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On Certain Concircular Flatness Conditions on Almost Cosymplectic Manifolds with Schouten-Van Kampen Connection

Snethemba H. Maduna * and Ntokozo Khuzwayo

School of Agriculture and Science, College of Agruculture, Engineering and Science, University of KwaZulu Natal, Private Bag X01, Scottsville, 3209

* Correspondence: smadunah@gmail.com/ncinoncino@gmail.com

Abstract

The objective of this paper is to study a Schouten-Van Kampen connection $\widehat{\nabla}$ on almost cosymplectic manifolds. We establish geometric properties of almost cosymplectic manifolds with respect to a $\widehat{\nabla}$ with flatness conditions to be ϕ- concircular and ξ-concircular, pseudo-concircular curvature tensor.

Keywords: almost cosymplectic manifold; schouten-van kampen connection; concircular flatness; scalar curvature; einstein; ricci curvature

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1. Introduction

In the framework of topological and geometric properties, Sasakian and f-Kenmotsu manifolds with respect to the Schouten-van Kampen connection have been extensively studied. Numerous scholars have recently focused on studying the geometric structure of Sasakian, Kenmotsu manifolds with a Schouten-Van Kampen connection (see[7],[11], [15], [21] and many more). This connection was first introduced in the early 1930's, by D. Van Kampen and T.A. Schouten[10] to study non-holomorphic manifolds, i.e, non-complex manifolds. The Schouten-Van Kampen connection maintains two complementary distributions on a differentiable manifold with an affine connection through parallelism. It is also used to study Riemannian manifold hyperdistributions [18]. This connection applied to an almost (para) contact metric structure was studied in 2014 by G. Olszak[15]. He stated the geometrical properties of the Schouten-Van Kampen connection and how they relate to the structure of almost (para) contact metric manifolds. Yildiz (2017) investigated this connection to f-Kenmotsu manifolds[21].

There are still unanswered questions and possible directions for research in the study of some classes of almost contact manifolds accepting Schouten-Van Kampen connection, despite the massive work that has been done already. Schouten-Van Kampen connections and their applicability to almost cosymplectic manifolds are a focus on this paper.

It is known that the form of the Riemannian manifolds with a concircular curvature tensor that vanishes is of constant curvature, hence the concircular curvature tensor is known to be a measure of the failure of a Riemannian manifold to be of constant curvature[20]. Introduced by K. Yano in the 1940's[20], concircular transformation has been studied by many authors. Its application is mainly in physics, specifically Einstein's theory of gravity.The concircular transformation preserves geodesic circles and the scalar curvature is constant in the concircular flat space-time[1]. For example, [6] proved that if M^n is concircular flat space-time satisfying $f(R)$ gravity, then ξ is Killing if and only if M^n admits matter collineation with respect to ξ.

This article is arranged as follows: In Section 1, we state the basic and relevant properties of almost cosymplectic manifolds that will be used throughout the paper. In Section 2, curvature relations of almost cosymplectic manifolds with Schouten-Van Kampen connection to the Levi-Civita coonection are derived. In Section 3, we derive the concircular curvature conditions with respect to Schouten-Van

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Kampen connection. We also consider a variety of concircular curvature transformations(ξ-concircular, pseudoconcircular, ϕ-concircular) to almost cosymplectic manifolds. Lastly, an example is added.

2. Preliminaries

Let (M, g, η, ξ, ϕ) be an almost contact metric smooth $(2p+1)$-dimensional manifold, with g, η, ξ, and ϕ representing a $(0,2)$-Riemannian metric tensor, 1-form, a vector field and $(1,1)$-tensor field, respectively[2], satisfying

$$\&\phi^2 D = -D + \eta(D)\xi, \tag{1}$$

$$\eta(D) = g(D, \xi), \tag{2}$$

$$\&\eta(\phi E) = 0, \tag{3}$$

$$\phi\xi = 0, \tag{4}$$

$$\&g(D, E) = g(\phi D, \phi E) + \eta(D)\eta(E), \tag{5}$$

for any vector field D, E.
A fundamental 2-form of (M, g, η, ξ, ϕ) is defined by

$$\Phi(D, E) = g(D, \phi E), \tag{6}$$

with a rank$(\phi) = 2p$.
(M, g, η, ξ, ϕ) is almost cosymplectic if the 1-forms η and the 2-form Φ are closed, i.e. $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differentiation operator [8]. A normal almost cosymplectic manifold (M, g, η, ξ, ϕ) is cosymplectic. The normality condition is defined by the vanishing of the torsion tensor field N_ϕ[2]. A mathematician I. Vaisman characterized the cosymplectic manifold (M, g, η, ξ, ϕ) by $\nabla\phi = 0$, where ∇ is the Levi-Civita(Riemannian) connection of (M, g, η, ξ, ϕ).

The Levi-Civita connection is given by the *Koszul formula*

$$\begin{aligned} 2g(\nabla_D Y, F)\& = \&D(g(E, F)) + E(g(F, D)) - F(g(D, E)) \\ \& \quad \& + g([D, E], F) + g([F, D], E) - g([E, F], D), \end{aligned} \tag{7}$$

for any vector fields D, E, F. On an almost cosymplectic manifold (M, ϕ, ξ, η, g) we have

$$2g((\nabla_D \phi)E, F) = g(N(E, F), \phi D), \tag{8}$$

for any vector fields D, E. In an affine connection, the Riemannian curvature tensor is defined by

$$R(D, E)F = \nabla_D \nabla_E F - \nabla_E \nabla_D F - \nabla_{[D,E]} F, \tag{9}$$

for any vector fields D, E, F.

The symmetric Ricci tensor (Ric) is a tensor field of type $(0,2)$ defined by [12]

$$Ric(E, F) = \text{trace}\{D \mapsto R(D, E)F\}. \tag{10}$$

Its components are given as

$$Ric\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \sum_i R^i_{kij}. \tag{11}$$

A well known example of an almost cosymplectic manifold is a product of an almost Kähler manifold and 1-dimensional manifold.

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In almost cosymplectic manifold, the following properties exist[13]:

$$\nabla_D\xi \& = -h'D, \tag{12}$$
$$g(\ell\phi, D) + g(\phi\ell, D) \& = -2g(\phi h^2, D), \tag{13}$$
$$g(\ell, D) \& = -g(h^2, D) + g(\phi\nabla_\xi h, D), \tag{14}$$
$$\text{trace}\,\ell \& = -\text{trace}(h^2) = constant, \tag{15}$$

for any vector field D. The ℓ represents a Jacobi operator with respect to the Reeb vector field ξ and $\mathfrak{R}$ represents the Riemannian curvature tensor.
The symmetric tensor $hD = \frac{1}{2}(\mathcal{L}_\xi\phi)E$, is the Blair tensor and its satisfy the following properties [15] :

$$g(h\xi, D) \& = 0 \tag{16}$$
$$g(h\phi D, E) \& = -g(\phi hD, E) \tag{17}$$
$$h'D \& = \phi hD \tag{18}$$
$$\text{trace}h \& = 0, \text{ and trace}h' = 0, \tag{19}$$

for any vector D, E, where $\mathcal{L}$ represents the Lie differentiation.

Also, in almost cosymplectic manifolds, the following relations exist [9]:

$$\&\mathfrak{R}(D,E)\xi = -\eta(D)\ell E + \eta(E)\ell D, \tag{20}$$
$$\eta(\mathfrak{R}(D,E)F) = \eta(D)g(\ell F, E) - \eta(E)g(\ell F, D), \tag{21}$$
$$(\nabla_D\eta)E = -g(h'D, E), \tag{22}$$
$$Ric(D,\xi) = \eta(D)\text{trace}\,\ell, \tag{23}$$
$$Q\xi = (\text{trace}\,\ell)\xi, \tag{24}$$
$$\mathfrak{R}ic(\phi D, \phi E) = \mathfrak{R}ic(D,E) - \eta(D)\eta(E)\text{trace}\ell, \tag{25}$$

for any vector fields D, E and F.

A manifold (M, g, η, ξ, ϕ) is called a η-Einstein manifold if its Ricci tensor $\mathfrak{R}ic$ is given by [4]

$$\mathfrak{R}ic(D,E) = \omega\eta(D)\eta(E) + \rho g(D,E), \tag{26}$$

for any vector fields D, E. Its scalar curvature tensor is given by $\tau = \rho(2p+1) + \omega$, where ρ and $\omega \neq 0$ are smooth functions on (M, g, ξ, η, ϕ). If $\omega = 0$, then we have an Einstein manifold.

3. Schouten-Van Kampen Connection $\widehat{\nabla}$

Associated with the Riemannian connection ∇ is the Schouten-Van Kampen (SvK) connection $\widehat{\nabla}$ that is associated to the horizontal $\mathcal{H}$ and vertical distribution $\mathcal{V}$. Olszak formulated this connection as follows:

$$\widehat{\nabla}_D E = \eta(E)h'D - g(h'D, E)\xi + \nabla_D E, \tag{27}$$

for any vector fields D, E (see [15] for more information). The torsion tensor $\widehat{T}$ of SvK connection $\widehat{\nabla}$ is non-symmetric. From (27), we have

$$\widehat{\nabla}_D\xi \& = 0, \quad \text{for any vector } D. \tag{28}$$

Let (M, g, η, ξ, ϕ) be an almost cosymplectic manifold such that $h'D \neq 0$ and admits the SvK connection $\widehat{\nabla}$.

For any vectors D, E and F, the SvK curvature is

$$\widehat{\mathfrak{R}}(D,E)F\& = -\eta(E)\eta(F)\ell D + \eta(D)\eta(F)\ell E + g(h'E,F)h'D$$
$$\& - g(h'D,F)h'E + \eta(E)g(\ell D,F)\xi - \eta(D)g(\ell E,F)\xi + \mathfrak{R}(D,E)F. \quad (29)$$

where $\mathfrak{R}$ and $\widehat{\mathfrak{R}}$ are curvature tensors with respect to the Riemannian connection ∇ and the Schouten-Van Kampen connection $\widehat{\nabla}$ respectively. Let $\widehat{\mathfrak{Ric}}$ and $\mathfrak{Ric}$ represent the Ricci curvature tensors with respect to the connections $\widehat{\nabla}$ and ∇, respectively. Then, we have

$$\widehat{\mathfrak{Ric}}(D,E)\& = -\eta(E)\eta(D)\text{trace}\ell - g(h'D,h'E) - g(\ell D,E) + \mathfrak{Ric}(D,E). \quad (30)$$

for any vector fields D and E. The Ricci $\widehat{\mathfrak{Ric}}$ associated to the SvK connection $\widehat{\nabla}$ will be called the *Schouten-Van-Kampen Ricci tensor*. The Ricci $\widehat{\mathfrak{Ric}}$ is not symmetric and satisfies $\widehat{\mathfrak{Ric}}(D,\xi) = 0$.

Using (15), the Scalar curvature τ and SvK Scalar curvature $\widehat{\tau}$ are related as follows

$$\widehat{\tau} = \tau - \text{trace}(\ell). \quad (31)$$

Lemma 1. *Let (M,g,η,ξ,ϕ) be an almost cosymplectic manifold such that $h'D \neq 0$ with SvK connection $\widehat{\nabla}$, then*

1. $\widehat{\mathfrak{R}}(D,E)\xi = 0$,
2. $\widehat{\mathfrak{R}}(D,\xi)E = 0$,
3. $\widehat{\ell}D = 0$, *for any vector fields D and E.*

Proof. The proof comes directly from (29). □

4. Concircular Curvature Tensor $\widehat{C}$

Definition 1. *With the SvK connection, the **concircular curvature tensor**[20] on (M,g,η,ξ,ϕ) is characterized by*

$$\widehat{C}(D,E)F = \frac{\widehat{\tau}}{2p(2p+1)}\{g(E,F)D - g(D,F)E\} + \widehat{\mathfrak{R}}(D,E)F, \quad (32)$$

for any vector fields D, E, F.

From the definition of concircular curvature tensor and (29) we get

$$g(Co\&(D,E)F,G) = -\eta(E)\eta(F)g(\ell D,G) + \eta(D)\eta(F)g(\ell E,G) + g(\mathfrak{R}(D,E)F,G)$$

$$\& + g(h'E,F)g(h'D,G) - g(h'D,F)g(h'E,G) + \eta(E)g(\ell D,F)\eta(G)$$
$$\& - \eta(D)g(\ell E,F)\eta(G) - \tfrac{\tau+\text{trace}(h^2)}{2p(2p+1)}\{g(E,F)g(D,G) - g(D,F)g(E,G)\},$$

and

$$g(\widehat{Co}\&(E,D)F,G) = -\eta(D)\eta(F)g(\ell E,G) + \eta(E)\eta(F)g(\ell D,G) + g(\mathfrak{R}(E,D)F,G)$$
$$\& + g(h'D,F)g(h'E,G) - g(h'E,F)g(h'D,G) + \eta(D)g(\ell E,F)\eta(G)$$
$$\& - \eta(E)g(\ell D,F)\eta(G) - \tfrac{\tau+\text{trace}(h^2)}{2p(2p+1)}\{g(D,F)g(E,G) - g(E,F)g(D,G)\}.$$

Combining the two relations (33) and (33), we have

$$g(\widehat{Co}(E,D)F,G) = -g(\widehat{Co}(D,E)F,G). \quad (33)$$

Also,

$$g(\widehat{Co}\&(D,E)G,F) = -\eta(E)\eta(G)g(\ell D,F) + \eta(D)\eta(G)g(\ell E,F) + g(\mathfrak{R}(D,E)G,F)$$
$$\& + g(h'E,G)g(h'D,F) - g(h'D,G)g(h'E,F) + \eta(E)g(\ell D,G)\eta(F)$$
$$- \eta(D)g(\ell E,G)\eta(F) - \frac{\tau+\text{trace}(h^2)}{2p(2p+1)}\{g(E,G)g(D,F) - g(D,G)g(E,F)\}. \quad (34)$$

Combining the two relations (33) and (34) we have

$$g(\widehat{Co}(D,E)F,G) = -g(\widehat{Co}(D,E)G,F). \tag{35}$$

Also,

$$g(\widehat{Co}\&(F,G)D,E) = -\eta(G)\eta(D)g(\ell F,E) + \eta(F)\eta(D)g(\ell G,E) + g(\mathfrak{R}(F,G)D,E)$$
$$+g(h'G,D)g(h'F,E) - g(h'F,D)g(h'G,E) + \eta(G)g(\ell F,D)\eta(E)$$
$$-\eta(F)g(\ell G,D)\eta(E)$$
$$\frac{\tau+\text{trace}(h^2)}{2p(2p+1)}\{g(G,D)g(F,E) - g(F,D)g(G,E)\}.$$

Combining the two relations (33) and (36) we have

$$g(\widehat{Co}(D,E)F,G) = g(\widehat{Co}(F,G)D,E). \tag{36}$$

Also,

$$g(\widehat{Co}\&(E,F)D,G) = -\eta(F)\eta(D)g(\ell E,G) + \eta(E)\eta(D)g(\ell F,G) + g(\mathfrak{R}(E,F)D,G)$$
$$\& + g(h'F,D)g(h'E,G) - g(h'E,D)g(h'F,G) + \eta(F)g(\ell E,D)\eta(G)$$
$$-\eta(E)g(\ell F,D)\eta(G)$$
$$-\frac{\tau+\text{trace}(h^2)}{2p(2p+1)}\{g(F,D)g(E,G) - g(E,D)g(F,G)\},$$

and

$$g(\widehat{Co}\&(F,D)E,G) = -\eta(D)\eta(E)g(\ell F,G) + \eta(F)\eta(E)g(\ell D,G) + g(\mathfrak{R}(F,D)E,G)$$
$$\& + g(h'D,E)g(h'F,G) - g(h'F,E)g(h'D,G) + \eta(D)g(\ell F,E)\eta(G)$$
$$-$$
$$\eta(F)g(\ell D,E)\eta(G) - \frac{\tau+\text{trace}(h^2)}{2p(2p+1)}\{g(D,E)g(F,G) - g(F,E)g(D,G)\}.$$

Combing relations (33), (37) and (37) we have the first Bianchi identity. Therefore, we have the following proposition

Proposition 1. *The concircular curvature tensor admitting the SvK connection on almost cosymplectic manifolds satisfies all the following properties*

a) $g(\widehat{Co}(E,D)F,G) = -g(\widehat{Co}(D,E)F,G)$,
b) $g(\widehat{Co}(D,E)G,F) = -g(\widehat{Co}(D,E)F,G)$,
c) $g(\widehat{Co}(F,G)D,E) = g(\widehat{Co}(D,E)F,G)$,
d) $g(\widehat{Co}(F,D)E,G) + g(\widehat{Co}(E,F)D,G) + g(\widehat{Co}(D,E)F,G) = 0$,

for any vector fields D, E, F and G.

ξ-Concircular Flat Almost Cosymplectic Manifolds

Definition 2. *An almost cosymplectic manifold with the SvK connection $\widehat{\nabla}$ is said to be ξ-**concircular flat** if*

$$\widehat{Co}(D,E)\xi = 0, \tag{37}$$

for any vector fields D, E.

From (32), (30) and (31) we have

$$\widehat{Co}(D,E)F\& = -\eta(E)\eta(F)\ell D + \eta(D)\eta(F)\ell E + g(h'E,F)h'D + \mathfrak{R}(D,E)F$$
$$\& - g(h'D,F)h'E + \eta(E)g(\ell D,F)\xi - \eta(D)g(\ell E,F)\xi$$
$$\& - \frac{\tau+\text{trace}(h^2)}{2p(2p+1)}\{g(E,F)D - g(D,F)E\}.$$

Letting $F = \xi$ we have

$$\widehat{Co}(D,E)\xi\& = -\eta(E)\ell D + \eta(D)\ell E + \mathfrak{R}(D,E)\xi - \frac{\tau+\text{trace}(h^2)}{2p(2p+1)}\{\eta(E)D - \eta(D)E\}. \tag{38}$$

Using $\mathfrak{R}(D,E)\xi = \eta(E)\ell D - \eta(D)\ell E$, we have

$$\widehat{Co}(D,E)\xi\& = -\frac{\tau+\text{trace}(h^2)}{2p(2p+1)}\{\eta(E)D - \eta(D)E\}, \tag{39}$$

for any vector fields D, E, therefore we have the following lemma

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Lemma 2. *An almost cosymplectic manifold (M, ϕ, ξ, η, g) such that $h' \neq 0$ with an SvK connection $\widehat{\nabla}$ is ξ-concircular flat if has a constant scalar curvature $\tau = \mathrm{trace}(h^2)$.*

From the above definition (32), we let $\widehat{Co} = 0$ and $G = D = \xi$, then

$$g(\widehat{\mathfrak{R}}(\xi, E)F, \xi)\& = \frac{\widehat{\tau}}{2p(2p+1)}\{g(E, F) - \eta(F)\eta(E)\}. \tag{40}$$

Since $g(\widehat{\mathfrak{R}}(\xi, E)F, \xi) = g(\widehat{\mathfrak{R}}(F, \xi)\xi, E)$ then,

$$g(\widehat{\ell}F, E) = \frac{\widehat{\tau}}{2p(2p+1)}\{g(E, F) - \eta(F)\eta(E)\}. \tag{41}$$

Using (31) we have

$$0 = \frac{\tau + \mathrm{trace}(h^2)}{2p(2p+1)}\{g(E, F) - \eta(F)\eta(E)\}. \tag{42}$$

Letting $E = QE$ we then have

$$\begin{aligned} 0\& = \tfrac{\tau+\mathrm{trace}(h^2)}{2p(2p+1)}\{g(QY, G) - \eta(G)\eta(QY)\}, \\ \& = \tfrac{\tau+\mathrm{trace}(h^2)}{2p(2p+1)}\{\mathfrak{R}ic(F, G) + \eta(G)\eta(F)\mathrm{trace}(h^2)\}, \\ \&\mathfrak{R}ic(E, F) = -\mathrm{trace}(h^2)\eta(E)\eta(F), \end{aligned} \tag{43}$$

for any vector fields E, F. Then;

Theorem 1. *A ξ-concircular flat almost cosymplectic manifold (M, g, η, ξ, ϕ) including the SvK connection with $h' \neq 0$, is a specialized type of an η-Einstein manifold.*

Assume that (M, g, η, ξ, ϕ) is cosymplectic (equivalently, normal with $\nabla\phi = 0$, $N_\phi = 0$, $\nabla\xi = 0$ and $h = 0$). Consequently,

$$(\nabla_W\phi)E = \nabla_W\phi E - \phi\nabla_W E = \nabla_W\phi E + \xi(\phi E)h'D - g(h'D, \phi E)\xi = 0.$$

Therefore, we conclude that ϕ commutes with $\widehat{Co}$.
Hence,

$$\phi(\widehat{Co}(D, E)F) - \widehat{Co}(D, E)\phi F = \frac{\widehat{\tau}}{2p(2p+1)}\Big[g(E, F)\phi D - g(D, F)\phi E\Big].$$

Thus, by the nongeneracy of g, we must have $\widehat{\tau}$ as required.
In addition, by setting $F = \xi$ in (32), we obtain

$$\widehat{Co}(D, E)\xi = \widehat{\mathfrak{R}}(D, E)\xi - \frac{\widehat{\tau}}{2p(2p+1)}\big(\eta(E)D - \eta(D)E\big) = 0 + 0 = 0,$$

so (2) holds.
Conversely, assume that (1) and (2) holds. Then from the definition of $\widehat{Co}$, we have $\widehat{\tau} = 0$. In particular,

$$0 = \widehat{\tau} = \tau - \mathrm{trace}(l) = \tau + \mathrm{trace}(h^2).$$

Therefore, $\mathrm{trace}(h^2) = -\tau \leq 0$. However,

$$\mathrm{trace}(h^2) = \sum_{i=1}^{2p+1} ||he_i||^2 \geq 0,$$

which implies $h = 0$ as required. Then

Theorem 2. *Let (M, ϕ, ξ, η, g) be an almost cosymplectic manifold with nonnegative scalar curvature $\tau \geq 0$ including the SvK connection. Then (M, g, η, ξ, ϕ) is cosymplectic (normal) if and only if the following two conditions hold:*

(1) $\widehat{Co}$ *commutes with ϕ, i.e.,*

$$\widehat{Co}(D, E)\,\phi F \;=\; \phi\big(\widehat{Co}(D, E)F\big) \qquad \text{for all } D, E, F,$$

(2) $\widehat{Co}$ *is* ξ*-concircular flat, i.e.,*

$$\widehat{Co}(D,E)\,\xi \;=\; 0, \qquad \textit{for all } D, E.$$

Remark 1. *Condition (1) can be equivalently written as* $[C,\phi] = 0$ *acting on vector fields.*

Pseudoconcircular Flat Almost Cosymplectic Manifolds

Definition 3. *The **pseudoconcircular flat curvature tensor** including the SvK connection is,*

$$g(\widehat{Co}(\phi D, E)F, \phi G) = 0, \tag{44}$$

for any vector fields D, E, F, G.

From the above definition (32),

$$g(\widehat{Co}(\phi D, E)F, \phi G)\& = g(\widehat{\mathfrak{R}}(\phi D, E)F, \phi G) - \frac{\widehat{\tau}}{2p(2p+1)}\{g(E,F)g(\phi D, \phi G)\& - g(\phi D, F)g(E, \phi G)\}. \tag{45}$$

Taking the orthonormal basis $\{E_i\} \in M$ and a summing, we have

$$\begin{gathered}\sum_{i=1}^{2p+1} \& g(\widehat{Co}(\phi E_i, E)F, \phi E_i) = \sum_{i=1}^{2p+1} g(\widehat{\mathfrak{R}}(\phi E_i, F)G, \phi E_i) \\ \& - \frac{\widehat{\tau}}{2p(2p+1)}\{\sum_{i=1}^{2p+1} g(F,G)g(\phi E_i, \phi E_i) - \sum_{i=1}^{2p+1} g(\phi E_i, G)g(F, \phi E_i)\}, \\ \& = \widehat{\mathfrak{R}ic}(E,F) - g(\widehat{\mathfrak{R}}(\xi, E)F, \xi) \\ \& - \frac{\widehat{\tau}}{2p(2p+1)}\{g(E,F)(2p+1) - g(E,F) + \eta(E)\eta(F)\}.\end{gathered} \tag{46}$$

Using the pseudoconcircular flatness condition and also (30), (31) we have

$$\begin{gathered}0\& = \mathfrak{R}ic(F,G) - g(G, \ell F) + \eta(G)\eta(F)\mathrm{trace}(h^2) - g(h'G, h'F) - g(\mathfrak{R}(\xi, F)G, \xi) \\ \& + g(G, \ell F) - \frac{\tau + \mathrm{trace}(h^2)}{2p(2p+1)}\{g(F,G)(2p+1) - g(F,G) + \eta(F)\eta(G)\}, \\ \& = \mathfrak{R}ic(F,G) + \eta(G)\eta(F)\mathrm{trace}(h^2) - g(h'G, h'F) - g(\ell F, G) \\ \& - \frac{\tau + \mathrm{trace}(h^2)}{2p(2p+1)}\{2ng(F,G) + \eta(F)\eta(G)\},\end{gathered} \tag{47}$$

which equates to

$$\begin{aligned}\mathfrak{R}ic(E,F)\& = -\eta(F)\eta(E)\mathrm{trace}(h^2) + g(h'F, h'E) + g(\ell E, F) \\ \& + \frac{\tau + \mathrm{trace}(h^2)}{2p(2p+1)}\{2pg(E,F) + \eta(E)\eta(F)\}.\end{aligned} \tag{48}$$

Taking into account (14), we have

$$\begin{aligned}\mathfrak{R}ic(E,F)\& = -\eta(G)\eta(F)\mathrm{trace}(h^2) + \frac{\tau + \mathrm{trace}(h^2)}{2p(2p+1)}\{2ng(F,G) + \eta(F)\eta(G)\}, \\ = \left(\frac{\tau + \mathrm{trace}(h^2)}{2p+1}\right)g(E,F) - \\ \left(\frac{\{2p(2p+1)-1\}\mathrm{trace}(h^2) - \tau}{2p(2p+1)}\right)\eta(F)\eta(G),\end{aligned} \tag{49}$$

therefore, we have

Theorem 3. *Let* (M, ϕ, ξ, η, g) *be a pseudoconcircular flat almost cosymplectic manifold with an SvK connection* $\widehat{\nabla}$ *such that* $h' \neq 0$ *. If* $\nabla_\xi h = 0$*, then* (M, g, η, ξ, ϕ) *is a* η*-Einstein manifold.*

Letting $E = F = \xi$ in (48), we have

$$\begin{aligned}\mathfrak{R}ic(\xi,\xi)\& = -\eta(\xi)\eta(\xi)\mathrm{trace}(h^2) + g(h'\xi, h'\xi) + g(\ell\xi, \xi) \\ + \frac{\tau + \mathrm{trace}(h^2)}{2p(2p+1)}\{2ng(\xi,\xi) + \eta(\xi)\eta(\xi)\},\end{aligned} \tag{50}$$

hence $\tau = -\mathrm{trace}(h^2)$, therefore we have the following;

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Corollary 1. *Let (M, ϕ, ξ, η, g) be a pseudoconcircular flat almost cosymplectic manifold such that $h' \neq 0$ and including the SvK connection $\widehat{\nabla}$ with $\nabla_\xi h = 0$, then $\tau = -\text{trace}(h^2)$.*

ϕ-Concircular flat almost cosymplectic manifolds

Definition 4. *An almost cosymplectic manifold with SvK connection $\widehat{\nabla}$ is said to be **ϕ-concircular flat** if*

$$g(\widehat{Co}(\phi D, \phi E)\phi F, \phi G) = 0, \tag{51}$$

for any vector fields D, E, F, G.

Taking the inner product g with respect to D in the definition of concircular (32), we have,

$$g(\widehat{Co}(D,E)F,G) = \&g(\widehat{\mathfrak{R}}(D,E)F,G) - \frac{\widehat{\tau}}{2p(2p+1)}\{g(E,F)g(D,G) - g(D,F)g(E,G)\}. \tag{52}$$

Computing each vector in (52) with ϕ,

$$\begin{gathered} g(\widehat{Co}(\phi D, \phi E)\&\phi F, \phi G) = g(\widehat{\mathfrak{R}}(\phi D, \phi E)\phi F, \phi G) \\ \& - \tfrac{\widehat{\tau}}{2p(2p+1)}\{g(\phi E, \phi F)g(\phi F, \phi G) \\ \& - g(\phi D, \phi F)g(\phi E, \phi G)\}. \end{gathered} \tag{53}$$

Using the definition of ϕ-concircular flat and (29), (31), we have

$$\begin{gathered} 0\& = g(\mathfrak{R}(\phi D, \phi E)\phi F, \phi G) + g(\phi F, h'\phi E)g(h'\phi D, \phi G) \\ \& - \eta(\phi E)\eta(\phi F)g(\ell\phi D, \phi G) + \eta(\phi D)\eta(\phi F)g(\ell\phi E, \phi G) \\ \& + \eta(\phi E)g(\ell\phi D, \phi F)g(\xi, \phi G) - \eta(\phi X W g(\ell\phi E, \phi F)g(\xi, \phi G) \\ \& - g(\phi F, h'\phi D)g(h'\phi E, \phi G) - \tfrac{\tau + \text{trace}(h^2)}{2p(2p+1)}\{g(\phi E, \phi F)g(\phi D, \phi G) \\ \& - g(\phi D, \phi F)g(\phi E, \phi G)\}. \end{gathered}$$

Using (1), (5), we have

$$\begin{gathered} 0\& = g(\mathfrak{R}(\phi D, \phi E)\phi F, \phi G) + g(\phi F, hE)g(hD, \phi G) - g(\phi F, hD)g(hE, \phi G) \\ \& - \tfrac{\tau + \text{trace}(h^2)}{2p(2p+1)}\{g(\phi E, \phi F)g(\phi D, \phi G) - g(\phi D, \phi F)g(\phi E, \phi G)\}. \end{gathered} \tag{54}$$

Taking the orthonormal bases $\{E_i\}$ and summing over $1 \leq i \leq 2p+1$, we have

$$\begin{gathered} \textstyle\sum_{i=1}^{2p+1} g(\mathfrak{R}\&(\phi E_i, \phi F)\phi G, \phi E_i) = -\sum_{i=1}^{2p+1} g(\phi G, hY)g(hE_i, \phi E_i) \\ \& + \textstyle\sum_{i=1}^{2p+1} g(\phi G, hE_i)g(hY, \phi E_i) + \tfrac{\tau + \text{trace}(h^2)}{2p(2p+1)}\{\sum_{i=1}^{2p+1} g(\phi F, \phi G)g(\phi E_i, \phi E_i) \\ \& - \textstyle\sum_{i=1}^{2p+1} g(\phi E_i, \phi G)g(\phi F, \phi E_i)\}, \end{gathered} \tag{55}$$

which equates to

$$\begin{gathered} \&\mathfrak{R}ic(F,G) - g(\mathfrak{R}(\xi, F)G, \xi) = g(hZ, hY) + \tfrac{\tau + \text{trace}(h^2)}{2p(2p+1)}\{g(\phi F, \phi G)(2p+1)\}, \\ \&\mathfrak{R}ic(E,F) = g(\ell E, F) + g(hF, hE) + \tfrac{\tau + \text{trace}(h^2)}{2p} g(\phi E, \phi F). \end{gathered} \tag{56}$$

Using proposition $\ell E = -h^2 E$ which implies that $\nabla_\xi h = 0$ we have

$$\mathfrak{R}ic(E,F) = \frac{\tau + \text{trace}(h^2)}{2p}\{g(E,F) - \eta(E)\eta(F)\}. \tag{57}$$

Hence, we have the following;

Theorem 4. *Let (M, ϕ, ξ, η, g) be a ϕ-concircular flat almost cosymplectic manifold with an SvK connection $\widehat{\nabla}$ such that $h' \neq 0$. If $\nabla_\xi h = 0$, then (M, g, η, ξ, ϕ) is an η-Einstein with respect to Levi-Civita connection of a certain type.*

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Let $F = G = E_i$ and sum over $1 \leq i \leq 2p+1$ we have

$$\& \sum_{i=1}^{2p+1} \mathfrak{R}ic(E_i, E_i) = \tau = -\text{trace}(h^2) + \text{trace}(h^2) + \frac{\tau + \text{trace}(h^2)}{2p}(2p+1). \tag{58}$$

Using (15), we have

$$\&\tau = -\text{trace}(h^2) + \text{trace}(h^2) + \frac{\tau + \text{trace}(h^2)}{2p}(2p+1), \&\tau = \frac{\tau + \text{trace}(h^2)}{2p}(2p+1), \&\tau = -(2p+1)\{\text{trace}(h^2)\}. \tag{59}$$

Therefore, we have the following;

Corollary 2. *Let (M, ϕ, ξ, η, g) be an almost cosymplectic manifold such that $h' \neq 0$. If (M, g, η, ξ, ϕ) admits an SvK connection $\widehat{\nabla}$ and with $\nabla_\xi h = 0$. Then, (M, g, η, ξ, ϕ) is ϕ-concircular symmetric if the scalar curvature is constant or (M, ϕ, ξ, η, g) Einstein .*

Example 1. *Let*

$$e_1 = \frac{1}{(x_3+1)^2}\frac{\partial}{\partial x_1}, \quad e_2 = (x_3 - 1)\frac{\partial}{\partial x_2}, \quad and \quad \xi = e_3 = (x_3^2 - 1)\frac{\partial}{\partial x_3} \tag{60}$$

be three linear independent vectors at each point in $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_3 \neq 1$ or $x_3 \neq -1\}$. Let g be the Riemannian metric on M defined by $g(e_1, e_2) = g(e_1, \xi) = g(e_2, \xi) = 0$ and $g(\xi, \xi) = g(e_2, e_2) = g(e_1, e_1) = 1$. That is, the form of Riemannian metric becomes

$$g = (x_3+1)^2 dx_1^2 + \frac{1}{(x_3-1)^2}dx_2^2 + \frac{1}{(x_3^2-1)^2}dx_3^2. \tag{61}$$

Define ϕ as $\phi e_1 = e_2, \phi e_2 = -e_1$, and $\phi\xi = 0$. Using the above calculations, then $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_3 \neq 1$ or $x_3 \neq -1\}$ is an almost contact metric manifold. To calculate the Levi-Civita connections, we first calculate the Lie-brackets

$$[e_1, \xi] = 2(x_3 - 1)e_1, \quad [e_2, \xi] = -(x_3+1)e_2, \quad [e_1, e_2] = 0. \tag{62}$$

Since $g([e_i, \xi], \xi) = 0$, then using $\eta(e_i) = 0$ for $i = 1, 2$ and $\eta(\xi) = 1$ in $d\eta(D, E) = D(\eta(E)) - \eta([D, E]) - E(\eta(D))$, we then have $d\eta(e_i, e_j) = -\eta([e_i, e_j]) = 0 \ \forall\ i, j \in \{1, 2, 3\}$. Since $\Phi(e_i, \xi) = 0$ for all $i = 1, 2$, then $\Phi([e_i, e_j], e_k) = 0,\ \forall\ i, j, k \in \{1, 2, 3\}$, then $d\Phi = 0$. Therefore, M is an almost cosymplectic manifold.
The Koszulu formula for orthonormal basis vector is

$$2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) - g([e_i, e_k], e_j) - g([e_j, e_k], e_i), \ \forall\ i, j, k \in \{1, 2, 3\} \tag{63}$$

We compute the following using the Lie-brackets;

$$\begin{gathered}\&\nabla_{e_1} e_1 = -2(x_3 - 1)\xi, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1}\xi = 2(x_3 - 1)e_2, \\ \&\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = (x_3+1)\xi, \quad \nabla_{e_2}\xi = -(x_3+1)e_2, \\ \&\nabla_\xi e_1 = 0, \quad \nabla_\xi e_2 = 0, \ and \ \nabla_\xi \xi = 0,\end{gathered} \tag{64}$$

which can results in $h'e_1 = -2(x_3 - 1)e_2$, $h'e_2 = (x_3+1)e_2$, and $h'\xi = 0$. Then, by the characterization of the Schouten-van Kampen connections we have

$$\begin{gathered}\&\widehat{\nabla}_{e_1} e_1 = -2(x_3 - 1)\xi, \quad \widehat{\nabla}_{e_1} e_2 = 2(x_3 - 1)\xi, \quad \widehat{\nabla}_{e_1}\xi = 0, \\ \&\widehat{\nabla}_{e_2} e_1 = 0, \quad \widehat{\nabla}_{e_2} e_2 = 0, \quad \widehat{\nabla}_{e_2}\xi = 0, \\ \&\widehat{\nabla}_\xi e_1 = 0, \quad \widehat{\nabla}_\xi e_2 = 0, \ and \ \widehat{\nabla}_\xi \xi = 0.\end{gathered} \tag{65}$$

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We compute the Riemannian curvature tensors

$$
\begin{aligned}
&\mathfrak{R}(e_1,e_1)e_1 = 0,\ \ \mathfrak{R}(e_2,e_2)e_2 = 0,\ \ \mathfrak{R}(\xi,\xi)\xi = 0,\\
&\mathfrak{R}(e_2,e_1)e_1 = 2(x_3^2-1)e_2,\ \ \mathfrak{R}(e_1,e_2)e_1 = -2(x_3^2-1)e_2,\ \ \mathfrak{R}(e_1,e_1)e_2 = 0,\\
&\mathfrak{R}(\xi,e_1)e_1 = -4(x_3-1)^2\xi,\ \ \mathfrak{R}(e_1,\xi)e_1 = 4(x_3-1)^2\xi,\ \ \mathfrak{R}(e_1,e_1)\xi = 0,\\
&\mathfrak{R}(\xi,e_2)e_2 = -(x_3+1)^2\xi,\ \ \mathfrak{R}(e_2,\xi)e_2 = (x_3+1)^2\xi,\ \ \mathfrak{R}(e_2,e_2)\xi = 0,\\
&\mathfrak{R}(e_1,e_2)e_2 = 2(x_3^2-1)e_2,\ \ \mathfrak{R}(e_2,e_1)e_2 = -2(x_3^2-1)e_2,\ \ \mathfrak{R}(e_2,e_2)e_1 = 0,\\
&\mathfrak{R}(e_1,\xi)\xi = -4(x_3-1)^2e_2,\ \ \mathfrak{R}(\xi,e_1)\xi = 4(x_3-1)^2e_2,\ \ \mathfrak{R}(\xi,\xi)e_1 = 0,\\
&\mathfrak{R}(e_2,\xi)\xi = -(x_3+1)^2e_2,\ \ \mathfrak{R}(\xi,e_2)\xi = (x_3+1)^2e_2,\ \ \mathfrak{R}(\xi,\xi)e_2 = 0,\\
&\mathfrak{R}(e_1,e_2)\xi = -2(x_3^2-1)\xi,\ \ \mathfrak{R}(e_2,\xi)e_1 = 0,\ \ \mathfrak{R}(\xi,e_1)e_2 = 0,
\end{aligned}
\tag{66}
$$

The Jacobi operator is $\ell e_1 = -4(x_3-1)^2e_2,\ \ell e_2 = -(x_3+1)^2e_2,\ \ell\xi = 0$. *With the above, we compute the SvK curvature tensor*

$$
\begin{aligned}
&\widehat{\mathfrak{R}}(e_1,e_1)e_1 = 0,\ \ \widehat{\mathfrak{R}}(e_2,e_2)e_2 = 0,\ \ \widehat{\mathfrak{R}}(\xi,\xi)\xi = 0,\\
&\widehat{\mathfrak{R}}(e_2,e_1)e_1 = 2(x_3^2-1)e_2,\ \ \widehat{\mathfrak{R}}(e_1,e_2)e_1 = -2(x_3^2-1)e_2,\ \ \widehat{\mathfrak{R}}(e_1,e_1)e_2 = 0,\\
&\widehat{\mathfrak{R}}(\xi,e_1)e_1 = -4(x_3-1)^2\xi,\ \ \widehat{\mathfrak{R}}(e_1,\xi)e_1 = 4(x_3-1)^2\xi,\ \ \widehat{\mathfrak{R}}(e_1,e_1)\xi = 0,\\
&\widehat{\mathfrak{R}}(\xi,e_2)e_2 = -(x_3+1)^2\xi,\ \ \widehat{\mathfrak{R}}(e_2,\xi)e_2 = (x_3+1)^2\xi,\ \ \widehat{\mathfrak{R}}(e_2,e_2)\xi = 0,\\
&\widehat{\mathfrak{R}}(e_1,e_2)e_2 = 2(x_3^2-1)e_2,\ \ \widehat{\mathfrak{R}}(e_2,e_1)e_2 = -2(x_3^2-1)e_2,\ \ \widehat{\mathfrak{R}}(e_2,e_2)e_1 = 0,\\
&\widehat{\mathfrak{R}}(e_1,\xi)\xi = -4(x_3-1)^2e_2,\ \ \widehat{\mathfrak{R}}(\xi,e_1)\xi = 4(x_3-1)^2e_2,\ \ \widehat{\mathfrak{R}}(\xi,\xi)e_1 = 0,\\
&\widehat{\mathfrak{R}}(e_2,\xi)\xi = -(x_3+1)^2e_2,\ \ \widehat{\mathfrak{R}}(\xi,e_2)\xi = (x_3+1)^2e_2,\ \ \widehat{\mathfrak{R}}(\xi,\xi)e_2 = 0,\\
&\widehat{\mathfrak{R}}(e_1,e_2)\xi = -2(x_3^2-1)\xi,\ \ \widehat{\mathfrak{R}}(e_2,\xi)e_1 = 0,\ \ \widehat{\mathfrak{R}}(\xi,e_1)e_2 = 0.
\end{aligned}
\tag{67}
$$

Hence $M = \{(x_1,x_2,x_3) \in \mathbb{R}^3$ *such that* $x_3 \neq 1$ *or* $x_3 \neq -1\}$ *is an almost cosymplectic manifold with a Schouten-Van Kampen connection.*

5. Conclusions

In this paper, we investigated the geometric effect of the Schouten-Van Kampen connection on almost cosymplectic manifolds with a special prominence on various classes of the concircular curvature tensor. The notions of ξ-concircular flat, pseudoconcircular flat and ϕ-concircular flat on almost cosymplectic manifolds with the Schouten-Van Kampen connection have been examined in this paper. We showed that under the effect of concircular transformation conditions, the almost cosymplectic manifolds with certain conditions are η-Einstein manifolds. This study showed that each of these curvature conditions have an impact on the geometry of the manifold. The concircular transformation tensors examined grantees the existence of a constant scalar curvature tensor. The results or equations generated can be used in physics such as in general relativity to explore the geometries of spacetime and cosmology models. This study can be extended further to understand the effects of conhamornic curvature tensor, conformal curvature tensor, ϕ-symmetriy and Ricci solitions on such manifolds with the adapted Schouten-Van Kampen connection.

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