclusion $AxBy + Cz \subset [p,q]$ are drawn for all $x, y, z \in [p,q]$.

Step III: Next we show that A is completely continuous on $[p,q]$.

Now the cone $\mathcal{P}$ in $\mathcal{X}$ is normal, so the order interval $[p,q]$ is norm-bounded. Hence there exists a constant $L > 0$ such that $\|\vartheta\| \leq L$ for all $x \in [p,q]$. As f is continuous on compact $J_k \times [-L,L]$, $k = 0,1,2,...,m$, it attains its maximum, say M. Therefore, for any subset ϖ of $[p,q]$ we have

$$\begin{aligned}\|A(\varpi)\|_{\mathscr{P}} &= \sup\{\|A\vartheta\| : \vartheta\in\varpi\}\\&=\sup\left\{\sup_{t\in J_k}|f(t,\vartheta(t))| : \vartheta\in\varpi\right\}\\&\leq\sup\left\{\sup_{t\in J_k}|f(t,\vartheta)| : \vartheta\in[-L,L]\right\}\\&\leq M,\quad k=0,1,2,...,m.\end{aligned}$$

This shows that $A(\varpi)$ is a uniformly bounded subset of $\mathcal{X}$.

Next we note that the function $A(\varpi)$ is quasi-equicontinuous set on $\mathcal{X}$. For any $t^{'}$, $t^{''} \in J_k$, $k = 0,1,...,m$. We have

$$|f(t^{'},\vartheta)-f(t^{''},\vartheta)|\to 0\quad as\, t\to t^{''}$$

for all $\vartheta \in [-L,L]$. Similarly, for any $\vartheta, \zeta \in [-L,L]$

$$|f(t,\vartheta)-f(t,\zeta)|\to 0\quad as\,\vartheta\to\zeta$$

for all $t \in J_k$. Hence, for any $t^{'}, t^{''} \in J_k$ and for any $\vartheta \in \varpi$ one has

$$\begin{aligned}|A\vartheta(t^{'}) - A\vartheta(t^{''})| &= |f(t^{'}, \vartheta(t^{'})) - f(t^{''}, \vartheta(t^{''}))| \\ &\leq |f(t^{'}, \vartheta(t^{'})) - f(t^{''}, \vartheta(t^{'}))| + |f(t^{''}, \vartheta(t^{'})) - f(t^{''}, \vartheta(t^{''}))| \\ &\to 0 \quad as\, t^{'} \to t^{''}.\end{aligned}$$

This shows that $A(\varpi)$ is an quasi-equicontinuous set in $\mathcal{X}$. Now, $A(\varpi)$ is a uniformly bounded and quasi-equicontinuous set in $\mathcal{X}$. Hence, an application of Lemma 2.3 yields that A is a completely continuous operator on $[p, q]$.

Step IV: we show that B is completely continuous operator.

We show that B is compact multi-valued operator on $[p, q]$. To finish, we shall show that $B(\varpi)$ is a uniformly bounded and quasi-equicontinuous set in $\mathcal{X}$ for any subset ϖ of $[p, q]$.

Let $b \in B(\varpi)$ be arbitrary. Then there is an element $a \in \varpi$ and a $v \in S_{Q,a}$ such that

$$\begin{aligned}b(t) =& \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s) ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} v(s) ds \\ &+ \sum_{i=1}^{k} \frac{I_i(a(t_i))}{\mathfrak{f}(t_i, a(t_i))} + \frac{a_0 - \hat{\mathfrak{h}}}{\hat{\mathfrak{f}}}, \quad t \in J_k,\ k = 0, 1, ..., m.\end{aligned}$$

Now by hypotheses $(H3)$, $(H5)$ we have

$$\begin{aligned}|b(t)| =& \Bigg| \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s) ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} v(s) ds \\ &+ \sum_{i=1}^{k} \frac{I_i(a(t_i))}{\mathfrak{f}(t_i, a(t_i))} + \frac{a_0 - \hat{\mathfrak{h}}}{\hat{\mathfrak{f}}} \Bigg| \\ \leq& \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |v(s)| ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |v(s)| ds \\ &+ \sum_{i=1}^{k} \frac{|I_i(a(t_i))|}{|\mathfrak{f}(t_i, a(t_i))|} + \frac{|a_0 - \hat{\mathfrak{h}}|}{|\hat{\mathfrak{f}}|} \\ \leq& \frac{(m+1)}{\Gamma(\alpha+1)} \|\phi\|_{L^1} + (m+1)G + \frac{|a_0 - \hat{\mathfrak{h}}|}{|\hat{\mathfrak{f}}|} := K.\end{aligned}$$

where $t \in J_k, k = 0, 1, ..., m$. If we take the supremum about t, we get $\|b\| \leq K$. which show that $B(\varpi)$ is a uniform bounded set in $\mathcal{X}$.

Next, we prove that $B(\varpi)$ is a quasi-equicontinuous set in $\mathcal{X}$. Let $t, \tau \in (t_{k-1}, t_k] \cap [0, 1]$ be arbitrary, $k = 0, 1, ..., m$.

$$\begin{aligned}|b(t) - b(\tau)| =& \left| \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s) ds - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{\tau} (\tau - s)^{\alpha-1} v(s) ds \right| \\ \leq& \frac{\|\phi\|_{L^1}}{\Gamma(\alpha)} \left(\left| \int_{t_k}^{\tau} (t-s)^{\alpha-1} - (\tau-s)^{\alpha-1} ds \right| + \left| \int_{\tau}^{t} (t-s)^{\alpha-1} ds \right| \right) \\ =& \frac{\|\phi\|_{L^1}}{\Gamma(\alpha)} \left(\frac{|(t-s)^{\alpha} - (\tau-s)^{\alpha}|}{\alpha} \Bigg|_{t_k}^{\tau} + \frac{|(\tau-t)^{\alpha}|}{\alpha} \right).\end{aligned}$$

Now notice that $0 < \alpha \leq 1$, which means we will have

$$|(t-s)^{\alpha} - (\tau-s)^{\alpha}| \leq |(t-\tau)^{\alpha}|.$$

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So,

$$|b(t)-b(\tau)| \leq \frac{2|(t-\tau)^{\alpha}|}{\alpha} \to 0 \quad as\, t \to \tau.$$

Hence, $B(\varpi)$ is a quasi-equicontinuous subset of $\mathcal{X}$, Thus, B is compact in view of Lemma 2.3. Consequently, B is completely continuous operator.

Step V: Finally we show that the operator C is a contraction on $[p,q]$.

Let $x, y \in [p,q]$. Then by hypothesis $(H2)$,

$$\begin{aligned}
\|Cx - Cy\| &= \sup_{t\in J} |h(t,x(t)) - h(t,y(t))| \\
&\leq \sup_{t\in J} l(t)|x(t)-y(t)| \\
&\leq \|l\|\,\|x-y\|,
\end{aligned}$$

where $\|l\| < 1/2$. This shows that C is a contraction on $[p,q]$ with a contraction constant $\|l\| < 1/2$.

Now an application of Lemma 2.2 yields that the operator inclusion $\vartheta \in A\vartheta B\vartheta + C\vartheta$ has a solution $\bar{\vartheta}$, and consequently the (1.1) has a solution $\bar{\vartheta}$ in $[p,q]$. This completes the proof. □

4. An Example

In this section, we present one example to illustrate our main result.

We consider the problem of the following impulsive hybrid fractional differential inclusion:

$$\begin{cases} {}^{c}D^{1/2}\left(\frac{\vartheta(t)-h(t,\vartheta(t))}{f(t,\vartheta(t))}\right) \in \mathcal{Q}(t,\vartheta(t)), \quad t \in J := [0,1] \setminus \{t_1, t_2, t_3\}, \\ \Delta\vartheta(t_k) = I_k(\vartheta(t_k)), \quad k = 1,2,3, \\ \vartheta(0) = 1/15. \end{cases} \tag{4.1}$$

We consider $J = [0,1]$ and the points $t_1 = 1/4$, $t_2 = 1/2$, $t_3 = 3/4$. Furthermore, we represent the functions h, f, I_k as follows:

$$h(t,\vartheta) = \frac{1}{20\pi}(\sin\vartheta + \pi),$$

$$f(t,\vartheta) = \begin{cases} \frac{1}{2}, & \vartheta \leq 0, \\ \frac{1+\vartheta}{2}, & 0 < \vartheta < 1, \\ 1, & \vartheta \geq 1, \end{cases}$$

$$I_k(\vartheta) = \begin{cases} 0, & \vartheta \leq 0, \\ \frac{1}{(k+1)^2}\frac{\vartheta}{1+\vartheta}, & \vartheta \geq 0,\, k = 1,2,3, \end{cases}$$

and $\mathcal{Q} : [0,1] \times R \to 2^{\mathbb{R}^+} \setminus \{\emptyset\}$ is multi-valued map given by:

$$\mathcal{Q}(t,\vartheta(t)) = \left[0, \frac{1}{22}(\tanh\vartheta + 1)\right],$$

for arbitrary $\vartheta \in \mathbb{R}$, $t \in J$. Following a comprehensive analysis and calculation, it can be concluded that the following results are all in accordance with the conditions set as the theorem's criteria.

(i) $f(t,x) - f(t,y) \geq 0$ when $x \geq y$;

(ii) $|h(t,x) - h(t,y)| \leq \frac{1}{20\pi}|x-y|$;

(iii) $I_k(\vartheta)/f(t,\vartheta)$ is nondecreasing, and $0 \leq \frac{I_k(\vartheta)}{f(t,\vartheta)} \leq \frac{1}{4}$, $k = 1,2,3$;

(iv) multi-valued map $\mathcal{Q} : J \times \mathbb{R} \to \mathscr{P}_{cp,cv}(\mathbb{R}^+)$ and $\mathcal{Q}(t,\vartheta)$ is right monotone increasing;

(v) $0 \leq \mathcal{Q}(t,\vartheta) \leq \frac{1}{11}$, $t \in J$, $\vartheta \in \mathbb{R}$;

(vi) We can know that $p(t) = 0$, $t \in J$, is a strictly lower solution of IVP (4.1), and $q(t) = 1$, $t \in J$, is a strictly upper solution of IVP (4.1) by Definition 3.2.

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With the above analysis, we can take $l(t) = \frac{1}{20\pi}$, $G = \frac{1}{4}$, $\phi(t) = \frac{1}{11}$ in Theorem 3.1, and we can derive $\| l \| < \frac{1}{2}$. Therefore, all conditions of Theorem 3.1 are satisfied. Then we can conclude that (4.1) has a solution $\bar{\vartheta}$. □

5. Conclusion

In this paper, we mainly study a class of fractional hybrid differential inclusion with impulses in an ordered Banach algebra. The existence result of a solution for the initial value problem (1.1) has been obtained by applying related hybrid fixed point theorem of multi-valued map. Our work implies that fractional orders have some influence on nonlinear hybrid differential inclusions. In addition to this, our conclusions are influenced to some extent by the impulse effect. Furthermore, our results not only take advantage of relevant hybrid fixed point theorem, but also extend some of theoretical results of fractional order inclusions systems. These advantages and new issues that may arise will encourage us to continue to study more deeply in this direction.

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