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Weakly sdf-Absorbing Submodules Over Commutative Rings

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Abstract: Let R be a commutative ring with identity and M a unital R-module. A proper submodule N of M is called a weakly square-difference factor absorbing submodule (briefly, weakly sdf-absorbing submodule) if for all $a, b \in R$ and $m \in M$, the condition $0 \neq (a^2 - b^2)m \in N$ implies that either $(a+b)m \in N$ or $(a-b)m \in N$. In this paper, we investigate various characterizations and properties of weakly sdf-absorbing submodules in several module constructions. To construct new examples, we explore connections of this class with idealization rings and amalgamation modules. Examples illustrating the distinction between weakly sdf-absorbing and sdf-absorbing submodules are also provided.

Keywords: kweakly prime submodule, weakly classical prime submodule, sdf-absorbing ideal, sdf-absorbing submodule, weakly sdf-absorbing submodules

MSC: Primary 13A15, Secondary 13F05

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1. Introduction

All rings are assumed to be commutative with identity and all modules are unital. Let R be a ring and M an R-module. Several classical notions arise throughout our study. An element $m \in M$ is called a torsion element if there exists $r \in R \setminus \{0\}$ such that $rm = 0$. The set of all torsion elements of M is denoted by $T(M)$, and M is called torsion-free if $T(M) = \{0\}$. An R-module M is said to be a multiplication module if every submodule N of M is of the form IM for some ideal I of R; in this case, $N = (N :_R M)M$. Given submodules N, K of M and an ideal I of R, the residual ideal of N by K is defined as $(N :_R K) = \{r \in R : rK \subseteq N\}$, while the residual submodule of N by I is $(N :_M I) = \{m \in M : Im \subseteq N\}$. The annihilators $\mathrm{Ann}_M(I) = \{m \in M : Im = 0\}$ and $\mathrm{Ann}_R(N) = \{r \in R : rN = 0\}$ are also frequently used. An R-module M is called faithful if $\mathrm{Ann}_R(M) = \{0\}$.

The notion of primeness plays a central role in commutative algebra, serving as a foundation for ideal and module theory. This importance has motivated a wide range of generalizations, such as weakly prime, 2-absorbing, (m,n)-prime and semiprime ideals and submodules, which aim to preserve core structural properties of primeness while extending their applicability in broader algebraic contexts (see [3],[6],[7],[9],[10],[15]-[19],[22],[23],[25]).

One of the newest generalizations of prime ideals is the concept of square-difference factor absorbing ideals, recently introduced and studied by Anderson, Badawi, and Coykendall [6]. A proper ideal I of a ring R is said to be a (resp. weakly) square-difference factor absorbing ideal if whenever $0 \neq a, b \in R$ with $a^2 - b^2 \in I$ (resp. $0 \neq a^2 - b^2 \in I$), then $a + b \in I$ or $a - b \in I$. In their paper, many characterizations and results are studied, along with several illustrative examples that deepen the understanding of this class of ideals.

Building on this concept in module theory, the notion of square-difference factor absorbing submodules (sdf-absorbing submodules) was recently introduced in [19]. A proper submodule N

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of M is called an sdf-absorbing submodule of M if for $m \in M$ and $a, b \in R \setminus \operatorname{Ann}_R(m)$, whenever $(a^2 - b^2)m \in N$, then $(a+b)m \in N$ or $(a-b)m \in N$. The concept of sdf-absorbing submodules naturally generalizes that of classical prime submodules, which were introduced by Behboodi and Koohy in 2004: a proper submodule P of M is classical prime if $rsm \in P$ implies either $rm \in P$ or $sm \in P$ for all $r, s \in R$, $m \in M$.

Motivated by these concepts, we introduce and investigate weakly square-difference factor absorbing submodules (briefly, weakly sdf-absorbing submodules). A proper submodule N of M is said to be weakly sdf-absorbing if whenever $a, b \in R$ and $m \in M$ such that $0 \neq (a^2 - b^2)m \in N$, then $(a+b)m \in N$ or $(a-b)m \in N$. Unlike the sdf-absorbing case, this definition does not require any restriction on a and b, which ensures that the zero submodule is always weakly sdf-absorbing, regardless of whether $a, b \in \operatorname{Ann}_R(m)$ or not.

In this paper, we establish numerous properties and characterizations of weakly sdf-absorbing submodules, supported by illustrative examples and counterexamples. We also explore their relationships with some other well-known classes of submodules. In Section 2, we investigate several fundamental properties of weakly sdf-absorbing submodules, analyzing their structural characteristics and exploring various equivalent formulations and applications. Examples of weakly sdf-absorbing submodules which are not sdf-absorbing submodule are given in Examples 1, 2. Several characterizations of this class of submodules are investigated (see Theorems 1, 2, 5, 7 and Corollary 1). The position of weakly sdf-absorbing submodules within the broader landscape of known submodule classes is examined and illustrated (see Theorem 6, Example 5). Moreover, under a certain condition, we characterize modules in which every proper submodule is weakly sdf-absorbing.

In Section 3, we thoroughly examine the behavior of weakly sdf-absorbing submodules under various module constructions, including homomorphic images, quotient modules, localizations, and finite direct products of R-modules.

The final section focuses on weakly sdf-absorbing submodules (and ideals) within trivial ring extensions and amalgamation modules. Our findings enable the construction of novel and original examples of such submodules.

By $char(R)$, $J(R)$, $U(R)$ and $Z(R)$, we denote the smallest integer which satisfies $n.1_R = 0$, the Jacobson radical of R, the set of unit elements of R and the set of zero-divisors of R, respectively.

2. Characterizations of Weakly sdf-Absorbing Submodules

Inspired by the definition of weakly sdf-absorbing ideals and the extension of sdf-absorbing ideals to submodules, we define a weakly sdf-absorbing submodule as follows:

Definition 1. *Let R be a ring and let M be an R-module. A proper submodule N of M is called a weakly square-difference factor absorbing submodule (briefly, weakly sdf-absorbing submodule) if whenever $a, b \in R$ and $m \in M$ such that $0 \neq (a^2 - b^2)m \in N$, then $(a+b)m \in N$ or $(a-b)m \in N$.*

The concept of weakly sdf-absorbing submodules generalizes the notion of sdf-absorbing (and so prime) submodules from [19] and weakly sdf-absorbing (and so weakly prime) ideals from [6]. We note that the condition $0 \neq (a^2 - b^2)m \in N$ ensures that $N = \{0\}$ is always weakly sdf-absorbing, regardless of the annihilator condition $a, b \notin \operatorname{Ann}_R(m)$ needed in sdf-absorbing submodules definition. It is clear that a proper submodule N of the R-module R is a weakly sdf-absorbing submodule if and only if it is a weakly sdf-absorbing ideal of R. While the zero submodule is weakly sdf-absorbing in a non-reduced module, it is shown in [19] that it is not sdf-absorbing. For example, by [19, Theorem 3], the zero submodule of $\mathbb{Z}_n$ is not sdf-absorbing when $n \notin \{4, 9, p, 2q\}$ where p is any prime and q is any odd prime.

The following is an example of a non-trivial weakly sdf-absorbing submodule that is not sdf-absorbing:

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Example 1. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_4 \times \mathbb{Z}_4$ and let $N = \langle \bar{2} \rangle \times \langle \bar{0} \rangle$. Let $a, b \in \mathbb{Z}$ and $m = (m_1, m_2) \in M$ such that $0 \neq (a^2 - b^2)m = ((a^2 - b^2)m_1, (a^2 - b^2)m_2) \in N$. Then $(a^2 - b^2)m_1 \in \langle \bar{2} \rangle$ and $(a^2 - b^2)m_2 = \bar{0}$. Since clearly $a^2 - b^2 \notin 4\mathbb{Z}$ and $m_1 \neq \bar{0}$, then $a^2 - b^2 \in 2\mathbb{Z}$ and so either $(a + b) \in 2\mathbb{Z}$ or $(a - b) \in 2\mathbb{Z}$. Moreover, $a^2 - b^2 \notin 4\mathbb{Z}$ also implies $m_2 \in \langle \bar{2} \rangle$. Therefore, $(a + b)m \in N$ or $(a - b)m \in N$ and N is weakly sdf-absorbing in M. On the other hand, N is not sdf-absorbing in M since for example, $(2^2 - 0^2)(\bar{1}, \bar{1}) = (\bar{0}, \bar{0}) \in N$ but $2(\bar{1}, \bar{1}) = (\bar{2}, \bar{2}) \notin N$.*

Next, we present an example of an infinite-dimensional weakly sdf-absorbing $Q[x]$-module that is not sdf-absorbing, contrasting with the finite torsion $\mathbb{Z}$-module $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Example 2. *Let $R = \mathbb{Q}[x]$ and consider the R-module $M = \mathbb{Q}[x]/\langle x^2 \rangle \times \mathbb{Q}[x]/\langle x^2 \rangle$ where $f \cdot (\bar{p}, \bar{q}) = (f\bar{p}, f\bar{q})$ for $f \in R$ and $(\bar{p}, \bar{q}) \in M$. Let $N = \langle \bar{x} \rangle \times \langle \bar{0} \rangle$. Then N is a weakly sdf-absorbing submodule of M. Indeed, let $f, g \in R$ and $m = (\bar{p}, \bar{q}) \in M$ such that $0 \neq (f^2 - g^2)m = ((f^2 - g^2)\bar{p}, (f^2 - g^2)\bar{q}) \in N$. Then $(f^2 - g^2)\bar{q} = \bar{0}$ and so $\bar{q} = \bar{0}$. Now, $(f^2 - g^2)\bar{p} \in \langle \bar{x} \rangle$ implies $(f^2 - g^2)p \in \langle x \rangle$ and so $(f + g) \in \langle x \rangle$ or $(f - g) \in \langle x \rangle$ or $p \in \langle x \rangle$ since $\langle x \rangle$ is a prime ideal of R. It follows that $(f + g)m \in N$ or $(f - g)m \in N$ as needed. On the other hand, take $f = x$, $g = 0$ and $m = (\bar{1}, \bar{1})$. Then $(f^2 - g^2)m = 0 \in N$ but $(f + g)m = (f - g)m = (\bar{x}, \bar{x}) \notin N$ and thus, N is not sdf-absorbing in M.*

Next, we give an example from [26] of an R-module containing no nonzero weakly sdf-absorbing submodules.

Example 3. *Let p be a fixed prime number and consider the submodule $E(p) = \left\{ \frac{r}{p^n} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} : r \in \mathbb{Z} \text{ and } n \in \mathbb{N} \cup \{0\} \right\}$ of the $\mathbb{Z}$-module $M = \mathbb{Q}/\mathbb{Z}$. For each $t \in \mathbb{N} \cup \{0\}$, $G_t = \left\{ r(\frac{1}{p^t} + \mathbb{Z}) \in E(p) : r \in \mathbb{Z} \right\}$ is a submodule of $E(p)$. Moreover, each proper submodule of $E(p)$ is equal to G_t for some $t \in \mathbb{N} \cup \{0\}$, [26, Example 7.10]. Now, for any submodule G_t of $E(p)$, we have $0 \neq (p^2 - 0^2)(\frac{1}{p^{t+2}} + \mathbb{Z}) \in G_t$ but $p(\frac{1}{p^{t+2}} + \mathbb{Z}) \notin G_t$.*

Building on the sdf-absorbing submodule framework from [19], we present equivalent characterizations of weakly sdf-absorbing submodules.

Theorem 1. *Let N be a proper submodule of an R-module M. The following statements are equivalent.*

1. N is a weakly sdf-absorbing submodule of M.
2. For every $a, b \in R$, we have $(N :_M a^2 - b^2) = (0 :_M a^2 - b^2) \cup (N :_M a + b) \cup (N :_M a - b)$.
3. For $a, b \in R$ and a cyclic submodule L of M, whenever $\{0\} \neq (a^2 - b^2)L \subseteq N$, then $(a + b)L \subseteq N$ or $(a - b)L \subseteq N$.

Proof. (1) $\Rightarrow$ (2) Suppose N is a weakly sdf-absorbing submodule of M and let $a, b \in R$. Let $m \in (N :_M a^2 - b^2) \backslash (0 :_M a^2 - b^2)$, so that $0 \neq (a^2 - b^2)m \in N$. Then by assumption, either $(a + b)m \in N$ or $(a - b)m \in N$ and so $(N :_M a^2 - b^2) \subseteq (0 :_M a^2 - b^2) \cup (N :_M a + b) \cup (N :_M a - b)$. The other containment holds trivially and so the equality holds.

(2) $\Rightarrow$ (3) Let $a, b \in R$ and $L = Rm$ be a cyclic submodule of M such that $\{0\} \neq (a^2 - b^2)L \subseteq N$. Then $0 \neq (a^2 - b^2)m \in N$ and so $m \in (N :_M a^2 - b^2) \backslash (0 :_M a^2 - b^2)$. By assumption, $m \in (N :_M a + b)$ or $m \in (N :_M a - b)$ and so $(a + b)m \in N$ or $(a - b)m \in N$. It follows that $(a + b)L = (a + b)Rm \subseteq N$ or $(a - b)L = (a - b)Rm \subseteq N$ as required.

(3) $\Rightarrow$ (1) Let $a, b \in R$ and $m \in M$ such that $0 \neq (a^2 - b^2)m \in N$. The claim follows by taking $L = Rm$ in (3). □

Following [24], a ring R is called a um-ring if for any R-module M, if M equals to a finite union of submodules, then it must equals to one of them. The known examples of um-rings include finite products of fields and Artinian principal ideal rings. Under the assumption that R is a um-ring, we

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can establish another characterization of weakly sdf-absorbing submodules by using any arbitrary submodule L in Theorem 1 (3) .

Theorem 2. *Let R be a um-ring and N be a proper submodule of an R-module M. Then N is weakly sdf-absorbing in M if and only if for $a, b \in R$ and a submodule L of M, whenever $\{0\} \neq (a^2 - b^2)L \subseteq N$, then $(a+b)L \subseteq N$ or $(a-b)L \subseteq N$.*

Proof. The proof is similar to that of Theorem 1 where in (2) $\Rightarrow$ (3), the um-property of R and $L \subseteq (N :_M a^2 - b^2) \backslash (0 :_M a^2 - b^2)$ imply either $(a+b)L \subseteq N$ or $(a-b)L \subseteq N$. □

In the following theorem, for a weakly sdf-absorbing submodule N of M, we establish conditions under which $(N :_R L)$ is a weakly sdf-absorbing ideal of R for a submodule $L \subsetneq N$, linking module and ideal structures.

Theorem 3. *Let N be a submodule of an R-module M.*

1. If N is a weakly sdf-absorbing submodule of M, then $(N :_R L)$ is a weakly sdf-absorbing ideal of R for every faithful cyclic submodule $L \nsubseteq N$ of M. In particular, if M is cyclic and faithful, then $(N :_R M)$ is a weakly sdf-absorbing ideal of R.
2. If $(N :_R L)$ is a weakly sdf-absorbing ideal of R for every cyclic submodule $L \nsubseteq N$ of M, then N is a weakly sdf-absorbing submodule of M.

Proof. (1) Suppose that N is a weakly sdf-absorbing submodule of M and let $L \nsubseteq N$ be a faithful cyclic submodule of M. Then clearly, $(N :_R L)$ is proper in R. Let $a, b \in R$ such that $0 \neq a^2 - b^2 \in (N :_R L)$. Then L is faithful implies $\{0\} \neq (a^2 - b^2)L \subseteq N$ and by Theorem 1, we have $(a+b)L \subseteq N$ or $(a-b)L \subseteq N$. Hence, $a+b \in (N :_R L)$ or $a - b \in (N :_R L)$ as needed. The "in particular" statement is obvious.

(2) Let $a, b \in R$ and $m \in M$ such that $0 \neq (a^2 - b^2)m \in N$. Then $0 \neq (a^2 - b^2) \in (N :_R Rm)$. If $Rm \subseteq N$, then obviously $(a+b)m \in N$ or $(a-b)m \in N$. If $Rm \nsubseteq N$, then by assumption, $(N :_R Rm)$ is a weakly sdf-absorbing ideal of R and so $a + b \in (N :_R Rm)$ or $a - b \in (N :_R Rm)$. Therefore, $(a+b)m \in N$ or $(a-b)m \in N$ and we are done by Theorem 1. □

In general, if M is a non-faithful module and N is a weakly sdf-absorbing submodule, then $(N :_R M)$ need not be a weakly sdf-absorbing ideal in R. Moreover, $(N :_R M)$ is weakly sdf-absorbing in R does not imply that N is a weakly sdf-absorbing submodule of M.

Example 4.

1. The submodule $\langle \bar{0} \rangle$ is weakly sdf-absorbing in the $\mathbb{Z}$-module $\mathbb{Z}_4$ but $(\langle \bar{0} \rangle :_{\mathbb{Z}} \mathbb{Z}_4) = 4\mathbb{Z}$ is not weakly sdf-absorbing in $\mathbb{Z}$ since $0 \neq 2^2 - 0^2 \in 4\mathbb{Z}$ but $2 \notin 4\mathbb{Z}$.
2. Consider the submodule $N = \mathbb{Z}$ of the $\mathbb{Z}$-module $M = \mathbb{Q}$. Then $(N :_{\mathbb{Z}} M) = \{0\}$ is a weakly sdf-absorbing ideal of $\mathbb{Z}$. But, N is not weakly sdf-absorbing in M since $0 \neq (3^2 - 1^2).\frac{1}{8} = 1 \in N$ but $4.\frac{1}{8} \notin N$ and $2.\frac{1}{8} \notin N$.
3. Consider the submodule $N = \{0\} \times \langle \bar{4} \rangle$ of the non multiplication $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_{12}$. Then $(N :_{\mathbb{Z}} M) = \{0\}$ is a (weakly) sdf-absorbing ideal of $\mathbb{Z}$ but N is not a weakly sdf-absorbing submodule of M. Indeed, $0 \neq (2^2 - 0^2)(0, \bar{1}) \in N$ but $2.(1, \bar{1}) \notin N$.

However, under certain conditions on R and M, N is weakly sdf-absorbing in M provided that $(N :_R M)$ is weakly sdf-absorbing in R. Recall that an R-module M is called a principal ideal multiplication module if for every submodule N of M, there exists an element $r \in R$ such that $N = rM$, [8].

Theorem 4. *Let N be a submodule of a torsion-free R-module M such that $(N :_R M)$ is a weakly sdf-absorbing ideal of R. If M is cyclic or M is a principal ideal multiplication module, then N is a weakly sdf-absorbing submodule of M.*

Proof. Suppose $M = Rm$ for some $m \in M$. Let $a, b \in R$ and let $rm \in M$ such that $0 \neq (a^2 - b^2)rm \in N$. Then $((ar)^2 - (br)^2)M = (a^2 - b^2)r^2Rm \subseteq N$ and so $(ar)^2 - (br)^2 \in (N :_R M)$. If $(ar)^2 - (br)^2 = 0$, then $(a^2 - b^2)r^2m = 0$ and as $r \neq 0$ and M is torsion-free, $(a^2 - b^2)rm = 0$, a contradiction. Thus, $(ar)^2 - (br)^2 \neq 0$ and by assumption, $(a + b)r \in (N :_R M)$ or $(a - b)r \in (N :_R M)$. It follows that $(a + b)rm \in N$ or $(a + b)rm \in N$ and N is weakly sdf-absorbing in M. Next, suppose M is a principal ideal multiplication module. Let $a, b \in R$ and $L = rM$ for $0 \neq r \in R$ such that $\{0\} \neq (a^2 - b^2)rM = (a^2 - b^2)L \subseteq N$. Then $\{0\} \neq (a^2 - b^2)r^2M \subseteq N$ since otherwise, M being torsion-free and $r \neq 0$ imply $(a^2 - b^2)L = \{0\}$, a contradiction. Thus, $0 \neq (ar)^2 - (br)^2 = (a^2 - b^2)r^2 \in (N :_R M)$. Again by assumption, $(a + b)r \in (N :_R M)$ or $(a - b)r \in (N :_R M)$. Thus, $(a + b)L \subseteq N$ or $(a - b)L \subseteq N$ and the result follows by Theorem 1. □

Note that in Example 4(2), the torsion-free $\mathbb{Z}$-module $\mathbb{Q}$ is neither cyclic nor multiplication.

Proposition 1. *Let N be a submodule of an R-module M. If N is a weakly sdf-absorbing submodule of M and I is an ideal of R such that $\mathrm{Ann}_M(I) = \{0\}$, then either $(N :_M I) = M$ or $(N :_M I)$ is a weakly sdf-absorbing submodule of M.*

Proof. Let I be an ideal of R and suppose that $(N :_M I)$ is proper in M. Let $a, b \in R$ and $m \in M$ such that $0 \neq (a^2 - b^2)m \in (N :_M I)$. Then $(a^2 - b^2)Im \subseteq N$. If $(a^2 - b^2)Im = 0$, then by assumption, $(a^2 - b^2)m = 0$, a contradiction. Thus, $(a^2 - b^2)Im \neq 0$ and so Theorem 1 implies that $(a + b)Im \subseteq N$ or $(a - b)Im \subseteq N$. Therefore, $(a + b)m \in (N :_M I)$ or $(a - b)m \in (N :_M I)$ and $(N :_M I)$ is weakly sdf-absorbing in M. □

The converse of Proposition 1 need not be true. For example, consider the submodule $N = \langle \bar{4} \rangle$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_{24}$ and the ideal $I = 2\mathbb{Z}$ of $\mathbb{Z}$. Then clearly $(N :_M I) = \langle \bar{2} \rangle$ is a (weakly) sdf-absorbing submodule of M. On the other hand, N is not weakly sdf-absorbing since $0 \neq (4^2 - 2^2).\bar{1} \in N$ but $6.\bar{1} \notin N$ and $2.\bar{1} \notin N$. Furthermore, we shall later provide an example (Example 9) to demonstrate that the condition $\mathrm{Ann}_M(I) = \{0\}$ in Proposition 1 is essential.

Lemma 1. *[1] Let N be a finitely generated faithful multiplication R-module M and I be an ideal of R. Then*

1. $(IN :_R M) = I(N :_R M)$.
2. *If I is finitely generated faithful multiplication, then*
   (a) $(IN :_M I) = N$.
   (b) *Whenever $N \subseteq IM$, then $(JN :_M I) = J(N :_M I)$ for any ideal J of R.*

By using Lemma 1, we present the following corollary of Theorems 3 and 4.

Corollary 1. *Let M be a torsion-free cyclic principal ideal multiplication R-module and I be an ideal of R. Then I is a weakly sdf-absorbing ideal of R if and only if IM is an sdf-absorbing submodule of M.*

Proof. By Lemma 1(1), $(IM :_R M) = I$ and the proof follows directly by Theorems 3 and 4. □

Theorem 5. *Let I be a finitely generated multiplication ideal of a ring R with $\mathrm{Ann}_M(I) = \{0\}$ and N a submodule of a cyclic faithful multiplication R-module M.*

1. If IN is a weakly sdf-absorbing submodule of M, then either I is a weakly sdf-absorbing ideal of R or N is a weakly sdf-absorbing submodule of M.

2. If R is a PID, then N is a weakly sdf-absorbing submodule of IM if and only if $(N :_M I)$ is a weakly sdf-absorbing submodule of M.

Proof. We note that $\text{Ann}_M(I) = \{0\}$ clearly implies that I is faithful in R.

(1) Suppose IN is a weakly sdf-absorbing submodule of M. If $N = M$, then $I = I(N :_R M) = (IN :_R M)$ is a weakly sdf-absorbing ideal of R by Theorem 3. Suppose that N is proper in M and let $a, b \in R$, $m \in M$ such that $0 \neq (a^2 - b^2)m \in N$. Then $0 \neq (a^2 - b^2)Im \subseteq IN$ since $\text{Ann}_M(I) = \{0\}$. By Theorem 1, $(a + b)Im \subseteq IN$ or $(a - b)Im \subseteq IN$ and so by Lemma 1, $(a + b)m \in (IN :_M I) = N$ or $(a - b)m \in (IN :_M I) = N$.

(2) Suppose N is a weakly sdf-absorbing submodule of IM. Then $(N :_M I)$ is proper in M since otherwise $IM = N$, a contradiction. Let $m \in M$ and $a, b \in R$ such that $0 \neq (a^2 - b^2)m \in (N :_M I)$. Then Im is a submodule of IM and $0 \neq (a^2 - b^2)Im \subseteq N$ since $\text{Ann}_M(I) = 0$. By Theorem 1, $(a + b)Im \subseteq N$ or $(a - b)Im \subseteq N$ and so $(a + b)m \in (N :_M I)$ or $(a - b)m \in (N :_M I)$ as needed. Conversely, suppose $(N :_M I)$ is a weakly sdf-absorbing submodule of M. Let $a, b \in R$ and $m' \in IM$ such that $0 \neq (a^2 - b^2)m' \in N$. Then Lemma 1 implies

$$0 \neq (a^2 - b^2)(\langle m' \rangle :_M I) = (\left\langle (a^2 - b^2)m' \right\rangle :_M I) \subseteq (N :_M I).$$

Since R is a PID, it is well-known that every submodule of M is cyclic and so $(\langle m' \rangle :_M I)$ is cyclic. It follows by Theorem 1 that $(a + b)(\langle m' \rangle :_M I) \subseteq (N :_M I)$ or $(a - b)(\langle m' \rangle :_M I) \subseteq (N :_M I)$. Therefore, again Lemma 1 implies

$$(a + b)m' \in (I\left\langle (a + b)m' \right\rangle :_M I) = I(\left\langle (a + b)m' \right\rangle :_M I) \subseteq I(N :_M I) = (IN :_M I) = N$$

or similarly, $(a - b)m' \in N$. Hence, N is a weakly sdf-absorbing submodule of IM. □

Even under the assumptions of Theorem 5, if I is a weakly sdf-absorbing ideal of R and N is a weakly sdf-absorbing submodule of an R-module M, then IN need not be weakly sdf-absorbing in M. For example, $I = 2\mathbb{Z}$ is a (weakly) sdf-absorbing ideal of $\mathbb{Z}$ and by [19, Proposition 4.], $N = \langle \overline{10} \rangle$ is a (weakly) sdf-absorbing submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{40}$. But, $IN = \langle \overline{20} \rangle$ is not weakly sdf-absorbing in $\mathbb{Z}_{40}$ since $0 \neq (2^2 - 0^2).\bar{5} \in IN$ but $2.\bar{5} \notin IN$.

According to [25], a proper submodule N of an R-module M is called semiprime if for $a \in R$ and $m \in M$, we have $a^2m \in N$ implies $am \in N$. More general, N is called weakly semiprime if $0 \neq a^2m \in N$ implies $am \in N$. Also, recall from [22] that a proper submodule P of an R-module M is called a weakly classical prime submodule if for $a, b \in R$ and $m \in M$ such that $0 \neq abm \in N$, we have $am \in N$ or $bm \in N$.

Next, we give some conditions under which weakly sdf-absorbing submodules are weakly classical prime or weakly semiprime.

Theorem 6. *Let N be a weakly sdf-absorbing submodule of an R-module M. Then*

1. N is weakly semiprime in M. The converse is true if $char(R) = 2$.
2. If $2 \in U(R)$, then N is a weakly classical prime submodule of M.
3. If M is torsion-free and N is a maximal weakly sdf-absorbing submodule with respect to inclusion, then N is weakly classical prime.

Proof. (1) Suppose N is a weakly sdf-absorbing submodule of M and let $a \in R$ and $m \in M$ such that $0 \neq a^2m \in N$. Then $0 \neq (a^2 - 0^2)m \in N$ and by assumption, $am \in N$ as needed. Conversely, suppose N is weakly semiprime and $char(R) = 2$. Let $a, b \in R$, $m \in M$ such that $0 \neq (a^2 - b^2)m \in N$. Then $char(R) = 2$ implies $0 \neq (a + b)^2m \in N$ and by assumption, $(a + b)m \in N$. Thus, N is a weakly sdf-absorbing submodule of M.

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(2) Let $a, b \in R$ and $m \in M$ such that $0 \neq abm \in N$. Choose $r_1 = \frac{a+b}{2}$ and $r_2 = \frac{a-b}{2}$. Then $r_1, r_2 \in R$ with $0 \neq (r_1^2 - r_2^2)m = abm \in N$. By assumption, $am = (r_1 + r_2)m \in N$ or $bm = (r_1 - r_2)m \in N$ and so N is weakly classical prime in M.

(3) Let $a, b \in R$ and $m \in M$ such that $0 \neq abm \in N$ and suppose $am \notin N$. Then $m \notin (N :_M a)$ and so $(N :_M a)$ is proper in M. Since M is torsion-free, then $(0 :_M a) = 0$ and so $(N :_M a)$ is a weakly sdf-absorbing submodule of M by Proposition 1. Since $N \subseteq (N :_M a)$ and N is a maximal weakly sdf-absorbing submodule with respect to inclusion, then $N = (N :_M a)$. Thus, $bm \in (N :_M a) = N$ and N is a weakly classical prime submodule of M. □

The following diagram illustrates the relationship of weakly sdf-absorbing submodules with the classes of submodules above:

$$\text{weakly classical prime} \underset{2\in U(R)}{\overset{\Longrightarrow}{\Longleftarrow}} \text{weakly sdf-absorbing} \underset{char(R)=2}{\overset{\Longrightarrow}{\Longleftarrow}} \text{weakly semi-prime}$$

By the next examples, we verify that the arrows are irreversible in general.

Example 5.

1. *The submodule $N = \langle \bar{6} \rangle$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_{24}$ is (weakly) sdf-absorbing, [19, Proposition 4.]. But N is not weakly classical prime since $0 \neq 2.3.\bar{1} \in N$, $2.\bar{1} \notin N$ and $3.\bar{1} \notin N$.*
2. *If $char(R) \neq 2$, then we may find a weakly semiprime submodule that is not weakly sdf-absorbing. For example, the radical submodule $N = \langle \overline{30} \rangle$ is (weakly) semiprime in the $\mathbb{Z}$-module $M = \mathbb{Z}_{60}$ but N is not weakly sdf-absorbing in $\mathbb{Z}_{60}$ since $0 \neq (4^2 - 1^2).\bar{2} \in N$ but $5.\bar{2} \notin N$ and $3.\bar{2} \notin N$.*

Definition 2. *Let N be a weakly sdf-absorbing submodule of R-module M. For $a, b \in R$ and $m \in M$, we call (a, b, m) an sdf-absorbing triple-zero of N if $(a^2 - b^2)m = 0$, $(a + b)m \notin N$ and $(a - b)m \notin N$.*

In the case of a module R over itself, the sdf-absorbing triple-zero $(a, b, 1)$ of N (denoted by (a, b)) is called an sdf-absorbing double-zero of N.

Analogues to [9, Lemma 3.10], we have the following result.

Theorem 7. *Let N be a weakly sdf-absorbing submodule of an R-module M. For $a, b \in R$ and a submodule L of M such that (a, b, m) is not sdf-absorbing triple-zero of N for all $m \in L$, whenever $(a^2 - b^2)L \subseteq N$, then $(a - b)L \subseteq N$ or $(a + b)L \subseteq N$.*

Proof. Suppose N is a weakly sdf-absorbing submodule of M. Let $a, b \in R$ and L be a submodule of M such that (a, b, m) is not sdf-absorbing triple-zero of N for all $m \in L$. Assume on the contrary that $(a^2 - b^2)L \subseteq N$, but $(a - b)L \nsubseteq N$ and $(a + b)L \nsubseteq N$. Then there exist l_1, $l_2 \in L$ such that $(a - b)l_1 \notin N$ and $(a + b)l_2 \notin N$. If $(a^2 - b^2)l_1 \neq 0$, then $(a + b)l_1 \in N$ as N is weakly sdf-absorbing and $(a - b)l_1 \notin N$. Now, if $(a^2 - b^2)l_1 = 0$, then we conclude again $(a + b)l_1 \in N$ as (a, b, l_1) is not sdf-absorbing triple-zero of N and $(a - b)l_1 \notin N$. If $(a^2 - b^2)l_2 \neq 0$ or $(a^2 - b^2)l_2 = 0$, then by using a similar argument, we get $(a - b)l_2 \in N$. Since $(a^2 - b^2)(l_1 + l_2) \in N$, we have the following cases:

Case I: $(a^2 - b^2)(l_1 + l_2) = 0$. Since $(a, b, l_1 + l_2)$ is not sdf-absorbing triple-zero of N, we have either $(a - b)(l_1 + l_2) \in N$ or $(a + b)(l_1 + l_2) \in N$. Thus, $(a - b)l_1 \in N$ or $(a + b)l_2 \in N$, a contradiction.

Case II. $(a^2 - b^2)(l_1 + l_2) \neq 0$. Since N is weakly sdf-absorbing, then again either $(a - b)(l_1 + l_2) \in N$ or $(a + b)(l_1 + l_2) \in N$. Thus, $(a - b)l_1 \in N$ or $(a + b)l_2 \in N$, a contradiction. Therefore, $(a - b)L \subseteq N$ or $(a + b)L \subseteq N$. □

Following a similar line of reasoning as in [9, Theorem 2.3], we establish the following result.

Proposition 2. *Let N be a weakly sdf-absorbing submodule of R-module M and (a, b, m) be a an sdf-absorbing triple-zero of N. Then the following assertions hold.*

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1. $(a^2 - b^2)N = 0$.
2. $2(a+b)(N :_R M)m = 0$.
3. $2(a-b)(N :_R M)m = 0$.
4. $4(N :_R M)^2 m = 0$.
5. $2(a+b)(N :_R M)N = 0$.
6. $2(a-b)(N :_R M)N = 0$.

Proof. (1) Suppose $(a^2 - b^2)N \neq 0$ and choose $x \in N$ such that $(a^2 - b^2)x \neq 0$. Then $(a^2 - b^2)(x + m) \neq 0$ and so $(a+b)(x+m) \in N$ or $(a-b)(x+m) \in N$. Thus, $(a+b)m \in N$ or $(a+b)m \in N$, a contradiction. Therefore, $(a^2 - b^2)N = 0$.

(2) Suppose $2(a+b)(N :_R M)m \neq 0$ and choose $r \in (N :_R M)$ such that $2(a+b)rm \neq 0$. Then $[(a+r)^2 - (b-r)^2]m = [(a^2 - b^2) + 2(a+b)r]m = 2(a+b)rm \in N \backslash \{0\}$. Since N is weakly sdf-absorbing in M, then $(a+b)m \in N$ or $(a-b+2r)m \in N$. Since $2rm \in N$, then $(a+b)m \in N$ or $(a-b)m \in N$, a contradiction. Hence, $2(a+b)(N :_R M)m = 0$.

(3) Similar to (2).

(4) Suppose $4(N :_R M)^2 m \neq 0$ so that there are $r_1, r_2 \in (N :_R M)^2$ such that $4r_1 r_2 m \neq 0$. Then by using parts (2) and (3),

$$\begin{aligned}[(a + (r_1 + r_2))^2 - (b - (r_1 - r_2))^2]m &= [(a^2 - b^2) + 2(a+b)r_1 + 2(a-b)r_2 + 4r_1 r_2]m \\ &= 4r_1 r_2 m \in N \backslash \{0\}\end{aligned}$$

Thus, $(a + b + 2r_2)m \in N$ or $(a - b + 2r_1)m \in N$. Therefore, $(a+b)m \in N$ or $(a-b)m \in N$, a contradiction. Hence, $4(N :_R M)^2 m = 0$.

(5) Suppose there are $r \in (N :_R M)$ and $x \in N$ such that $2(a+b)rx \neq 0$. Then by using parts (1) and (2), we have $[(a+r)^2 - (b-r)^2](m+x) = 2(a+b)rx \in N \backslash \{0\}$. Thus, $(a+b)(m+x) \in N$ or $(a-b+2r)(m+x) \in N$. It follows that $(a+b)m \in N$ or $(a+b)m \in N$, a contradiction. Therefore, $2(a+b)(N :_R M)N = 0$.

(6) Similar to (5). □

Recall that for an R-module M, the set of zero divisors on M is $Z_R(M) = \{r \in R : rm = 0 \text{ for some } 0 \neq m \in M\}$. Following [2], A submodule N of an R-module M is called a nilpotent submodule if $(N :_R M)^k N = 0$ for some positive integer k, and we say that $m \in M$ is nilpotent if Rm is a nilpotent submodule of M. The following theorem is analogous to [6, Theorem 5.8].

Theorem 8. *Let N be a submodule of a non-zero R-module M. If N is a weakly sdf-absorbing submodule that is not sdf-absorbing, then $(N :_R M) \subseteq Z_R(M)$ and $4(N :_R M)^2 N = 0$. If moreover $4 \notin Z_R(M)$, then N is nilpotent.*

Proof. Suppose N is a weakly sdf-absorbing submodules that is sdf-absorbing and let $r \in (N :_R M)$. If $N = \{0\}$, then clearly $(N :_R M) \subseteq Z_R(M)$. Suppose N is non-zero and choose $a, b \in R$ and $0 \neq m \in M$ such that $(a^2 - b^2)m \in N$ but $(a+b)m \notin N$ and $(a-b)m \notin N$. Then N being weakly sdf-absorbing implies $(a^2 - b^2)m = 0$. Now, $(a^2 - (b+r)^2)m = (a^2 - b^2 - 2br - r^2)m = (-2br - r^2)m \in N$. If $(a^2 - (b+r)^2)m \neq 0$, then by assumption $(a + (b+r))m \in N$ or $(a - (b+r))m \in N$. So, $(a+b)m \in N$ or $(a-b)m \in N$, a contradiction. Therefore, $-(2br + r^2)m = (a^2 - (b+r)^2)m = 0$. Similarly, $(2br - r^2)m = (a^2 - (b-r)^2)m = 0$. Thus, $(2br + r^2)m = (2br - r^2)m = 0$ and so $2r^2 m = 0$. Suppose $r \notin Z_R(M)$. Then $2rm = 0$ and so $r^2 m = 2brm = 0$. Hence, again $r \notin Z_R(M)$ implies $rm = 0$, a contradiction. Therefore, $r \in Z_R(M)$ and $(N :_R M) \subseteq Z_R(M)$ as needed. Now, since N is weakly sdf-absorbing submodules that is not sdf-absorbing, then clearly we can find an sdf-absorbing triple-

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zero (a,b,m) of N. Suppose $4(N:_R M)^2N \neq 0$ and choose $r_1, r_2 \in (N:_R M)$ and $x \in N$ such that $4r_1r_2x \neq 0$. Then by Proposition **??**,

$$[(a+(r_1+r_2))^2-(b-(r_1-r_2))^2](m+x)$$
$$=[(a^2-b^2)+2(a+b)r_1+2(a-b)r_2+4r_1r_2](m+x)=4r_1rx \in N\backslash\{0\}$$

Since N is weakly sdf-absorbing in M, then $(a+b+2r_2)(m+x) \in N$ or $(a-b+2r_1)(m+x) \in N$. Therefore, $(a+b)m \in N$ or $(a-b)m \in N$, a contradiction. Hence, $4(N:_R M)^2N = 0$. If $4 \notin Z_R(M)$, then we conclude that $(N:_R M)^2N = 0$ and so N is nilpotent. □

Proposition 3. *If a submodule N of an R-module M is weakly sdf-absorbing that is not sdf-absorbing, then* $\sqrt{4(N:_R M)} = \sqrt{(0:_R M)}$. *If moreover M is multiplication, the* $M-rad(4N) = M-rad(0)$.

Proof. Suppose N is weakly sdf-absorbing that is not sdf-absorbing. Then by Theorem 8, we have $4(N:_R M)^2N = 0$. Thus,

$$4(N:_R M)^3 = 4(N:_R M)^2(N:_R M)$$
$$\subseteq (4(N:_R M)^2N:_R M) = (0:_R M)$$

Therefore, $\sqrt{4(N:_R M)} = \sqrt{4(N:_R M)^3} \subseteq \sqrt{(0:_R M)}$ and since the other containment is obvious, we get $\sqrt{4(N:_R M)} = \sqrt{(0:_R M)}$. Now. suppose M is multiplication. Then $4N^3 = 4(N:_R M)^2N = 0$ and so $M-rad(4N) = M-rad(0)$ as needed. □

In the following theorem, we describe modules in which every proper submodule is weakly sdf-absorbing submodule.

Theorem 9. *Let M be an R-module. If every proper submodule of M is weakly sdf-absorbing, then* $4J(R)^2M = 0$. *Moreover, the converse is true if R is quasi-local and* $4 \in U(R)$.

Proof. Suppose that every proper submodule of M is weakly sdf-absorbing but $4J(R)^2M \neq 0$. Choose $r_1, r_2 \in J(R)$ and $m \in M$ such that $4r_1r_2m \neq 0$ and let $N = R(4r_1r_2)m$. Then $[(r_1+r_2)^2-(r_1-r_2)^2]m = 4r_1r_2m \in N\backslash\{0\}$. By assumption, N is weakly sdf-absorbing in M and so either $2r_1m \in N$ or $2r_2m \in N$. If $2r_1m \in N$, then $2r_1m = 4rr_1r_2m$ for some $r \in R$ and so $(1-2rr_2)(2r_1)m = 0$. Since $(1-2rr_2) \in U(R)$, then $2r_1m = 0$. If $2r_2m \in N$, then by a similar argument, we get $2r_2m = 0$. In both cases, we conclude $4r_1r_2m = 0$, a contradiction. Therefore, $4J(R)^2M = 0$. Now, suppose R is quasi-local with $4J(R)^2M = 0$ and $4 \in U(R)$. Then $J(R)^2M = 0$. Let $a,b \in R$ and let $m \in M$ such that $0 \neq (a^2-b^2)m \in N$. If $a+b \in U(R)$ or $a-b \in U(R)$, then clearly $(a-b)m \in N$ or $(a+b)m \in N$. If $a+b$ and $a-b$ are non-units, then $a+b, a-b \in J(R)$ and so $(a^2-b^2)m \in J(R)^2M = 0$, a contradiction. Therefore, N is a weakly sdf-absorbing submodule of M. □

3. Weakly sdf-Absorbing Submodules in Module Extensions

In this section, we study properties and characterizations of weakly sdf-absorbing submodules in quotient modules, localizations, and direct product of modules.

Proposition 4. *Let* $f: M \to M'$ *be an R-module homomorphism.*

1. If f is a monomorphism and N' is a weakly sdf-absorbing submodule of M' with $f^{-1}(N') \neq M$, then $f^{-1}(N')$ is a weakly sdf-absorbing submodule of M.
2. If f is an epimorphism and N is a weakly sdf-absorbing submodule of M containing $Ker(f)$, then $f(N)$ is a weakly sdf-absorbing submodule of M'.

Proof. (1) Suppose that $0 \neq (a^2-b^2)m \in f^{-1}(N')$ for some $a,b \in R$ and $m \in M$. Then, $0 \neq (a^2-b^2)f(m) = f((a^2-b^2)m) \in N'$ as f is a monomorphism. Since N' is weakly sdf-absorbing

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in M', we have either $(a+b)f(m_1) = f((a+b)m) \in N'$ or $(a-b)f(m) = f((a-b)m) \in N'$. Thus, $(a+b)m \in f^{-1}(N')$ or $(a-b)m \in f^{-1}(N')$ and $f^{-1}(N')$ is weakly sdf-absorbing in M.

(2) Suppose that $0 \neq (a^2-b^2)m' \in f(N)$ for some $a, b \in R$ and $m' = f(m) \in M'$ where $m \in M$. Then, $0 \neq (a^2-b^2)m \in N$ as N contains $Ker(f)$ and so either $(a+b)m \in N$ or $(a-b)m \in N$. Thus, $(a+b)m' \in f(N)$ or $(a-b)m' \in f(N)$, and $f(N)$ is weakly sdf-absorbing in M'. □

We illustrate in the next example that the conditions "f is a monomorphism" and "f is an epimorphism" in Proposition 4 are crucial

Example 6. *Let $f : \mathbb{Z}_8 \longrightarrow \mathbb{Z}_8$ be a $\mathbb{Z}$-module homomorphism defined by $f(x) = 2x$. Then f is neither one-to-one nor onto. Now, $N' = \langle\bar{0}\rangle$ and $N = \langle\bar{2}\rangle$ are clearly weakly sdf-absorbing submodule of $\mathbb{Z}_8$ but $f^{-1}(N') = f(N) = \langle\bar{4}\rangle$ is not weakly sdf-absorbing in $\mathbb{Z}_8$ as $0 \neq (2^2-0^2)\cdot\bar{1} \in \langle\bar{4}\rangle$ but $2\cdot\bar{1} \notin \langle\bar{4}\rangle$.*

Also, the condition "$Ker(f) \subseteq N$" is essential in Proposition 4(2)

Example 7. *Let $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ be a $\mathbb{Z}$-module homomorphism defined by $\varphi(f(x)) = f(0)$. Then the submodule $N = \langle x+18\rangle$ of $\mathbb{Z}[x]$ is clearly a (weakly) sdf-absorbing submodule of $\mathbb{Z}[x]$. But, $\varphi(N) = 18\mathbb{Z}$ is not weakly sdf-absorbing in $\mathbb{Z}$ by [6, Example 2.8]. Note that $Ker(\varphi) = \langle x\rangle \nsubseteq N$.*

In view of Proposition 4, we conclude the following result:

Corollary 2. *Let M be a nonzero R-module and $K \subseteq N$ be submodules of M.*

1. If K is weakly sdf-absorbing in M, then $K \cap M$ is weakly sdf-absorbing in N.
2. If N is weakly sdf-absorbing in M, then N/K is weakly sdf-absorbing in M/K.
3. If N/K is a weakly sdf-absorbing submodule of M/K and K is an sdf-absorbing submodule of M, then N is a (weakly) sdf-absorbing submodule of M.

Proof. (1) This follows by Proposition 4(1) considering the natural injection $i : N \to M$ defined by $i(m) = m$ for all $m \in N$.

(2) Take the canonical epimorphism $\pi : M \to M/K$ defined by $\pi(m) = m + K$ in Proposition 4(2).

(3) Let $a, b \in R$ and $m \in M$ such that $0 \neq (a^2-b^2)m \in N$. If $(a^2-b^2)m \in K$, then we have either $(a+b)m \in K \subseteq N$ or $(a-b)m \in K \subseteq N$. If $(a^2-b^2)m \notin K$, then $K \neq (a^2-b^2)(m+K) \in N/K$. It follows that either $(a+b)(m+K) \in N/K$ or $(a-b)(m+K) \in N/K$. Therefore, $(a+b)m \in N$ or $(a-b)m \in N$, as required. □

Let N be a submodule of an R-module M. By $Z_N(M)$, we denote the set $\{r \in R : rm \in N$ for some $m \in M\backslash N\}$.

Proposition 5. *Let S be a multiplicatively closed subset of a ring R and N a proper submodule of an R-module M satisfying $(N :_R M) \cap S = \emptyset$. If N is a weakly sdf-absorbing submodule of M, then $S^{-1}N$ is a weakly sdf-absorbing submodule of $S^{-1}M$. The converse part also holds if $Z_N(M) \cap S = Z(M) \cap S = \emptyset$.*

Proof. We note that $S^{-1}N \neq S^{-1}M$ since otherwise $S^{-1}(N :_R M) = (S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}R$ which contradicts the assumption that $(N :_R M) \cap S = \emptyset$. Suppose that $0 \neq ((\frac{a}{s})^2 - (\frac{b}{t})^2)\frac{m}{u} \in S^{-1}N$ for some $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ and $\frac{m}{u} \in S^{-1}M$. Then there exists $v \in S$ such that $0 \neq v(t^2a^2 - s^2b^2)m \in N$. By assumption, we have either $(ta+sb)vm \in N$ or $(ta-sb)vm \in N$. Thus, $(\frac{a}{s} + \frac{b}{t})\frac{m}{u} \in S^{-1}N$ or $(\frac{a}{s} - \frac{b}{t})\frac{m}{u} \in S^{-1}N$ and $S^{-1}N$ is weakly sdf-absorbing in $S^{-1}M$. Conversely, suppose that $0 \neq (a^2-b^2)m \in N$ for some $a, b \in R$ and $m \in M$. Then $((\frac{a}{1})^2 - (\frac{b}{1})^2)\frac{m}{1} \in S^{-1}N$. If $((\frac{a}{1})^2 - (\frac{b}{1})^2)\frac{m}{1} = 0$, then there exists $w \in S$ such that $w(a^2-b^2)m = 0$ and $Z(M) \cap S = \emptyset$ implies $(a^2-b^2)m = 0$, a contradiction. Thus, $0 \neq ((\frac{a}{1})^2 - (\frac{b}{1})^2)\frac{m}{1} \in S^{-1}N$ which implies either $(\frac{a}{1} + \frac{b}{1})\frac{m}{1} \in S^{-1}N$ or $(\frac{a}{1} - \frac{b}{1})\frac{m}{1} \in S^{-1}N$. Thus, there exist $u, v \in S$ such that $u(a+b)m \in N$ or $v(a-b)m \in N$. By the assumption $Z_N(M) \cap S = \emptyset$, we have either $(a+b)m \in N$ or $(a-b)m \in N$, we are done. □

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Next, we characterize weakly sdf-absorbing submodules in the Cartesian product of modules.

Proposition 6. *Let N_1,N_2 be nonzero proper submodules of R-modules M_1 and M_2, respectively. Then*

1. If $N_1 \times N_2$ is a weakly sdf-absorbing submodule of $M_1 \times M_2$, then N_1 and N_2 are sdf-absorbing in M_1 and M_2, respectively.
2. If N_1 and N_2 are sdf-absorbing in M_1 and M_2, respectively and $2 \in (N_1 :_R M_1)$ or $2 \in (N_2 :_R M_2)$, then $N_1 \times N_2$ is a weakly sdf-absorbing submodule of $M_1 \times M_2$.

Proof. (1) Suppose $N = N_1 \times N_2$ is a weakly sdf-absorbing submodule of M and let $a, b \in R$, $m \in M$ such that $(a^2 - b^2)m \in N_1$. Since $N_2 \neq \{0\}$, there is a nonzero element $n \in N_2$. Then $(0,0) \neq (a^2 - b^2)(m,n) \in N_1 \times N_2$ implies either $(a+b)(m,n) \in N_1 \times N_2$ or $(a-b)(m,n) \in N_1 \times N_2$. Thus, $(a+b)m \in N_1$ or $(a-b)m \in N_1$ and N_1 is an sdf-absorbing submodule of M_1. It is clear similarly that N_2 is an sdf-absorbing submodule of M_2.

(2) From [19, Proposition 11(3)], N is a sdf-absorbing submodule of M, thus it is a weakly sdf-absorbing submodule of M. □

In the following, we discuss the case when one or both of the submodules in the direct product is non-proper or zero.

Proposition 7. *Let N_1,N_2 be proper submodules of R-modules M_1 and M_2.*

1. If $N_1 \times M_2$ is a weakly sdf-absorbing submodule of $M_1 \times M_2$, then N_1 is a weakly sdf-absorbing submodule of M_1.
2. If $M_1 \times N_2$ is a weakly sdf-absorbing submodule of$M_1 \times M_2$, then N_2 is a weakly sdf-absorbing submodule of M_2.
3. If $N_1 \times \{0\}$ is a weakly sdf-absorbing submodule of $M_1 \times M_2$, then N_1 is a weakly sdf-absorbing submodule of M_1. The converse holds if M_2 is torsion-free.
4. If $\{0\} \times N_2$ is a weakly sdf-absorbing submodule of $M_1 \times M_2$, then N_2 is a weakly sdf-absorbing submodule of M_2. The converse holds if M_1 is torsion-free.

Proof. (1) and (2) are straightforward.

(3) Suppose that $a, b \in R$, $m \in M_1$ such that $0 \neq (a^2 - b^2)m \in N_1$. Then $(0,0) \neq (a^2 - b^2)(m,0) \in N_1 \times \{0\}$ implies either $(a+b)(m,0) \in N_1 \times N_2$ or $(a-b)(m,0) \in N_1 \times N_2$. Hence, $(a+b)m \in N_1$ or $(a-b)m \in N_1$ and N_1 is a weakly sdf-absorbing submodule of M_1. Conversely, let M_2 be a torsion-free module. Suppose that $(0,0) \neq (a^2 - b^2)(m_1, m_2) \in N_1 \times \{0\}$ for some $a, b \in R$, $(m_1, m_2) \in M_1 \times M_2$. Since $(a^2 - b^2)m_2 = 0$, $a^2 - b^2 \neq 0$ and M_2 is torsion-free, we conclude $m_2 = 0$. Since $0 \neq (a^2 - b^2)m_1 \in N_1$, we have either $(a+b)m \in N_1$ or $(a-b)m \in N_1$. Therefore, $(a+b)(m_1, m_2) \in N_1 \times \{0\}$ or $(a-b)(m_1, m_2) \in N_1 \times \{0\}$, as needed.

(4) Similar to (3). □

A general characterization for weakly sdf-absorbing submodules of Cartesian product of finitely many R-modules is as following:

Theorem 10. *Let $N_1, N_2, ..., N_k$ be nonzero proper submodules of R-modules $M_1, M_2, ..., M_k$ where $k \geq 2$. Let $M = \times_{i=1}^{k} M_i$ and $N = \times_{i=1}^{k} N_i$. Suppose that $N_1, ..., N_t$ are proper for some $1 \leq t \leq k$ and $N_j = M_j$ for all $t < j \leq k$. Then N is a weakly sdf-absorbing submodule of M if and only if N_i's are sdf-absorbing submodules of M_i for all $i \in \{1, ..., t\}$ and at most for one of $i \in \{1, ..., t\}$, $2 \notin (N_i :_{R_i} M_i)$.*

4. Weakly Sdf-Absorbing Submodules of Amalgamation Modules

Let R be a ring and M be an R-module. The idealization ring $R(+)M$ of M in R is defined as the set $\{(r, m) : r \in R, m \in M\}$ with the usual componentwise addition and multiplication defined as $(r, m)(s, n) = (rs, rn + sm)$. It can be easily verified that $R(+)M$ is a commutative ring with identity

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$(1_R, 0)$. If I is an ideal of R and N is a submodule of M, then $I(+)N = \{(r,m) : r \in I, m \in N\}$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. In this case, $I(+)N$ is called a homogeneous ideal of $R(+)M$, see [4].

In [6, Theorem 4.16], the authors completely determined the nonzero sdf-absorbing ideals of $R(+)M$. They proved that for a nonzero proper ideal I of R, and a submodule N of an an R-module M, $I(+)N$ is sdf-absorbing in $R(+)M$ if and only if I is sdf-absorbing in R and $N = M$.

In the next theorem, we justify a condition under which $I(+)N$ is a weakly sdf-absorbing ideal of $R(+)M$.

Theorem 11. *Let I be a proper ideal of a ring R and N be a a submodule of an R-module M.*

1. If $I(+)N$ is a weakly sdf-absorbing ideal of $R(+)M$, then I is a weakly sdf-absorbing ideal of R. The converse is true if $N = M$ and $I \nsubseteq \sqrt{0}$.
2. If I is a weakly sdf-absorbing ideal of R such that $a, b \in \mathrm{Ann}_R(M)$ for any sdf-absorbing double zero (a,b) of I, then $I(+)M$ is a weakly sdf-absorbing ideal of $R(+)M$.

Proof. (1) Suppose $I(+)N$ is weakly sdf-absorbing in $R(+)M$ and let $a, b \in R$ such that $0 \neq a^2 - b^2 \in I$. Then $(0,0) \neq (a,0)^2 - (b,0)^2 \in I(+)N$ and by assumption, $(a+b, 0) \in I(+)N$ or $(a-b, 0) \in I(+)N$. Hence, $a+b \in I$ or $a-b \in I$ as needed. Now, if $N = M$ and $I \nsubseteq \sqrt{0}$, then N is an sdf-absorbing ideal of R by [6, Theorem 5.8]. Hence, $I(+)N$ is an (a weakly) sdf-absorbing ideal of $R(+)M$ by [6, Theorem 4.16].

(2) Let $(a, m_1), (b, m_2) \in R(+)M$ such that $(0,0) \neq (a, m_1)^2 - (b, m_2)^2 \in I(+)M$. Then $(a^2 - b^2, 2am_1 - 2bm_2) \in I(+)M$ and $a^2 - b^2 \in I$. If $a^2 - b^2 \neq 0$, then $a+b \in I$ or $a-b \in I$ since I is weakly sdf-absorbing in R and so $(a+b, m_1+m_2) \in I(+)M$ or $(a-b, m_1-m_2) \in I(+)M$. Suppose $a^2 - b^2 = 0$ but $a+b \notin I$ and $a-b \notin I$, then (a,b) is an sdf-absorbing double zero of I. By assumption, $a, b \in \mathrm{Ann}_R(M)$ and so $(a^2-b^2, 2am_1 - 2bm_2) = (0,0)$, a contradiction. Thus, $a+b \in I$ or $a-b \in I$ and so again $(a+b, m_1+m_2) \in I(+)M$ or $(a-b, m_1-m_2) \in I(+)M$. Therefore, $I(+)M$ is weakly sdf-absorbing in $R(+)M$. □

Remark 1. *(1) If $char(R) = 2$ and I is a weakly sdf-absorbing ideal of R, then by following the proof of Theorem 11(2), we clearly conclude that $I(+)M$ is a weakly sdf-absorbing ideal of $R(+)M$.*

(2) The condition "$a, b \in \mathrm{Ann}_R(M)$ for any sdf-absorbing double zero (a,b) of I" in Theorem 11(2) can not be discarded. For example, the ideal $\langle\bar{0}\rangle(+)\mathbb{Z}_8$ of the idealization ring $\mathbb{Z}_8(+)\mathbb{Z}_8$ is not weakly sdf-absorbing. Indeed, $(\bar{3}, \bar{1})^2 - (\bar{1}, \bar{2})^2 = (\bar{0}, \bar{2}) \in \langle\bar{0}\rangle(+)\mathbb{Z}_8 \backslash \{(0,0)\}$ but $(\bar{4}, \bar{3}) \notin \langle\bar{0}\rangle(+)\mathbb{Z}_8$ and $(\bar{2}, \bar{7}) \notin \langle\bar{0}\rangle(+)\mathbb{Z}_8$. Note that $(3,1)$ is not sdf-absorbing double zero of $\langle\bar{0}\rangle$.

In [6, Remark 4.18], it is proved that if N is a non-zero proper submodule of an R-module M, then $\{0\}(+)N$ is never an sdf-absorbing ideal of $R(+)M$. Also, it is proved in [19, Proposition 12.] that if N is a proper submodule of an R-module M and $\langle 0\rangle(+)N$ is an sdf-absorbing ideal of $R(+)M$, then $N = \{0\}$ is an sdf-absorbing submodule of M. However, this need not be true in the weakly sdf-absorbing case as we can see in the following example.

Example 8. *Let $R = M = \mathbb{Z}_2[x]/\langle x^4\rangle$, Then $\langle\bar{0}\rangle(+)\langle\bar{x}^2\rangle$ is not an sdf-absorbing ideal of $R(+)M$ by [6, Remark 4.18]. However, $\langle\bar{0}\rangle(+)\langle\bar{x}^2\rangle$ is a weakly sdf-absorbing ideal of $R(+)M$. Indeed, if $(a, m_1), (b, m_2) \in R(+)M$ such that $(a^2 - b^2, 2am_1 - 2bm_2) = (a, m_1)^2 - (b, m_2)^2 \in \langle\bar{0}\rangle(+)\langle\bar{x}^2\rangle$, then $char(R) = 2$ implies $(a, m_1)^2 - (b, m_2)^2 = 0$. Moreover, $N = \langle\bar{x}^2\rangle$ is not a weakly sdf-absorbing submodule of M since for example, $\bar{0} \neq (\bar{x}^2 - \bar{0}^2).\bar{1} \in N$ but $\bar{x} \notin N$.*

In fact, if R is a ring of characteristic 2 and M is any R-module, then the ideal $\langle 0\rangle(+)N$ in $R(+)M$ is weakly sdf-absorbing but not sdf-absorbing for every proper submodule N of M. This provides a wide class of examples of weakly sdf-absorbing ideals that fail to be sdf-absorbing.

We now return to Proposition 1 and present an example illustrating that the condition $\text{Ann}_M(I) = \{0\}$ is essential for ensuring that $(N :_M I)$ is weakly sdf-absorbing submodule of an R-module M provided N itself has this property.

Example 9. *Let $R = M = \mathbb{Z}_2[x]/\langle x^4\rangle$ and consider the idealization ring $R(+)M$. Define the submodule $N = \langle\bar{0}\rangle(+)\langle\bar{x}^2\rangle$ and the ideal $I = \langle\bar{x}^2\rangle(+)\langle\bar{x}\rangle$. As shown in Example 8, N is a weakly sdf-absorbing submodule of $R(+)M$. However, straightforward computation reveals that $(0 :_{R(+)M} I) = \langle\bar{x}^2\rangle(+)\left\langle\overline{x^2 - x}\right\rangle \neq (\bar{0},\bar{0})$ demonstrating that the annihilator of I is non-zero. On the other hand, we find that $(N :_{R(+)M} I) = \langle\bar{x}^2\rangle(+)M$ which is not weakly sdf-absorbing in $R(+)M$ since $(0,0) \neq (\bar{x}^2 - \bar{0}^2)(\bar{1},\bar{1}) \in (N :_{R(+)M} I)$ but $(\bar{x},\bar{x}) \notin (N :_{R(+)M} I)$.*

Let R be a ring, J an ideal of R and M an R-module. The amalgamated duplication of R along J is defined as

$$R \bowtie J = \{(r, r+j) : r \in R\,, j \in J\}$$

which is a subring of $R \times R$, see [12]. The duplication of the R-module M along the ideal J denoted by $M \bowtie J$ is defined recently in [11] as

$$M \bowtie J = \{(m, m') \in M \times M : m - m' \in JM\}$$

which is an $(R \bowtie J)$-module with scalar multiplication defined by $(r, r+j)(m, m') = (rm, (r+j)m')$ for $r \in R$, $j \in J$ and $(m, m') \in M \bowtie J$. Many properties and results concerning this kind of modules can be found in [11].

Let N be a submodule of an R-module M and J be an ideal of R. Then clearly

$$N \bowtie J = \{(a, b) \in N \times M : a - b \in JM\}$$

and

$$\bar{N} = \{(b, a) \in M \times N : b - a \in JM\}$$

are submodules of $M \bowtie J$.

In general, let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. The subring

$$R_1 \bowtie^f J = \{(r, f(r) + j) : r \in R_1, j \in J\}$$

of $R_1 \times R_2$ is called the amalgamation of R_1 and R_2 along J with respect to f. In [13], the amalgamation of M_1 and M_2 along J with respect to φ is defined as

$$M_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1 \text{ and } m_2 \in JM_2\}$$

$$M_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1 \text{ and } m_2 \in JM_2\}$$

which is an $(R_1 \bowtie^f J)$-module with the scalar product defined as

$$(r, f(r) + j)(m_1, \varphi(m_1) + m_2) = (rm_1, \varphi(rm_1) + f(r)m_2 + j\varphi(m_1) + jm_2)$$

For submodules $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$, the sets

$$N_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2 : m_1 \in N_1\}$$

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and

$$\overline{N_2}^{\varphi} = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2 : \ \varphi(m_1) + m_2 \in N_2\}$$

are submodules of $M_1 \bowtie^{\varphi} JM_2$.

The above notation will be used throughout the rest of this section. In the following two theorems, we justify conditions under which the submodules $N_1 \bowtie^{\varphi} JM_2$ and $\overline{N_2}^{\varphi}$ are weakly sdf-absorbing in $M_1 \bowtie^{\varphi} JM_2$.

Theorem 12. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^{\varphi} JM_2$ defined as above and let N_1 be a submodule of M_1. The following are equivalent.*

1. $N_1 \bowtie^{\varphi} JM_2$ is a weakly sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$.
2. N_1 is a weakly sdf-absorbing submodule of M_1 and whenever (r_1, s_1, m_1) is an sdf-absorbing triple-zero of N_1, then $[(f(r_1) + j_1)^2 - (f(s_1) + j_2)^2](\varphi(m_1) + m_2) = 0_{M_2}$ for every $j_1, j_2 \in J$ and $m_2 \in JM_2$.

Proof. Firstly, we note that N_1 is a proper submodule of M_1 if and only if $N_1 \bowtie^{\varphi} JM_2$ is a proper submodule of $M_1 \bowtie^{\varphi} JM_2$.

(1) $\Longrightarrow$ (2) Suppose $N_1 \bowtie^{\varphi} JM_2$ is a weakly sdf-absorbing in $M_1 \bowtie^{\varphi} JM_2$ and let $r_1, s_1 \in R_1$, $m_1 \in M_1$ such that $0 \neq (r_1^2 - s_1^2)m_1 \in N_1$. Then $(r_1, f(r_1)), (s_1, f(s_1)) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1)) \in M_1 \bowtie^{\varphi} JM_2$. Compute:

$$\begin{aligned}(0,0) &\neq \left[(r_1, f(r_1))^2 - (s_1, f(s_1))^2\right](m_1, \varphi(m_1)) \\ &= ((r_1^2 - s_1^2)m_1, \varphi((r_1^2 - s_1^2)m_1)) \in N_1 \bowtie^{\varphi} JM_2\end{aligned}$$

By the weakly sdf-absorbing property, it follows that either

$$\begin{aligned}((r_1 + s_1)m_1, \varphi((r_1 + s_1)m_1)) &= (r_1 + s_1, f(r_1 + s_1))(m_1, \varphi(m_1)) \\ &= [(r_1, f(r_1)) + (s_1, f(s_1))](m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2\end{aligned}$$

or

$$\begin{aligned}((r_1 - s_1)m_1, \varphi((r_1 - s_1)m_1)) &= (r_1 - s_1, f(r_1 - s_1)(m_1, \varphi(m_1)) \\ &= [(r_1, f(r_1)) - (s_1, f(s_1))](m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2\end{aligned}$$

Therefore, either, $(r_1 + s_1)m_1 \in N_1$ or $(r_1 - s_1)m_1 \in N_1$, showing that N_1 is a weakly sdf-absorbing submodule of M_1. Now, suppose, toward a contradiction, that (r_1, s_1, m_1) is an sdf-absorbing triple-zero of N_1 and $[(f(r_1) + j_1)^2 - (f(s_1) + j_2)^2](\varphi(m_1) + m_2) \neq 0_{M_2}$ for some $j_1, j_2 \in J$ and $m_2 \in JM_2$. Then

$$\begin{aligned}(0,0) &\neq [(r_1, f(r_1) + j_1)^2 - (s_1, f(s_1) + j_2)^2](m_1, \varphi(m_1) + m_2) \\ &= [0, ((f(r_1) + j_1)^2 - (f(s_1) + j_2)^2)(\varphi(m_1) + m_2)] \in N_1 \bowtie^{\varphi} JM_2\end{aligned}$$

Again, applying the weakly sdf-absorbing property, we have:

$$(r_1 + s_1, f(r_1 + s_1) + j_1 + j_2)(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2$$

or

$$(r_1 - s_1, f(r_1 - s_1) + j_1 - j_2)(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2$$

This implies either $(r_1 + s_1)m_1 \in N_1$ or $(r_1 - s_1)m_1 \in N_1$ contradicting the assumption that (r_1, s_1, m_1) is an sdf-absorbing triple-zero of N_1. Hence, for all $j_1, j_2 \in J$ and $m_2 \in JM_2$, $[(f(r_1) + j_1)^2 - (f(s_1) + j_2)^2](\varphi(m_1) + m_2) = 0_{M_2}$.

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(2) $\Longrightarrow$ (1) Let $(r_1, f(r_1)+j_1), (s_1, f(s_1)+j_2) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1)+m_2) \in M_1 \bowtie^\varphi JM_2$ such that

$$(0,0) \neq [(r_1, f(r_1)+j_1)^2 - (s_1, f(s_1)+j_2)^2](m_1, \varphi(m_1)+m_2)$$
$$= [(r_1^2 - s_1^2)m_1, ((f(r_1)+j_1)^2 - (f(s_1)+j_2)^2)(\varphi(m_1)+m_2)] \in N_1 \bowtie^\varphi JM_2$$

If $(r_1^2 - s_1^2)m_1 \neq 0$, then by assumption, $(r_1+s_1)m_1 \in N_1$ or $(r_1-s_1)m_1 \in N_1$. Thus, the corresponding element

$$(r_1 \pm s_1, f(r_1 \pm s_1) + j_1 \pm j_2)(m_1, \varphi(m_1)+m_2) \in N_1 \bowtie^\varphi JM_2$$

If $(r_1^2 - s_1^2)m_1 = 0$, then $[(f(r_1)+j_1)^2 - (f(s_1)+j_2)^2](\varphi(m_1)+m_2) \neq 0_{M_2}$. Therefore, (r_1, s_1, m_1) is not an sdf-absorbing triple-zero of N_1 and so either $(r_1+s_1)m_1 \in N_1$ or $(r_1-s_1)m_1 \in N_1$. This implies again

$$(r_1 \pm s_1, f(r_1 \pm s_1) + j_1 \pm j_2)(m_1, \varphi(m_1)+m_2) \in N_1 \bowtie^\varphi JM_2$$

Thus, $N_1 \bowtie^\varphi JM_2$ is a weakly sdf-absorbing submodule of $M_1 \bowtie^\varphi JM_2$. □

Corollary 3. *Let N be a submodule of an R-module M and J be an ideal of R. Then $N \bowtie J$ is a weakly sdf-absorbing submodule of $M \bowtie J$ if and only if N is a weakly sdf-absorbing submodule of M and whenever (r,s,m) is an sdf-absorbing triple-zero of N, then $[(r+j_1)^2 - (s+j_2)^2](m+m_1) = 0$ for every $j_1, j_2 \in J$ and $m_1 \in JM$.*

Theorem 13. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ defined as in Theorem 12 where f and φ are epimorphisms and let N_2 be a submodule of M_2. The following are equivalent.*

1. $\overline{N_2}^\varphi$ is a weakly sdf-absorbing submodule of $M_1 \bowtie^\varphi JM_2$.
2. N_2 is a weakly sdf-absorbing submodule of M_2 and whenever $(f(r_1)+j_1, f(s_1)+j_2, \varphi(m_1)+m_2)$ is an sdf-absorbing triple-zero of N_2 for some $j_1, j_2 \in J$ and $m_2 \in JM_2$, then $(r_1^2 - s_1^2)m_1 = 0$.

Proof. (1) $\Rightarrow$ (2): Suppose $\overline{N_2}^\varphi$ is a weakly sdf-absorbing submodule of $M_1 \bowtie_J^\varphi M_2$. Clearly, N_2 is proper in M_2. Let $r_2, s_2 \in R_2$ and $m_2 \in M_2$ such that $0 \neq (r_2^2 - s_2^2)m_2 \in N_2$. Since f and φ are epimorphisms, there exist $r_1, s_1 \in R_1$ and $m_1 \in M_1$ such that $r_2 = f(r_1)$, $s_2 = f(s_1)$, and $m_2 = \varphi(m_1)$. Then:

$$[(r_1, r_2)^2 - (s_1, s_2)^2](m_1, m_2) = ((r_1^2 - s_1^2)m_1, (r_2^2 - s_2^2)m_2) \in \overline{N_2}^\varphi,$$

with $(0,0) \neq ((r_1^2 - s_1^2)m_1, (r_2^2 - s_2^2)m_2)$. Since $\overline{N_2}^\varphi$ is weakly sdf-absorbing, we have either:

$$(r_1+s_1, r_2+s_2)(m_1, m_2) \in \overline{N_2}^\varphi \quad \text{or} \quad (r_1-s_1, r_2-s_2)(m_1, m_2) \in \overline{N_2}^\varphi,$$

which implies $(r_2+s_2)m_2 \in N_2$ or $(r_2-s_2)m_2 \in N_2$. Hence, N_2 is weakly sdf-absorbing in M_2. Now, suppose $(f(r_1)+j_1, f(s_1)+j_2, \varphi(m_1)+m_2)$ is an sdf-absorbing triple-zero of N_2, and suppose, for contradiction, that $(r_1^2 - s_1^2)m_1 \neq 0$. Then:

$$[(r_1, f(r_1)+j_1)^2 - (s_1, f(s_1)+j_2)^2](m_1, \varphi(m_1)+m_2) = ((r_1^2 - s_1^2)m_1, 0) \in \overline{N_2}^\varphi.$$

Since $\overline{N_2}^\varphi$ is weakly sdf-absorbing, one of the following must hold:

$$(r_1+s_1, f(r_1+s_1)+j_1+j_2)(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^\varphi,$$

$$\text{or } (r_1-s_1, f(r_1-s_1)+j_1-j_2)(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^\varphi.$$

This implies:

$$(f(r_1+s_1)+j_1+j_2)(\varphi(m_1)+m_2) \in N_2 \quad \text{or} \quad (f(r_1-s_1)+j_1-j_2)(\varphi(m_1)+m_2) \in N_2,$$

which contradicts the assumption. Therefore, $(r_1^2 - s_1^2)m_1 = 0$.

(2)$\Longrightarrow$(1) Suppose (2) holds. We start by showing that $\overline{N_2}^{\varphi}$ is proper in $M_1 \bowtie^{\varphi} JM_2$. Suppose $\overline{N_2}^{\varphi} = M_1 \bowtie^{\varphi} JM_2$ and let $m_2 = \varphi(m_1) \in M_2$ for some $m_1 \in M_1$. Then $(m_1, m_2) \in M_1 \bowtie^{\varphi} JM_2 = \overline{N_2}^{\varphi}$ and so $m_2 \in N_2$. Thus, $N_2 = M_2$ which is a contradiction. Now, let $(r_1, f(r_1) + j_1), (s_1, f(s_1) + j_2) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2$ such that

$$\begin{aligned}(0,0) &\neq [(r_1, f(r_1) + j_1)^2 - (s_1, f(s_1) + j_2)^2](m_1, \varphi(m_1) + m_2) \\ &= [(r_1^2 - s_1^2)m_1, ((f(r_1) + j_1)^2 - (f(s_1) + j_2)^2)(\varphi(m_1) + m_2)] \in \overline{N_2}^{\varphi}\end{aligned}$$

If the second component is nonzero, then by assumption,

$$(f(r_1 + s_1) + j_1 + j_2)(\varphi(m_1) + m_2) \in N_2 \text{ or } (f(r_1 - s_1) + j_1 - j_2)(\varphi(m_1) + m_2) \in N_2$$

Hence the corresponding element is in $\overline{N_2}^{\varphi}$. If the second component is zero, then $(r_1^2 - s_1^2)m_1 \neq 0$. Thus $(f(r_1) + j_1, f(s_1) + j_2, \varphi(m_1) + m_2)$ is not an sdf-absorbing triple-zero of N_2 and so again either $(f(r_1 + s_1) + j_1 + j_2)(\varphi(m_1) + m_2) \in N_2$ or $(f(r_1 - s_1) + j_1 - j_2)(\varphi(m_1) + m_2) \in N_2$. Thus, again the corresponding element is in $\overline{N_2}^{\varphi}$. Therefore, $\overline{N_2}^{\varphi}$ is a weakly sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$. □

Corollary 4. *Let N be a submodule of an R-module M and J be an ideal of R . Then $\overline{N}$ is a weakly sdf-absorbing submodule of $M \bowtie J$ if and only if N is a weakly sdf-absorbing submodule of M and whenever $(r + j_1, s + j_2, m + m_1)$ is an sdf-absorbing triple-zero of N for some $j_1, j_2 \in J$ and $m_1 \in JM$, then $(r^2 - s^2)m = 0$.*

The following example demonstrates that, in the absence of the condition established in Corollary 3 (resp. Corollary 4), N is a weakly sdf-absorbing submodule of M does not necessarily imply $N \bowtie J$ (resp. $\overline{N}$) is a weakly sdf-absorbing submodule of $M \bowtie J$.

Example 10. *Let $R = \mathbb{Z}$, $J = 2\mathbb{Z}$ and $M = \mathbb{Z}_{12}$ so that $R \bowtie J = \{(r, r') : r - r' \in 2\mathbb{Z}\}$ and $M \bowtie J = \{(m, m') \in M \times M : m - m' \in \langle\bar{2}\rangle\}$. Then $N = \langle\bar{0}\rangle$ is a weakly sdf-absorbing submodule of M. On the other hand,*

1. $N \bowtie J = \{(\bar{0}, \bar{0}), (\bar{1}, \bar{1}), \cdots (\overline{11}, \overline{11})\}$ is not a weakly sdf-absorbing submodule of the $M \bowtie J$. Indeed, let $a = (7, 5)$, $b = (6, 0) \in R \bowtie J$ and $m = (\bar{1}, \bar{1}) \in M \bowtie J$. Then $(a^2 - b^2)m = (\bar{1}, \bar{1}) \in N \bowtie J \backslash \{(\bar{0}, \bar{0})\}$ but $(a + b)m = (a - b)m = (\bar{1}, \bar{5}) \notin N \bowtie J$. We note that clearly, $(4, 2, \bar{1})$ is an sdf-absorbing triple-zero of N and $j_1 = j_2 = 2 \in J$, but $((4 + j_1)^2 - (2 + j_2)^2)(\bar{1}) = 20.\bar{1} \neq 0$.
2. $\overline{N} = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{0}), (\bar{4}, \bar{0}), (\bar{6}, \bar{0}), (\bar{8}, \bar{0}), (\overline{10}, \bar{0})\}$ is not a weakly sdf-absorbing submodule of the $M \bowtie J$. Indeed, let $a = (6, 4)$, $b = (4, 2) \in R \bowtie J$ and $m = (\bar{1}, \bar{1}) \in M \bowtie J$. Then $(a^2 - b^2)m = (\bar{8}, \bar{0}) \in \overline{N} \backslash \{(\bar{0}, \bar{0})\}$ but $(a + b)m = (\overline{10}, \bar{6}) \notin \overline{N}$ and $(a - b)m = (\bar{2}, \bar{2}) \notin \overline{N}$. Note that for $j_1 = j_2 = 2 \in J$, the triple $(2 + j_1, 0 + j_2, \bar{1}) = (4, 2, \bar{1})$ is an sdf-absorbing triple-zero of N but $(2^2 - 0^2)(\bar{1}) \neq 0$.

Example 11. *In this example, we show that if the homomorphism $\varphi : M_1 \longrightarrow M_2$ is not an epimorphism, then the equivalence in Theorem 13 need not be holds.*

1. Let $R_1 = R_2 = \mathbb{Z}$, $M_1 = \mathbb{Z}_4$, $M_2 = \mathbb{Z}_8$, $J = 0$, $f = Id_{R_1}$ and $\varphi : M_1 \longrightarrow M_2$ defined by $\varphi(x) = 2x$. Then $R_1 \bowtie^f J = \{(r, r) : r \in \mathbb{Z}\}$ and $M_1 \bowtie^{\varphi} JM_2 = \{(m, 2m) : m \in \mathbb{Z}_4\} = \mathbb{Z}_4 \times \langle\bar{2}\rangle$. The submodule $N_2 = \langle\bar{4}\rangle$ of M_2 is not weakly sdf-absorbing since $0 \neq (2^2 - 0^2).\bar{1} \in N_2$ but $2.\bar{1} \notin N_2$. Now, consider $\overline{N_2}^{\varphi} = \{(m, 2m) : 2m \in N_2\} = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{4})\}$. Let $(a, a), (b, b) \in R_1 \bowtie^f J$ and $(m, 2m) \in M_1 \bowtie^{\varphi} JM_2$ such that $(\bar{0}, \bar{0}) \neq [(a, a)^2 - (b, b)^2)(m, 2m) \in \overline{N_2}^{\varphi}$. Then $(a^2 - b^2)m \in \{\bar{0}, \bar{2}\}$ and $2(a^2 - b^2)m \in \{\bar{0}, \bar{4}\}$. If $(a^2 - b^2)m = 0_{M_1}$, then $2(a^2 - b^2)m = 0_{M_2}$ and so $[(a, a)^2 - (b, b)^2)(m, 2m) = (\bar{0}, \bar{0})$, a contradiction. Therefore, $(a^2 - b^2)m = \bar{2}$ and $2(a^2 - b^2)m = \bar{4}$.

Clearly, $a^2 - b^2 \notin 4\mathbb{Z}$ and $a^2 - b^2 \neq 2, 6, 10, \cdots$. Thus, $a^2 - b^2 \notin 2\mathbb{Z}$ and so we must have $m = \bar{2}$. Hence, $(m, 2m) = (\bar{2}, \bar{4}) \in \overline{N_2}^{\varphi}$ and clearly $\overline{N_2}^{\varphi}$ is a weakly sdf-absorbing in $M_1 \bowtie^{\varphi} JM_2$.

2. Let $R_1 = M_1 = R_2 = \mathbb{Z}$, $M_2 = \mathbb{Z}_{12}$ and $J = \langle 0_{R_2} \rangle$. Define $\varphi : M_1 \to M_2$ by $\varphi(x) = 6x$ and let $N_2 = \langle \bar{6} \rangle$. Then N_2 is a (weakly) sdf-absorbing submodule of M_2 by [19, Proposition 4]. On the other hand $\overline{N_2}^{\varphi} = \{(m_1, 6m_1) : 6m_1 \in \langle \bar{6} \rangle\} = M_1 \bowtie^{\varphi} JM_2$ is not a weakly sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$.

Theorem 14. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^{\varphi} JM_2$ defined as in Theorem 12 where f and φ are epimorphisms and $Ann_{M_2}(J) = 0$. Let N_2 be a submodule of M_2. If $\overline{N_2}^{\varphi}$ is a weakly sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$ and $J \nsubseteq (N_2 :_{R_2} M_2)$, then $(N_2 :_{M_2} J)$ is a weakly sdf-absorbing submodule of M_2.*

Proof. Since $J \nsubseteq (N_2 :_{R_2} M_2)$, $(N_2 :_{M_2} J)$ is proper in M_2. Suppose $\overline{N_2}^{\varphi}$ is a weakly sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$. Let $r_2 = f(r_1), s_2 = f(s_1) \in R_2$ and $m_2 \in M_2$ such that $0 \neq (r_2^2 - s_2^2)m_2 \in (N_2 :_{M_2} J)$. Then $(r_2^2 - s_2^2)Jm_2 \subseteq N_2$ and $Ann_{M_2}(J) = 0$ implies $(r_2^2 - s_2^2)Jm_2 \neq 0$. Thus, $[(r_1, r_2)^2 - (s_1, s_2)^2](0, Jm_2) \subseteq \overline{N_2}^{\varphi} \backslash \{(0,0)\}$ and by assumption, we have either $[(r_1, r_2) + (s_1, s_2)](0, Jm_2) \subseteq \overline{N_2}^{\varphi}$ or $[(r_1, r_2) - (s_1, s_2)](0, Jm_2) \subseteq \overline{N_2}^{\varphi}$. Thus, $(r_2 + s_2)Jm_2 \subseteq N_2$ or $(r_2 - s_2)Jm_2 \subseteq N_2$. It follows that $(r_2 + s_2)m_2 \in (N_2 :_{M_2} J)$ or $(r_2 - s_2)m_2 \in (N_2 :_{M_2} J)$ and so $(N_2 :_{M_2} J)$ is a weakly sdf-absorbing submodule of M_2. □

However, the converse of Theorem 14 need not be true in general.

Example 12. *Let $R_1 = M_1 = R_2 = \mathbb{Z}$, $M_2 = \mathbb{Z}_{12}$ and $J = 2\mathbb{Z}$. Let $f : R_1 \to R_2$, $\varphi : M_1 \to M_2$ be the identity epimorphisms and $N_2 = \langle \bar{4} \rangle$. Then $(N_2 :_{M_2} J) = \langle \bar{2} \rangle$ is clearly a (weakly) sdf-absorbing submodule of M_2. But, N_2 is not a weakly sdf-absorbing submodule of M_2 since $0 \neq (2^2 - 0^2).\bar{1} \in N_2$ but $2.\bar{1} \notin N_2$. Thus, $\overline{N_2}^{\varphi}$ is not a weakly sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$ by Theorem 13.*

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