

Concept Paper

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Finding a Research Paper Which Meaningfully Averages Unbounded Sets

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Keywords: Expected Values; Sets; Hausdorff measure; Hausdorff dimension; Discretization; Segmentation; Partitions; Samples; Euclidean Distance; Choice Function



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Concept Paper

Finding a Published Research Paper Which Meaningfully Averages Unbounded Sets (v4)

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Abstract: Suppose $n \in \mathbb{N}$. We wish to meaningfully average 'sophisticated' unbounded sets (i.e., sets with positive n -d Hausdorff measure, in any n -d box of the n -d plane, where the measures do not equal the area of the boxes). We do this by taking an unique, *satisfying* extension of the expected value, w.r.t the Hausdorff measure in its dimension, on bounded sets taking finite values only. As of now, I'm unable to define this due to limited knowledge of advanced math and most people are too busy to help. Therefore, I'm wondering if anyone knows a research paper which solves my doubts. (Unlike previous versions with similar names, we made everything rigorous).

Keywords: expected values; sets; hausdorff measure; hausdorff dimension; discretization; segmentation; partitions; samples; euclidean distance; choice function

1. Intro

Let $n \in \mathbb{N}$. Suppose $A \subseteq \mathbb{R}^n$ is Borel, where:

- $\mathbf{U}(\cdot)$ is the set of all unbounded Borel subsets of a set
- $\mathbf{B}(\cdot) = \mathcal{P}(\cdot) \setminus \mathbf{U}(\cdot)$ is the set of all bounded Borel subsets of a set

In addition, $\dim_{\mathbf{H}}(\cdot)$ is the Hausdorff dimension and $\mathcal{H}^{\dim_{\mathbf{H}}(\cdot)}(\cdot)$ is the Hausdorff measure *in its dimension* on the Borel σ -algebra.

1.1. First special case of A

Suppose the n -dimensional box is B . We want an **explicit** A , such that:

1. $\mathcal{H}^n(A \cap B) > 0$ for all B
2. $\mathcal{H}^n(A \cap B) \neq \mathcal{H}^n(B)$ for all B .

This case of A is called A' .

1.1.1. Potential Answer

If $n = 1$, using this reddit post [1], define A such that:

Find some set $A \subseteq \mathbb{R}$, where $0 < \mathcal{H}^1(A \cap I) < \mathcal{H}^1(I)$ for every interval I .

Hence, construct a subset of $[0, 1]$ with this property (then copy and paste this set to get the desired set $A' \subset \mathbb{R}$):

We do this by constructing a strange map from $[0, 1] \rightarrow \mathbb{R}$. Take a real number $x \in [0, 1]$, expand that number in binary as $0.b_0b_1b_2\cdots$ and map the value to the series $\sum_{n=1}^{\infty} (2(b_n) - 1)/n$. It's possible using Khintchine's inequality [2, p.187-205] to show the sum converges for a.e. $x \in [0, 1]$. Thus, our desired set A' will just consist of those x for which the sum is positive.

The fact this set works is a little bit annoying to prove, but relies on Khintchine's inequality [2, p.187-205] and the divergence of the Harmonic series. Essentially, we want to show that for any initial sequence $b_0, b_1, \cdots, b_n$ of digits there is a positive probability that the final sum is positive and a positive probability that the final sum is negative.

Note, in case there is a research paper which can average A' into a finite number, consider the next example.

1.2. Second special case of A

Suppose the n -dimensional box is B . Is there an explicit A , such that:

1. $\mathcal{H}^n(A \cap B) > 0$ for all B
2. $\mathcal{H}^n(A \cap B) \neq \mathcal{H}^n(B)$ for all B
3. For all n -d boxes $\beta \subseteq B$:
 - (a) $\mathcal{H}^n(B \setminus \beta) > \mathcal{H}^n(\beta) \Rightarrow \mathcal{H}^n(A \cap (B \setminus \beta)) < \mathcal{H}^n(A \cap \beta)$
 - (b) $\mathcal{H}^n(B \setminus \beta) < \mathcal{H}^n(\beta) \Rightarrow \mathcal{H}^n(A \cap (B \setminus \beta)) > \mathcal{H}^n(A \cap \beta)$
 - (c) $\mathcal{H}^n(B \setminus \beta) = \mathcal{H}^n(\beta) \Rightarrow \mathcal{H}^n(A \cap (B \setminus \beta)) \neq \mathcal{H}^n(A \cap \beta)$

If explicit A exists, what is an example? If not, how does one modify Section 1.2, so that it exists? If such a set exists, this case of A is called A'' . (We have yet to prove or disprove A'' exists.)

1.3. Attempting to Analyze/Average A

The expected value of A , w.r.t the Hausdorff measure in its dimension, is:

$$\text{Avg}(A) = \frac{1}{\mathcal{H}^{\dim_H(A)}(A)} \int_A (x_1, \dots, x_n) d\mathcal{H}^{\dim_H(A)} \quad (1)$$

Nevertheless, we have the following problems:

Notation 1 (Problem 1). When $A = A'$ (Section 1.1), $\text{Avg}(A')$ is undefined.

1.3.1. Explanation of Problem 1

When $n = 1$, using Section 1.1 and Section 1.1.1, since $\mathcal{H}^1(A' \cap B) > 0$ for all B , notice $\dim_H(A') = 1$ and $\mathcal{H}^{\dim_H(A')}(A') = +\infty$. This makes $\text{Avg}(A')$ undefined due to division by infinity. Similarly, the most obvious extension of $\text{Avg}(A')$ or $\mathcal{A}(A')$ is also undefined; i.e., given the absolute value is $|| \cdot ||$, then the following is false since $\mathcal{H}^1(A' \cap B) \neq \mathcal{H}^1(B)$ for all B :

$$\exists(\mathcal{A}(A') \in \mathbb{R}) \forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(t \in \mathbb{N}) (t \geq N \Rightarrow ||\text{Avg}(A' \cap [-t, t]) - \mathcal{A}(A')|| < \epsilon)$$

Notation 2 (Problem 2). Suppose $\mathbf{U}(\mathbb{R}^n)$ is the set of all unbounded Borel subsets of $\mathbb{R}^n$ and $|| \cdot ||_n$ is the n -d Euclidean norm. If $\mathcal{U} \subseteq \mathbf{U}(\mathbb{R}^n)$ is the set of all $A \in \mathbf{U}(\mathbb{R}^n)$, where $||\text{Avg}(A)||_n$ is finite and the cardinality is $|\cdot|$, then $|\mathcal{U}| < |\mathbf{U}(\mathbb{R}^n)|$ (i.e., $\mathcal{U}$ is “measure zero” in $\mathbf{U}(\mathbb{R}^n)$)

1.3.2. Explanation of Problem 2

All unbounded A have means with a finite n -d norm when the symmetry lines of A intersect at one point. Thus, for any sequence of bounded sets $(F_r)_{r \in \mathbb{N}}$ whose set-theoretic limit (Section 1.3.3) is A , such that the lines of symmetry of A intersect at one point, $\text{Avg}((F_r), A)$ in the following statement ($|| \cdot ||$ is the absolute value):

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{\mathcal{H}^{\dim_H(F_r)}(F_r)} \int_{F_r} (x_1, \dots, x_n) d\mathcal{H}^{\dim_H(F_r)} - \text{Avg}((F_r), A) \right\| < \epsilon \right) \quad (2)$$

always exists and has the same value.

Since the cardinality of the set of all unbounded sets with one or more symmetry lines is $\mathfrak{c}$, the cardinality of all unbounded sets whose lines of symmetry intersect at one point is less than or equal to $\mathfrak{c}$. Hence, since the cardinality of the collection of all unbounded subsets of $\mathbb{R}^n$ is $2^{\mathfrak{c}}$, $|\mathbf{U}(\mathbb{R}^n)| = 2^{\mathfrak{c}}$ and $|\mathcal{U}| = \mathfrak{c}$. Thus, we conclude $\mathfrak{c} = |\mathcal{U}| < |\mathbf{U}(\mathbb{R}^n)| = 2^{\mathfrak{c}}$, which shows problem 2 is correct.

Therefore, suppose:

1.3.3. Set Theoretic Limit of a Sequence of Sets

The set-theoretic limit of a sequence of sets of bounded set $(F_r)_{r \in \mathbb{N}}$ is A when:

$$\limsup_{r \rightarrow \infty} F_r = \bigcap_{r \geq 1} \bigcup_{q \geq r} F_q$$

$$\liminf_{r \rightarrow \infty} F_r = \bigcup_{r \geq 1} \bigcap_{q \geq r} F_q$$

where:

$$\limsup_{r \rightarrow \infty} F_r = \liminf_{r \rightarrow \infty} F_r = A$$

which we represent with the notation:

$$F_r \rightarrow A$$

Now, consider an approach to solving problems 1 and 2:

1.3.4. Approach

Suppose $\mathbf{U}(\mathbb{R}^n)$ is the set of all unbounded Borel subsets of $\mathbb{R}^n$ and $\mathbf{B}(\mathbb{R}^n)$ is the set of all bounded Borel subsets of $\mathbb{R}^n$, where $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$ is an arbitrary set. In addition, suppose $\|\cdot\|_n$ is the n -d Euclidean norm. If $\mathcal{U} \subset \mathbf{U}(\mathbb{R}^n)$ is the set of all $A \in \mathcal{U}$, where for all $r \in \mathbb{N}$ and $F_r \in \mathcal{B}$, such that $F_r \rightarrow A$ (Section 1.3.3), $\text{Avg}((F_r), A)$ is unique, $\text{Avg}((F_r), A)$ is satisfying (Section 3), and $\|\text{Avg}((F_r), A)\|_n$ is finite:

1. $|\mathbf{U}(\mathbb{R}^n) \setminus \mathcal{U}| < |\mathbf{U}(\mathbb{R}^n)|$ (i.e., $\mathcal{U}$ is "almost everywhere" in $\mathbf{U}(\mathbb{R}^n)$)
2. If (1) isn't true, then $|\mathcal{U}| = |\mathbf{U}(\mathbb{R}^n) \setminus \mathcal{U}|$

1.3.5. Explanation of Approach

If $F_r \rightarrow A$ (Section 1.3.3) where we apply $\text{Avg}((F_r), A)$ (Section 1.3.2 Equation (2), depending on the $(F_r)_{r \in \mathbb{N}}$ chosen, $\text{Avg}((F_r), A)$ could be one of many values. Here is an example:

Example 1. Suppose $A = \mathbb{R}^2$. We define the bounded sequences of sets $(F_r^*)_{r \in \mathbb{N}} = ([-r, r] \times [-r, r])_{r \in \mathbb{N}}$ and $(F_j^{**})_{j \in \mathbb{N}} = ([-j+1, j+1] \times [-j-1, j-1])_{j \in \mathbb{N}}$, where $F_r^*, F_j^{**} \rightarrow A$ (Section 1.3.3). Note, the center of F_r^* , for all $r \in \mathbb{N}$, is $(0, 0)$ and the center of F_j^{**} , for all $j \in \mathbb{N}$, is $(1, -1)$. Thus, $\text{Avg}((F_r^*), A) = (0, 0)$ and $\text{Avg}((F_j^{**}), A) = (1, -1)$. Furthermore, depending on the (F_r^*) chosen, $\text{Avg}((F_r^*), A)$ can be any point in $\mathbb{R}^2$ (when it exists).

Therefore, we ask the following:

1.4. Question

Is there a unique way to define "satisfying" in the approach of Section 1.3.4, which solves problems 1 and 2 with applications?

2. Extending the Expected Value of A w.r.t the Hausdorff Measure

The following are two methods to finding the most generalized, satisfying extension of $\text{Avg}(A)$ in Equation (1) which we later use to answer the question in Section 1.4:

1. One way is defining a generalized, satisfying extension of the Hausdorff measure, on all A with positive & finite measure which takes positive, finite values for all Borel A . This can theoretically be done in the paper "A Multi-Fractal Formalism for New General Fractal Measures"[3], where in Equation 1 we replace the Hausdorff measure with the extended Hausdorff measure.
2. Another way is finding generalized, satisfying average of all A in the fractal setting. This can be done with the papers "Analogues of the Lebesgue Density Theorem

for Fractal Sets of Reals and Integers" [4] and "Ratio Geometry, Rigidity and the Scenery Process for Hyperbolic Cantor Sets" [5] where we take the expected value of A w.r.t the densities in [4,5].

Note, neither approach answers Section 1.4, since the n -d norms of the averages in the approach are not unique for unbounded sets (ex. 1). Therefore, we ask a leading question that uses Section 2 to guide readers to an interesting solution to Section 1.4.

3. Attempt to Define "Unique and Satisfying" in The Approach of Section 1.3

3.1. Leading Question

To define *satisfying* inside the approach of Section 1.3.4, we take the expected value of a sequence of bounded sets chosen by a choice function. To find the choice function, we ask the **leading question...**

Suppose $\mathbf{B}(\mathbb{R}^n)$ is the set of all bounded Borel subsets of $\mathbb{R}^n$, where there exists an arbitrary set $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, such that for all $r, j \in \mathbb{N}$:

- (A) $F_r^* \in \mathcal{B}$
- (B) $F_j^{**} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$
- (C) $\square$ is the logical symbol for "it's necessary"
- (D) Define C to be chosen center point of $\mathbb{R}^n$ (e.g., the origin)
- (E) Define E to be the fixed, expected rate of expansion of $(F_r^*)_{r \in \mathbb{N}}$ w.r.t center point C (e.g., $E = 1$)
- (F) Define $\mathcal{E}(C, F_r^*)$ to be actual rate of expansion of $(F_r^*)_{r \in \mathbb{N}}$ w.r.t center point C (Section 5.5)

Does there exist a unique choice function which chooses an unique set $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, where for all $r, v \in \mathbb{N}$, every $F_r^* \in \mathcal{B}$ is equivalent (Section 5.3, def. 1) to $F_v' \in \mathcal{B}$, such that:

1. $F_r^* \rightarrow A$ (Section 1.3.3)
2. For all $j \in \mathbb{N}$ and $F_j^{**} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$, where $F_j^{**} \rightarrow A$ (Section 1.3.3), the "measure" (Section 5.4.1, Section 5.4.2) of $(F_r^*)_{r \in \mathbb{N}}$ must increase at rate linear or superlinear to that of $(F_j^{**})_{j \in \mathbb{N}}$
3. If $\|\cdot\|_n$ is the n -d Euclidean norm, $\text{Avg}((F_r^*), A)$ is unique and $\|\text{Avg}((F_r^*), A)\|_n$ is finite (Section 5.2).
4. For some $F_r^* \in \mathcal{B}$ satisfying (1), (2), and (3), where for all $\mathcal{B}' \subset \mathbf{B}(\mathbb{R}^n)$, given $\star \mapsto \star \star \star$, $r \mapsto k$, and $\mathcal{B} \mapsto \mathcal{B}'$ in (1), (2), and (3), s.t. $\neg \square (\text{Avg}((F_r^*), A) = \text{Avg}((F_k^{***}), A))$, note that for all $k \in \mathbb{N}$, where $F_k^{***} \in \mathcal{B}'$ satisfies (1), (2), and (3):
 - If $d(X, Y)$ is the n -d Euclidean distance between $X, Y \in \mathbb{R}^n$, then $d(\text{Avg}((F_r^*), A), C) \leq d(\text{Avg}((F_k^{***}), A), C)$
 - If $r \in \mathbb{N}$, for any linear $k_1 : \mathbb{N} \rightarrow \mathbb{N}$, where $k = k_1(r)$ and the Big-O notation is $\mathcal{O}$, there exists a function $K : \mathbb{R} \rightarrow \mathbb{R}$, where the absolute value is $|\cdot|$ and (Section 3.1.E-F):

$$\begin{aligned} \|\mathcal{E}(C, F_r^*) - E\| &= \mathcal{O}(K(\|\mathcal{E}(C, F_k^{***}) - E\|)) \\ &= \mathcal{O}(K(\|\mathcal{E}(C, F_{k_1(r)}^{***}) - E\|)) \end{aligned}$$

such that:

$$0 \leq \lim_{x \rightarrow +\infty} K(x)/x < +\infty$$

In simpler terms, "the rate of divergence" of $\|\mathcal{E}(C, F_r^*) - E\|$ (Section 3.1.E-F) is *less than or equal to* "the rate of divergence" of $\|\mathcal{E}(C, F_k^{***}) - E\|$ (Section 3.1.E-F).

5. If $\mathbf{U}(\mathbb{R}^n)$ is the set of all unbounded Borel subsets of $\mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbf{U}(\mathbb{R}^n)$ is the set of all $A \in \mathbf{U}(\mathbb{R}^n)$, where $F_r^* \in \mathcal{B}$ satisfies (1), (2), (3) and (4), then:

- $|\mathbf{U}(\mathbb{R}^n) \setminus \mathcal{U}| < |\mathbf{U}(\mathbb{R}^n)|$ (i.e., $\mathcal{U}$ is “almost everywhere” in $\mathbf{U}(\mathbb{R}^n)$)
 - When $|\mathbf{U}(\mathbb{R}^n) \setminus \mathcal{U}| \not< |\mathbf{U}(\mathbb{R}^n)|$, replace the Hausdorff measure in $\text{Avg}((F_r^*), A)$ of criteria (3) with the generalized measures and densities in Section 2, where we want $|\mathbf{U}(\mathbb{R}^n) \setminus \mathcal{U}| < |\mathbf{U}(\mathbb{R}^n)|$
 - When neither are possible, then $|\mathcal{U}| = |\mathbf{U}(\mathbb{R}^n) \setminus \mathcal{U}|$
6. Out of all choice functions which satisfy (1), (2), (3), (4), and (5) we choose the one with the simplest form, meaning for each choice function fully expanded, we take the one with the fewest variables/numbers?

(In case this is unclear, see Section 5.)

I’m convinced $\text{Avg}((F_r^*), A)$, where all $F_r^* \in \mathcal{B}$ (Section 3.1) which answers the *leading question* isn’t *unique* nor *satisfying enough* to answer Section 1.3.4. Still, adjustments are possible by changing the criteria or by adding new criteria to Section 3.1.

4. Question Regarding My Work

Most don’t have time to address everything in my research, hence I ask the following:

Is there a research paper which already solves the ideas I’m working on? (Non-published papers, such as mine [6], don’t count.)

5. Clarifying Section 3

Let $A \subseteq \mathbb{R}^n$ be an unbounded, Borel set. Suppose $\dim_{\mathbf{H}}(\cdot)$ be the Hausdorff dimension and $\mathcal{H}^{\dim_{\mathbf{H}}(\cdot)}(\cdot)$ be the Hausdorff measure *in its dimension* on the Borel σ -algebra. See Section 3.1, once reading this section, and consider the following:

Is there a simpler version of the definitions below?

5.1. Set Theoretic Limit of a Sequence of Bounded Sets

Note, the set-theoretic limit of a sequence of sets of bounded set $(F_r)_{r \in \mathbb{N}}$ is A when:

$$\limsup_{r \rightarrow \infty} F_r = \bigcap_{r \geq 1} \bigcup_{q \geq r} F_q$$

$$\liminf_{r \rightarrow \infty} F_r = \bigcup_{r \geq 1} \bigcap_{q \geq r} F_q$$

where:

$$\limsup_{r \rightarrow \infty} F_r = \liminf_{r \rightarrow \infty} F_r = A$$

which we denote $F_r \rightarrow A$

Example 2. When $A = \mathbb{R}$, we define sequences of sets $(F_r^*)_{r \in \mathbb{N}} = ([-r, r])_{r \in \mathbb{N}}$ and $(F_j^{**})_{j \in \mathbb{N}} = ([-j-1, j-1])_{j \in \mathbb{N}}$, such that $F_r^*, F_j^{**} \rightarrow A$ (Section 5.1). We aren’t sure how to prove this, but assume readers should be able to verify.

5.2. Expected Value of Bounded Sequences of Sets

If $(F_r)_{r \in \mathbb{N}}$ is a sequence of bounded sets, where $F_r \rightarrow A$ (Section 5.1) and the absolute value is $|| \cdot ||$, the expected value of A w.r.t $(F_r)_{r \in \mathbb{N}}$ (when it exists) is a real number $\text{Avg}((F_r), A)$ which satisfies the following:

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{\mathcal{H}^{\dim_{\mathbf{H}}(F_r)}(F_r)} \int_{F_r} (x_1, \dots, x_n) d\mathcal{H}^{\dim_{\mathbf{H}}(F_r)} - \text{Avg}((F_r), A) \right\| < \epsilon \right) \quad (3)$$

Note, $\text{Avg}((F_r), A)$ can be extended using Section 2.

5.2.1. Example 1

Using example 2, suppose $A = \mathbb{R}$, $(F_r^*)_{r \in \mathbb{N}} = ([-r, r])_{r \in \mathbb{N}}$, and $\text{Avg}((F_r^*), A) = 0$. Hence:

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{\mathcal{H}^{\dim_{\mathcal{H}}(F_r^*)}(F_r^*)} \int_{F_r^*} (x_1, \dots, x_n) d\mathcal{H}^{\dim_{\mathcal{H}}(F_r^*)} - \text{Avg}((F_r^*), A) \right\| < \epsilon \right) = \quad (4)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{\mathcal{H}^{\dim_{\mathcal{H}}([-r, r])}([-r, r])} \int_{[-r, r]} x_1 d\mathcal{H}^{\dim_{\mathcal{H}}([-r, r])} - 0 \right\| < \epsilon \right) = \quad (5)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{\mathcal{H}^1([-r, r])} \int_{[-r, r]} x_1 d\mathcal{H}^1 \right\| < \epsilon \right) = \quad (6)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{2r} \int_{-r}^r x_1 dx_1 \right\| < \epsilon \right) = \quad (7)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{2r} \left(\frac{1}{2} x_1^2 \right)_{-r}^r \right\| < \epsilon \right) = \quad (8)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{2r} \left(\frac{1}{2} r^2 - \frac{1}{2} (-r)^2 \right) \right\| < \epsilon \right) = \quad (9)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{2r} \left(\frac{1}{2} r^2 - \frac{1}{2} r^2 \right) \right\| < \epsilon \right) = \quad (10)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left\| \frac{1}{2r} (0) \right\| < \epsilon \right) = \quad (11)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(r \in \mathbb{N}) (r \geq N \Rightarrow 0 < \epsilon) \quad (12)$$

Since Equation (12) is true, $\text{Avg}((F_r^*), A) = 0$

5.2.2. Example 2

Using example 2, suppose $A = \mathbb{R}$, $(F_j^{**})_{j \in \mathbb{N}} = ([-j-1, j-1])_{j \in \mathbb{N}}$, and $\text{Avg}((F_j^{**}), A) = -1$. Hence:

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{\mathcal{H}^{\dim_{\mathcal{H}}(F_j^{**})}(F_j^{**})} \int_{F_j^{**}} (x_1, \dots, x_n) d\mathcal{H}^{\dim_{\mathcal{H}}(F_j^{**})} - \text{Avg}((F_j^{**}), A) \right\| < \epsilon \right) = \quad (13)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{\mathcal{H}^{\dim_{\mathcal{H}}([-j-1, j-1])}([-j-1, j-1])} \int_{[-j-1, j-1]} x_1 d\mathcal{H}^{\dim_{\mathcal{H}}([-j-1, j-1])} - (-1) \right\| < \epsilon \right) = \quad (14)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{\mathcal{H}^1([-j-1, j-1])} \int_{[-j-1, j-1]} x_1 d\mathcal{H}^1 + 1 \right\| < \epsilon \right) = \quad (15)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{(j-1) - (-j-1)} \int_{-j-1}^{j-1} x_1 dx_1 + 1 \right\| < \epsilon \right) = \quad (16)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{j+j-1+1} \left(\frac{1}{2} x_1^2 \right)_{-j-1}^{j-1} + 1 \right\| < \epsilon \right) = \quad (17)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{2j} \left(\frac{1}{2} (j-1)^2 - \frac{1}{2} (-j-1)^2 \right) + 1 \right\| < \epsilon \right) = \quad (18)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{2j} \left(\frac{1}{2} (j-1)^2 - \frac{1}{2} (j+1)^2 \right) + 1 \right\| < \epsilon \right) = \quad (19)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{2j} \left(\frac{1}{2} (j^2 - 2j + 1) - \frac{1}{2} (j^2 + 2j + 1) \right) + 1 \right\| < \epsilon \right) = \quad (20)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{2j} \left(\frac{1}{2} j^2 - j + \frac{1}{2} - \frac{1}{2} j^2 - j - \frac{1}{2} \right) + 1 \right\| < \epsilon \right) = \quad (21)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) \left(j \geq N \Rightarrow \left\| \frac{1}{2j} (-2j) + 1 \right\| < \epsilon \right) = \quad (22)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) (j \geq N \Rightarrow \|-1 + 1\| < \epsilon) = \quad (23)$$

$$\forall(\epsilon > 0) \exists(N \in \mathbb{N}) \forall(j \in \mathbb{N}) (j \geq N \Rightarrow 0 < \epsilon) = \quad (24)$$

Since Equation (24) is true, $\text{Avg}((F_j^{**}), A) = -1$.

5.3. Defining Equivelant and Non-Equivelant Sequences of Bounded Functions

Definition 1 (Equivelant Sequences of Bounded Sets). The sequences of sets $(F_r^*)_{r \in \mathbb{N}}$ and $(F_v')_{v \in \mathbb{N}}$ are equivalent, if there exists $N' \in \mathbb{N}$, where for all $r \geq N'$, there exists $v \in \mathbb{N}$, such that:

$$\mathcal{H}^{\dim_H(F_r^*)}(F_r^* \Delta F_v') = 0$$

and for all $v \geq N'$, there exists $r \in \mathbb{N}$, such that:

$$\mathcal{H}^{\dim_H(F_v')}(F_r^* \Delta F_v') = 0$$

5.3.1. Explanation

Suppose $\mathcal{P}(\mathbb{R}^n)$ is the set of all subsets of $\mathbb{R}^n$. Hence, $(F_r^*)_{r \in \mathbb{N}}$ and $(F_v')_{v \in \mathbb{N}}$ are equivalent, when for all sets $A \in \mathcal{P}(\mathbb{R}^n)$, where $F_r^*, F_v' \rightarrow A$ (Section 5.1) and $\text{Avg}((F_r^*), A)$ or $\text{Avg}((F_v'), A)$ exist, the following is true:

$$\text{Avg}((F_r^*), A) = \text{Avg}((F_v'), A) \quad (\text{Section 5.2})$$

Thus, consider:

Theorem 1. If $(F_r^*)_{r \in \mathbb{N}}$ and (F_v') are equivalent, then for all $A \in \mathcal{P}(\mathbb{R}^n)$, where $F_r^*, F_v' \rightarrow A$ (Section 5.1):

$$\text{Avg}((F_r^*), A) = \text{Avg}((F_v'), A) \quad (\text{Section 5.2})$$

Hence, consider the following:

5.3.2. Example of Equivalent Sequences of Bounded Sets

Suppose:

$$(F_r^*)_{r \in \mathbb{N}} = ([-r-2, r+2])_{r \in \mathbb{N}}$$

$$(F_v')_{v \in \mathbb{N}} = ([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1]))_{v \in \mathbb{N}}$$

Now, suppose $N' = 3$. Thus, using def. 1, we prove:

1. For all $r \geq N' = 3$, there exists a $5 \leq v =: r+2 \in \mathbb{N}$, where:

$$\mathcal{H}^{\dim_H(F_r^*)}(F_r^* \Delta F_v') = 0 \quad (25)$$

which is proven with the following:

In Equation (25), since $F_r^* = [-r-2, r+2]$ is a 1-d interval, $\dim_H(F_r^*) = 1$. Hence,

$$\mathcal{H}^1([-r-2, r+2] \Delta ([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1]))) = \quad (26)$$

$$\mathcal{H}^1([-r-2, r+2] \Delta ([-(r+2), (r+2)] \cup (\mathbb{Q} \cap [-(r+2)-1, (r+2)+1]))) = \quad (27)$$

$$\mathcal{H}^1([-r-2, r+2] \Delta ([-(r+2), (r+2)] \cup (\mathbb{Q} \cap [-r-3, r+3]))) = \quad (28)$$

$$\mathcal{H}^1(\mathbb{Q} \cap ([r+2, r+3]) \cup (\mathbb{Q} \cap [-r-3, r-2])) = \quad (29)$$

$$0 \quad (30)$$

2. For all $v \geq N' = 3$, there exists a $1 \leq v-2 =: r \in \mathbb{N}$, where:

$$\mathcal{H}^{\dim_H(F_v')}(F_r^* \Delta F_v') = 0 \quad (31)$$

which is proven with the following:

In Equation (31), since $\dim_H(F'_v) = \dim_H([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1])) = 1$:

$$\mathcal{H}^1([-r-2, r+2] \Delta([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1]))) = \quad (32)$$

$$\mathcal{H}^1([-v-2, v+2] \Delta([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1]))) = \quad (33)$$

$$\mathcal{H}^1([-v, v] \Delta([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1]))) = \quad (34)$$

$$\mathcal{H}^1((\mathbb{Q} \cap [-v-1, -v]) \cup (\mathbb{Q} \cap [v, v+1])) = \quad (35)$$

$$0 \quad (36)$$

Since crit. (1) and (2) is true, using definition 1, we have shown $(F_r^*)_{r \in \mathbb{N}} = ([-r-2, r+2])_{r \in \mathbb{N}}$ and $(F'_v)_{v \in \mathbb{N}} = ([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1]))_{v \in \mathbb{N}}$ are equivalent.

Note, there is no $A \in \mathcal{P}(\mathbb{R}^n)$, where $F_r^*, F'_v \rightarrow A$ (Section 5.1), such that $\text{Avg}((F_r^*), A) \neq \text{Avg}((F'_v), A)$.

Definition 2 (Non-Equivalent Sequences of Bounded Sets). The sequences of sets $(F_r^*)_{r \in \mathbb{N}}$ and $(F'_v)_{v \in \mathbb{N}}$ are non-equivalent, if there exists $N' \in \mathbb{N}$, where for all $r \geq N'$, there is either $v \in \mathbb{N}$, such that:

$$\mathcal{H}^{\dim_H(F_r^*)}(F_r^* \Delta F'_v) \neq 0$$

or for all $v \geq N'$, there exists $r \in \mathbb{N}$, such that:

$$\mathcal{H}^{\dim_H(F'_v)}(F_r^* \Delta F'_v) \neq 0$$

5.3.3. Explanation

Suppose, $\mathcal{P}(\mathbb{R}^n)$ is the set of all subsets of $\mathbb{R}^n$. Hence, $(F_r^*)_{r \in \mathbb{N}}$ and (F'_v) are non-equivalent, when there exists a set $A \in \mathcal{P}(\mathbb{R}^n)$, where $F_r^*, F'_v \rightarrow A$ (Section 5.1) and $\text{Avg}((F_r^*), A)$ or $\text{Avg}((F'_v), A)$ exist, s.t. the following is true:

$$\text{Avg}((F_r^*), A) \neq \text{Avg}((F'_v), A) \quad (\text{Section 5.2})$$

Therefore, consider the following:

5.3.4. Example of Non-Equivalent Sequences of Bounded Sets

Suppose:

$$(F_r^*)_{r \in \mathbb{N}} = ([-r, r])_{r \in \mathbb{N}}$$

$$(F'_v)_{v \in \mathbb{N}} = ([-2v, 2v])_{v \in \mathbb{N}}$$

Now, suppose $N := 1$. Thus, using def. 2, we prove:

1. For all $v \geq N' = 1$, there exists a $3 \leq 2v + 1 =: r \in \mathbb{N}$, where:

$$\mathcal{H}^{\dim_H(F'_v)}(F_r^* \Delta F'_v) \neq 0$$

which is proven using the following:

Since $F'_v = [-2v, 2v]$ is a 1-d interval, $\dim_H(F'_v) = 1$. Therefore:

$$\mathcal{H}^1([-r, r] \Delta [-2v, 2v]) = \quad (37)$$

$$\mathcal{H}^1([- (2v+1), 2v+1] \Delta [-2v, 2v]) = \quad (38)$$

$$\mathcal{H}^1([-2v-1, 2v+1] \Delta [-2v, 2v]) = \quad (39)$$

$$\mathcal{H}^1([-2v-1, 2v+1] \Delta [-2v, 2v]) = \quad (40)$$

$$\mathcal{H}^1([-2v-1, -2v] \cup [2v, 2v+1]) = \quad (41)$$

$$1 + 1 \neq 0 \quad (42)$$

Since crit. (1) is true, using definition 2, we have shown $(F_r^*)_{r \in \mathbb{N}} = ([-r, r])_{r \in \mathbb{N}}$ and $(F'_v)_{v \in \mathbb{N}} = ([-2v, 2v])_{v \in \mathbb{N}}$ are non-equivalent.

5.3.5. Question

When:

$$(F_r^*)_{r \in \mathbb{N}} = ([-r, r])_{r \in \mathbb{N}}$$

$$(F'_v)_{v \in \mathbb{N}} = ([-2v, 2v])_{v \in \mathbb{N}}$$

How do we find $A \in \mathcal{P}(\mathbb{R}^n)$, where $F_r^*, F'_v \rightarrow A$ (Section 5.1), such that $\text{Avg}((F_r^*), A) \neq \text{Avg}((F'_v), A)$ (Section 5.2)?

5.4. Defining the "Measure"

5.4.1. Preliminaries

We define the "measure" of $(F_r)_{r \in \mathbb{N}}$ in Section 5.4.2. To understand this "measure", keep reading. (If the following steps are unclear, see Section 8 for examples.)

1. For every $r \in \mathbb{N}$, "over-cover" F_r with minimal, pairwise disjoint sets of equal $\mathcal{H}^{\dim_H(F_r)}$ measure. (We denote the equal measures ε , where the former sentence is defined $\mathbf{C}(\varepsilon, F_r, \omega)$: i.e., $\omega \in \Omega_{\varepsilon, r}$ enumerates all collections of these sets covering F_r . In case this step is unclear, see Section 8.1.)
2. For every ε, r and ω , take a sample point from each set in $\mathbf{C}(\varepsilon, F_r, \omega)$. The set of these points is "the sample" which we define $\mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)$: i.e., $\psi \in \Psi_{\varepsilon, r, \omega}$ enumerates all possible samples of $\mathbf{C}(\varepsilon, F_r, \omega)$. (In the case this is unclear, see Section 8.2.)
3. For every ε, r, ω and ψ ,
 - (a) Take a "pathway" of line segments: we start with a line segment from arbitrary point x_0 of $\mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)$ to the sample point with smallest n -dimensional Euclidean distance to x_0 (i.e., when more than one sample point has smallest n -dimensional Euclidean distance to x_0 , take either of those points). Next, repeat this process until the "pathway" intersects with every sample point once. (If this is unclear, see Section 8.3.1.)
 - (b) Take the set of the length of all segments in (3a), except for lengths that are outliers [7] (i.e., for any constant $C > 0$, the outliers are more than C times the interquartile range of the length of all line segments as $\varepsilon \rightarrow 0$). Define this $\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi))$. (In case this is unclear, see Section 8.3.2.)
 - (c) Multiply remaining lengths in the pathway by a constant so they add up to one (i.e., a probability distribution). This will be denoted $\mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))$. (In case this step is unclear, see Section 8.3.3.)
 - (d) Take the shannon entropy [8, p.61-95] of step (3c). We define this:

$$\mathbb{E}(\mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))) = \sum_{x \in \mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))} -x \log_2 x$$

which will be *shortened* to $E(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))$. (In case this is unclear, see Section 8.3.4)

- (e) Maximize the entropy w.r.t all "pathways". This we will denote:

$$E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi))) = \sup_{x_0 \in \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)} E(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))$$

(In case this step is unclear, see Section 8.3.5.)

4. Therefore, the **maximum entropy** of $(F_r)_{r \in \mathbb{N}}$ w.r.t ε , using (1) and (2) is:

$$E_{\max}(\varepsilon, r) = \sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))$$

5.4.2. What Am I Measuring?

Suppose we define two sequences of bounded sets that have a set theoretic limit of A : e.g., $(F_r^*)_{r \in \mathbb{N}}$ and $(F_j^{**})_{j \in \mathbb{N}}$, where for **constant** ε and *cardinality* $|\cdot|$

- (a) Using (2) and (33e) of section 5.4.1, suppose:

$$\begin{aligned} & \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} = \\ & \inf \left\{ \left| \mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi') \right| : j \in \mathbb{N}, \omega' \in \Omega_{\varepsilon, j}, \psi' \in \Psi_{\varepsilon, j, \omega'}, E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi'))) \geq E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi))) \right\} \\ & \text{then (using } \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} \text{) we get:} \end{aligned}$$

$$\bar{\alpha}(\varepsilon, r, \omega, \psi) = \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} / |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|$$

- (b) Also, using (2) and (33e) of section 5.4.1, suppose:

$$\begin{aligned} & \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} = \\ & \sup \left\{ \left| \mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi') \right| : j \in \mathbb{N}, \omega' \in \Omega_{\varepsilon, j}, \psi' \in \Psi_{\varepsilon, j, \omega'}, E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi'))) \leq E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi))) \right\} \\ & \text{then (using } \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} \text{) we also get:} \end{aligned}$$

$$\underline{\alpha}(\varepsilon, r, \omega, \psi) = \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} / |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|$$

1. If using $\bar{\alpha}(\varepsilon, r, \omega, \psi)$ and $\underline{\alpha}(\varepsilon, r, \omega, \psi)$ we have:

$$1 < \limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} \bar{\alpha}(\varepsilon, r, \omega, \psi), \liminf_{\varepsilon \rightarrow 0} \liminf_{r \rightarrow \infty} \inf_{\omega \in \Omega_{\varepsilon, r}} \inf_{\psi \in \Psi_{\varepsilon, r, \omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(F_r^*)_{r \in \mathbb{N}}$ increases at a rate **superlinear** to that of $(F_j^{**})_{j \in \mathbb{N}}$.

2. If using equations $\bar{\alpha}(\varepsilon, j, \omega, \psi)$ and $\underline{\alpha}(\varepsilon, j, \omega, \psi)$ (where we swap $(F_r^*)_{r \in \mathbb{N}}$ and $r \in \mathbb{N}$, in $\bar{\alpha}(\varepsilon, r, \omega, \psi)$ and $\underline{\alpha}(\varepsilon, r, \omega, \psi)$, with $(F_j^{**})_{j \in \mathbb{N}}$ and $j \in \mathbb{N}$) we get:

$$1 < \limsup_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon, j}} \sup_{\psi \in \Psi_{\varepsilon, j, \omega}} \bar{\alpha}(\varepsilon, j, \omega, \psi), \liminf_{\varepsilon \rightarrow 0} \liminf_{j \rightarrow \infty} \inf_{\omega \in \Omega_{\varepsilon, j}} \inf_{\psi \in \Psi_{\varepsilon, j, \omega}} \underline{\alpha}(\varepsilon, j, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(F_r^*)_{r \in \mathbb{N}}$ increases at a rate **sublinear** to that of $(F_j^{**})_{j \in \mathbb{N}}$.

3. If using equations $\bar{\alpha}(\varepsilon, r, \omega, \psi)$, $\underline{\alpha}(\varepsilon, r, \omega, \psi)$, $\bar{\alpha}(\varepsilon, j, \omega, \psi)$, and $\underline{\alpha}(\varepsilon, j, \omega, \psi)$, we **both** have:

- (a) $\limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} \bar{\alpha}(\varepsilon, r, \omega, \psi)$ or $\liminf_{\varepsilon \rightarrow 0} \liminf_{r \rightarrow \infty} \inf_{\omega \in \Omega_{\varepsilon, r}} \inf_{\psi \in \Psi_{\varepsilon, r, \omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi)$ are equal to zero, one or $+\infty$

- (b) $\limsup_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon, j}} \sup_{\psi \in \Psi_{\varepsilon, j, \omega}} \bar{\alpha}(\varepsilon, j, \omega, \psi)$ or $\liminf_{\varepsilon \rightarrow 0} \liminf_{j \rightarrow \infty} \inf_{\omega \in \Omega_{\varepsilon, j}} \inf_{\psi \in \Psi_{\varepsilon, j, \omega}} \underline{\alpha}(\varepsilon, j, \omega, \psi)$ are equal to zero, one or $+\infty$

then *what I'm measuring from* $(F_r^*)_{r \in \mathbb{N}}$ increases at a rate **linear** to that of $(F_j^{**})_{j \in \mathbb{N}}$.

Now consider the following examples:

5.4.3. Example of the “measure” of (F_r^*) converging super-linearly to that of (F_j^{**})

Using Section 5.4.2 consider the following:

Notation 3. Recall, if $\dim_H(F_r^*) = 2$:

1. When $|\cdot|$ is the cardinality, for all $\omega \in \Omega_{\varepsilon,r}$ and $\psi \in \Psi_{\varepsilon,r,\omega}$, $|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)| = \lceil \text{Area}(F_r^*)/\varepsilon \rceil$
2. For all $\omega \in \Omega_{\varepsilon,r}$ and $\psi \in \Psi_{\varepsilon,r,\omega}$, the largest $E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)))$ can be is $\log_2(\lceil \text{Area}(F_r^*)/\varepsilon \rceil - 1) = \log_2(|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)| - 1)$

Now consider:

1. $(F_r^*)_{r \in \mathbb{N}} = (\{x^2 + y^2 = 6r^2\})_{r \in \mathbb{N}}$
2. $(F_j^{**})_{j \in \mathbb{N}} = (\{x^2/(9j^2) + y^2/(4j^2) = 1\})_{j \in \mathbb{N}}$

where:

Notation 4. The area of F_r^* and F_j^{**} are:

1. $\text{Area}(F_r^*) = \pi(\sqrt{6}r)^2 = 6\pi r^2$
2. $\text{Area}(F_j^{**}) = 3j \cdot 2j \cdot \pi = 6\pi j^2$

Hence, we maximize $E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)))$ by doing the following:

Notation 5 (Procedure to Maximize $E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)))$). Consider the procedure below:

1. Cover the circle, with the same or larger-sized circle, which can be divided into minimum t “pie-slices” of equal area $\varepsilon > 0$. Notice, $t = \lceil \text{Area}(F_r^*)/\varepsilon \rceil = \lceil (6\pi r^2)/\varepsilon \rceil$.
2. Take the centroid of each slice
3. Out of all centroids in step 2, take the centroid with the largest x -coordinate: i.e., denote this point x_0 which is the start-point of the pathway of line segments in the resulting step
4. Take the distances between all pairs of consecutive centroids, starting with x_0 , rotating counter-clockwise or clockwise. Either-way, the end result should change by only a negligible amount.
5. Multiply the distances by a constant so they add up to 1 (i.e., a probability distribution)
6. Take the shannon entropy of the distribution using log base 2 in Section 5.4.1(33d)-(33e). (Note, since the “pie-slices” of step 1 are congruent and the distances of step 4 are equal, the entropy of the distribution is the largest possible amount (i.e., note 3 crit. 2 & note 4 crit. 1):

$$\log_2(\lceil \text{Area}(F_r^*)/\varepsilon \rceil - 1) = \log_2(\lceil (6\pi r^2)/\varepsilon \rceil - 1)$$

Repeat, steps 1-6 with (F_j^{**}) , where the circle is an ellipse (i.e., with this case the “pie-slices” of step 1 are non-congruent and the distances of step 4 are non-equivalent). Hence, we use programming:

Listing 1: Note 5 Steps (1)-(6) on $(F_j^{**})_{j \in \mathbb{N}}$

```
Clear["Global`*"]
```

```
(* We're using Mathematica *)
```

```
(*The table below is note 5, steps 1-6 on  $F_j^{**}$ *)
```

```
EntropyFj = Table[
```

```
{eps = 3; (*  $\varepsilon > 0$  *)
```

```
n = Ceiling[6 (j^2) Pi/eps]; (* total number of "slices" with equal  $\varepsilon > 0$  area*)
```

```
(* 'p' is the scaling transform for the same or larger sized ellipse covering  $(F_j^{**})$ 
```

```
which can be subdivided into minimum  $t$  pieces of equal area  $\varepsilon > 0$ , where  $t = \lceil (6\pi r^2)/\varepsilon \rceil$ *)
```

```
p = Sqrt[(eps/(6 (j^2) Pi))*Ceiling[(6 (j^2) Pi)/eps]];
```

```

(*'slices' is step 1 on  $F_j^{**}$ *)
slices =
  BoundaryDiscretizeRegion /@ (TransformedRegion[#,
    ScalingTransform[{3 j p, 2 j p}]] & /@ (Disk[{0, 0},
    1, #] & /@ Partition[Subdivide[0, 2 \[Pi], n], 2, 1]));

(*'s' is step 2 on  $F_j^{**}$ : take the centroid of each slice.*)
s = RegionCentroid /@ slices;

(*Below are all pathways for any  $x_0$  in s*)
ClearAll[pathway],
pathway[path_, points_] := Block[{$RecursionLimit = 2000},
  Module[{next}, If[points == {}, Return[path]];
    next = Nearest[points, Last@path,
      DistanceFunction -> EuclideanDistance];
    Level[Map[
      pathway[Append[path, #],
        Delete[points, FirstPosition[points, #]]] &, next], {-3}]],
  paths =
    Flatten[pathway[{s[[#]]}, Delete[s, #]] & /@ Range@Length@s, 1];
steps = Partition[#, 2, 1] & /@ paths;
dists = Apply[EuclideanDistance, steps, {2}];
ClearAll[outlierLimit],
outlierLimit[distances_] :=
  Module[{q = Quartiles[distances], iqr},
    iqr = First@Differences@q[[{1, 3}]]; q[[2]] + (3/2) iqr];

(*'outlimits' remove any outliers as mentioned in Section 5.4.1 (33b)*)
outLimits = outlierLimit /@ dists;
nonoutliers =
  Select[#1, LessThan[#2]] & @@@ Thread[{dists, outLimits}];

(*'normalize' is step 3, 4 & 5 on  $F_j^{**}$ : Take a pathway of line segments starting from the centroid
with the largest x-coordinate. Then, take the distances between pairs of consecutive centroids
clockwise or counter-clockwise and multiply the distances by a constant so they add up to one. *)

normalize = nonoutliers[[1]]/Total[nonoutliers[[1]]];

(*'entropy' is step 6 on  $F_j^{**}$ *)
entropy = -Total[normalize Log[2, normalize]], {j, 1, 9}] (*The table for 'EntropyFj' ends here.*)

(*Table of all entropy values*)
Table[EntropyFj[[j, 4]], {j, 1, 10}]

```

where the output of the code is:

Listing 2: Ouput of Code 1

```
{2.56866, 4.63026, 5.79356, 6.6299, 7.28059, 7.8061, 8.24799, 8.63693, 8.97455, 9.28048}
```

Further note, adding the following code:

Listing 3: Extra Code for Code 1

```

(*Below is the entropy of circle ( $F_r^*$ ) $_{r \in \mathbb{N}}$ *)
Entropyofcircle[r_] :=
  Entropyofcircle[r] = Log[2, Ceiling[6 (r^2) Pi/eps] - 1]

(*'ratio' compares the rate of increase of the entropy of ( $F_j^{**}$ ) $_{j \in \mathbb{N}}$  with the entropy
of ( $F_r^*$ ) $_{r \in \mathbb{N}}$ *)
ratio[j_] :=
  ratio[j] =
    N[(Entropyofcircle[j] - EntropyFj[[j, 4]])/(Entropyofcircle[j] -
      Entropyofcircle[j - 1])]

(*Table of values of Function ratio*)
ratiotable=Table[ratio[j], {j, 2, 10}]

ratiodifferences = Differences[ratiotable]

(*Here is the evidence the difference of consecutive 'ratio' values is
almost constant. 'Evidence1' should be almost .005 and 'Evidence 2' should approach zero.*)
Evidence1 = Mean[ratiodifferences]
Evidence2 = N[Total[ratiodifferences - Mean[ratiodifferences]]/Mean[ratiodifferences]]

```

we get the output is:

Listing 4: Output of Code 3

```
(*Output of 'ratiotable' in code 3*)
{0.00660547, 0.0118564, 0.0166815, 0.021556, 0.0267818, 0.0319121, 0.0363022, 0.0418554, 0.0462194}

(*Output of 'ratiodifferences' in code 3. Notice, the outputs are around .005*)
{0.00525097, 0.00482508, 0.00487451, 0.00522582, 0.00513024, 0.00439012, 0.00555315, 0.00436408}

(*Output of 'Evidence1' in code 3. Note, this is close to .005*)
0.00495175

(*Output of 'Evidence2'. Note, this is extremely close to zero.*)
1.05098*10^-15
```

Note, the output of ratiotable in code 5 can be written as:

$$\approx .005(j-2) + .006 \quad (43)$$

$$\approx .005j - .01 + .006 \quad (44)$$

$$\approx .005j - .004 \quad (45)$$

and is the same as:

$$\frac{\log_2\left(\left\lceil \frac{6j^2\pi}{\varepsilon} \right\rceil - 1\right) - E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi)))}{\log_2\left(\left\lceil \frac{6j^2\pi}{\varepsilon} \right\rceil - 1\right) - \log_2\left(\left\lceil \frac{6(j-1)^2\pi}{\varepsilon} \right\rceil - 1\right)} \quad (46)$$

Hence, with:

$$\frac{\log_2\left(\left\lceil \frac{6j^2\pi}{\varepsilon} \right\rceil - 1\right) - E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi)))}{\log_2\left(\left\lceil \frac{6j^2\pi}{\varepsilon} \right\rceil - 1\right) - \log_2\left(\left\lceil \frac{6(j-1)^2\pi}{\varepsilon} \right\rceil - 1\right)} \sim .005j - .004$$

we solve for $E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi)))$:

$$\frac{\log_2\left(\left\lceil 6j^2\pi/\varepsilon \right\rceil - 1\right) - E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi)))}{\log_2\left(\frac{\left\lceil 6j^2\pi/\varepsilon \right\rceil - 1}{\left\lceil 6(j-1)^2\pi/\varepsilon \right\rceil - 1}\right)} \sim .005j - .004 \quad (47)$$

$$\log_2\left(\left\lceil 6j^2\pi/\varepsilon \right\rceil - 1\right) - E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi))) \sim (.005j - .004) \log_2\left(\frac{\left\lceil 6j^2\pi/\varepsilon \right\rceil - 1}{\left\lceil 6(j-1)^2\pi/\varepsilon \right\rceil - 1}\right) \quad (48)$$

$$- E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi))) \sim (.005j - .004) \log_2\left(\frac{\left\lceil 6j^2\pi/\varepsilon \right\rceil - 1}{\left\lceil 6(j-1)^2\pi/\varepsilon \right\rceil - 1}\right) - \log_2\left(\left\lceil 6j^2\pi/\varepsilon \right\rceil - 1\right) \quad (49)$$

$$E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi))) \sim \log_2\left(\left\lceil 6j^2\pi/\varepsilon \right\rceil - 1\right) - (.005j - .004) \log_2\left(\frac{\left\lceil 6j^2\pi/\varepsilon \right\rceil - 1}{\left\lceil 6(j-1)^2\pi/\varepsilon \right\rceil - 1}\right) \quad (50)$$

since:

$$\lim_{j \rightarrow \infty} (.005j - .004) \log_2\left(\frac{\left\lceil 6j^2\pi/\varepsilon \right\rceil - 1}{\left\lceil 6(j-1)^2\pi/\varepsilon \right\rceil - 1}\right) \approx .014427 \text{ for } \varepsilon > 0 \quad (51)$$

Listing 5: Limit of 51

```
(*We're using Mathematica*)

Limit[(.005 j - .004) Log[
  2, (Ceiling[(6 (j^2) Pi)/eps] -
  1)/(Ceiling[(6 ((j - 1)^2) Pi)/eps] - 1)], j -> Infinity]

(*Output of code is .014427 if eps>0*)
```

Hence:

$$E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi))) \sim \log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - (.005j - .004) \log_2\left(\frac{\lceil 6j^2\pi/\varepsilon \rceil - 1}{\lceil 6(j-1)^2\pi/\varepsilon \rceil - 1}\right) \quad (52)$$

$$E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi))) \sim \log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - .014427 \quad (53)$$

Note, using Section 5.4.2 (0a) and Section 5.4.2 (1), take $|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{**}, \omega'), \psi')| = \lceil 6j^2\pi/\varepsilon \rceil$ and use note 3 crit. 2 to compute the following:

$$\begin{aligned} & \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} = \\ & \inf\left\{|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi')| : j \in \mathbb{N}, \omega' \in \Omega_{\varepsilon,j}, \psi' \in \Psi_{\varepsilon,j,\omega}, E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi')))) \geq E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)))\right\} = \\ & \inf\left\{\frac{3}{\pi^2}j^2 : j \in \mathbb{N}, \omega' \in \Omega_{\varepsilon,j}, \psi' \in \Psi_{\varepsilon,j,\omega}, \log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - .014427 \geq \log_2(\lceil 6r^2\pi/\varepsilon \rceil)\right\} = \end{aligned} \quad (54)$$

where:

1. For every $r \in \mathbb{N}$, we find a $j \in \mathbb{N}$, where $\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - .014427 \geq \log_2(\lceil 6r^2\pi/\varepsilon \rceil)$, but the absolute value of $\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - .014427 - \log_2(\lceil 6r^2\pi/\varepsilon \rceil)$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $j \in \mathbb{N}$ where:

$$\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - .014427 \geq \log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1) \quad (55)$$

$$\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) \geq \log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1) + .014427 \quad (56)$$

$$2^{\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1)} \geq 2^{\log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1) + .014427} \quad (57)$$

$$\lceil 6j^2\pi/\varepsilon \rceil - 1 \geq (\lceil 6r^2\pi/\varepsilon \rceil - 1)2^{.014427} \quad (58)$$

$$\lceil 6j^2\pi/\varepsilon \rceil \geq 1.0101(\lceil 6r^2\pi/\varepsilon \rceil - 1) + 1 \quad (59)$$

$$\lceil 6j^2\pi/\varepsilon \rceil = \lceil 1.0101(\lceil 6r^2\pi/\varepsilon \rceil - 1) + 1 \rceil \sim \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} \quad (60)$$

Finally, since $|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)| = \lceil 6\pi r^2/\varepsilon \rceil$, we wish to prove

$$1 < \limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \bar{\alpha}(\varepsilon, r, \omega, \psi) < +\infty$$

within Section 5.4.2 crit. 1:

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \bar{\alpha}(\varepsilon, r, \omega, \psi) = \limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \frac{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|}{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} \quad (61)$$

$$= \lim_{\varepsilon \rightarrow 0} \lim_{r \rightarrow \infty} \frac{\lceil 1.0101(\lceil 6r^2\pi/\varepsilon \rceil - 1) + 1 \rceil}{\lceil 6r^2\pi/\varepsilon \rceil} \quad (62)$$

where using mathematica, we get the limit is greater than one:

Listing 6: Limit of Equation (62)

```
Clear["Global`*"]
(*We use a double limit since the iterated limit takes too long to compute.
If the double limit exists, it's the same as the iterated limit.*)

Limit[Floor[Rationalize[1.0101, 0]* (Ceiling[(6 (r^2) Pi)/eps] - 1) + 1]/
Ceiling[(6 (r^2) Pi)/eps], {eps, r} -> {0, Infinity}]

(*The output is 1.0101*)
```

Also, using Section 5.4.2 (0b) and Section 5.4.2 (1), take $|S(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi')| = \lceil (6\pi j^2)/\varepsilon \rceil$ to compute the following:

$$\begin{aligned} & |S(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)| = \\ & \sup \left\{ |S(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi')| : j \in \mathbb{N}, \omega' \in \Omega_{\varepsilon, j}, \psi' \in \Psi_{\varepsilon, j, \omega}, E(\mathcal{L}(S(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi'))) \leq E(\mathcal{L}(S(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi))) \right\} = \\ & \sup \{ \lceil 6j^2\pi/\varepsilon \rceil : j \in \mathbb{N}, \omega' \in \Omega_{\varepsilon, j}, \psi' \in \Psi_{\varepsilon, j, \omega}, \log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - .014427 \leq \log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1) \} = \end{aligned} \quad (63)$$

where:

1. For every $r \in \mathbb{N}$, we find a $j \in \mathbb{N}$, where $\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - .014427 \leq \log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1)$, but the absolute value of $\log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1) - (\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - .014427)$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $j \in \mathbb{N}$ where:

$$\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) - .014427 \leq \log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1) \quad (64)$$

$$\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1) \leq \log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1) + .014427 \quad (65)$$

$$2^{\log_2(\lceil 6j^2\pi/\varepsilon \rceil - 1)} \leq 2^{\log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1) + .014427} \quad (66)$$

$$\lceil 6j^2\pi/\varepsilon \rceil - 1 \leq (\lceil 6r^2\pi/\varepsilon \rceil - 1) 2^{.014427} \quad (67)$$

$$\lceil 6j^2\pi/\varepsilon \rceil \leq 1.0101(\lceil 6r^2\pi/\varepsilon \rceil - 1) + 1 \quad (68)$$

$$\lceil 6j^2\pi/\varepsilon \rceil = \lfloor 1.0101(\lceil 6r^2\pi/\varepsilon \rceil - 1) + 1 \rfloor \sim |S(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)| \quad (69)$$

Finally, since $|S(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)| = \log_2(\lceil 6r^2\pi/\varepsilon \rceil - 1)$, we wish to prove

$$1 < \liminf_{\varepsilon \rightarrow 0} \liminf_{r \rightarrow \infty} \inf_{\omega \in \Omega_{\varepsilon, r}} \inf_{\psi \in \Psi_{\varepsilon, r, \omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty$$

within Section 5.4.2 crit. 1:

$$\liminf_{\varepsilon \rightarrow 0} \liminf_{r \rightarrow \infty} \inf_{\omega \in \Omega_{\varepsilon, r}} \inf_{\psi \in \Psi_{\varepsilon, r, \omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) = \liminf_{\varepsilon \rightarrow 0} \liminf_{r \rightarrow \infty} \inf_{\omega \in \Omega_{\varepsilon, r}} \inf_{\psi \in \Psi_{\varepsilon, r, \omega}} \frac{|S(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|}{|S(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} \quad (70)$$

$$= \lim_{\varepsilon \rightarrow 0} \lim_{r \rightarrow \infty} \frac{\lfloor 1.0101(\lceil 6r^2\pi/\varepsilon \rceil - 1) + 1 \rfloor}{\lceil 6r^2\pi/\varepsilon \rceil} \quad (71)$$

where using mathematica, we get the limit is greater than one:

Listing 7: Limit of Equation (71)

```
Clear["Global`*"]
(*We use a double limit since the iterated limit takes too long to compute.
If the double limit exists, it's the same as the iterated limit.*)

Limit[Floor[Rationalize[1.0101, 0]* (Ceiling[(6 (r^2) Pi)/eps] - 1) + 1]/
Ceiling[(6 (r^2) Pi)/eps], {eps, r} -> {0, Infinity}]

(*The output is 1.0101*)
```

Hence, since the limits in Equation (62) and Equation (71) are greater than one and less than $+\infty$: i.e.,

$$1 < \liminf_{\varepsilon \rightarrow 0} \liminf_{r \rightarrow \infty} \inf_{\omega \in \Omega_{\varepsilon, r}} \inf_{\psi \in \Psi_{\varepsilon, r, \omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi), \limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} \bar{\alpha}(\varepsilon, r, \omega, \psi) < +\infty \quad (72)$$

we have proven *what we're measuring from* $(F_r^*)_{r \in \mathbb{N}}$ increases at a rate **superlinear** to that of $(F_j^{**})_{j \in \mathbb{N}}$ (i.e., 5.4.2 crit. 1).

5.4.4. Example of The "Measure" from $(F_r^*)_{r \in \mathbb{N}}$ Increasing at a Rate Sub-Linear to that of $(F_j^{**})_{j \in \mathbb{N}}$

Using our previous example, we can use the following theorem:

Theorem 2. If what we're measuring from $(F_r^*)_{r \in \mathbb{N}}$ increases at a rate superlinear to that of $(F_j^{**})_{j \in \mathbb{N}}$, then what we're measuring from $(F_j^{**})_{r \in \mathbb{N}}$ increases at a rate **sublinear** to that of $(F_r^*)_{r \in \mathbb{N}}$

Hence, in our definition of super-linear (Section 5.4.2 crit. 1), swap F_r^* and $r \in \mathbb{N}$ for (F_j^{**}) and $j \in \mathbb{N}$ within $\bar{\alpha}(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}(\epsilon, r, \omega, \psi)$ (i.e., $\bar{\alpha}(\epsilon, j, \omega, \psi)$ and $\underline{\alpha}(\epsilon, j, \omega, \psi)$). Notice, thm. 2 is true when:

$$1 < \limsup_{\epsilon \rightarrow 0} \limsup_{j \rightarrow \infty} \sup_{\omega \in \Omega_{\epsilon, j}} \sup_{\psi \in \Psi_{\epsilon, j, \omega}} \bar{\alpha}(\epsilon, j, \omega, \psi), \liminf_{\epsilon \rightarrow 0} \liminf_{j \rightarrow \infty} \inf_{\omega \in \Omega_{\epsilon, j}} \inf_{\psi \in \Psi_{\epsilon, j, \omega}} \underline{\alpha}(\epsilon, j, \omega, \psi) < +\infty$$

5.4.5. Example of the "measure" of (F_r^*) converging linearly to that of (F_j^{**})

Using Section 5.4.2 consider the following:

Notation 6. Recall, if $\dim_H(F_r^*) = 2$:

1. When $|\cdot|$ is the cardinality, for all $\omega \in \Omega_{\epsilon, r}$ and $\psi \in \Psi_{\epsilon, r, \omega}$, $|\mathcal{S}(\mathbf{C}(\epsilon, F_r^*, \omega), \psi)| = \lceil \text{Area}(F_r^*) / \epsilon \rceil$
2. For all $\omega \in \Omega_{\epsilon, r}$ and $\psi \in \Psi_{\epsilon, r, \omega}$, the largest $E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\epsilon, F_r^*, \omega), \psi)))$ can be is $\log_2(\lceil \text{Area}(F_r^*) / \epsilon \rceil - 1) = \log_2(|\mathcal{S}(\mathbf{C}(\epsilon, F_r^*, \omega), \psi)| - 1)$

Now consider:

1. $(F_r^*)_{r \in \mathbb{N}} = ([-r, r] \times [-r, r])_{r \in \mathbb{N}}$
2. $(F_j^{**})_{j \in \mathbb{N}} = ([-j^2, j^2] \times [-j^2, j^2])_{j \in \mathbb{N}}$

where:

Notation 7. The area of F_r^* and F_j^{**} is:

1. $\text{Area}(F_r^*) = (r - (-r)) \cdot (r - (-r)) = 2r \cdot 2r = 4r^2$
2. $\text{Area}(F_j^{**}) = (j^2 - (-j^2)) \cdot (j^2 - (-j^2)) = 2j^2 \cdot 2j^2 = 4j^4$

Notice, since F_r^* is a square, using note 6 crit. 2 and note 7 crit. 1, it's simple to see:

$$E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\epsilon, F_r^*, \omega), \psi))) = \log_2(\lceil \text{Area}(F_r^*) / \epsilon \rceil - 1) = \log_2(\lceil 4r^2 / \epsilon \rceil - 1)$$

Hence, using Section 5.4.2 (0b) and note 6 crit. (2):

$$\begin{aligned} |\mathcal{S}(\mathbf{C}(\epsilon, F_r^*, \omega), \psi)| &= \\ \sup \{ |\mathcal{S}(\mathbf{C}(\epsilon, F_j^{**}, \omega'), \psi')| : j \in \mathbb{N}, \omega' \in \Omega_{\epsilon, j}, \psi' \in \Psi_{\epsilon, j, \omega'}, E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\epsilon, F_j^{**}, \omega'), \psi'))) \leq E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\epsilon, F_r^*, \omega), \psi))) \} &= \\ \sup \{ |\mathcal{S}(\mathbf{C}(\epsilon, F_j^{**}, \omega'), \psi')| : j \in \mathbb{N}, \omega' \in \Omega_{\epsilon, j}, \psi' \in \Psi_{\epsilon, j, \omega'}, E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\epsilon, F_j^{**}, \omega'), \psi'))) \leq \log_2(\lceil 4r^2 / \epsilon \rceil - 1) \} &= \end{aligned} \quad (73)$$

Note, since F_j^{**} is also a square, for all $\omega' \in \Omega_{\epsilon, j}$ and $\psi' \in \Psi_{\epsilon, j, \omega'}$

$$0 < E(\mathcal{L}(\mathcal{S}(\mathbf{C}(\epsilon, F_j^{**}, \omega'), \psi'))) \leq \log_2(\lceil \text{Area}(F_j^{**}) / \epsilon \rceil - 1) = \log_2(\lceil 4j^4 / \epsilon \rceil - 1) \quad (74)$$

Thus, using Equations (73) and (74):

1. For every $r \in \mathbb{N}$, we find a $j \in \mathbb{N}$, where $\log_2(\lceil 4j^4/\varepsilon \rceil - 1) \leq \log_2(\lceil 4r^2/\varepsilon \rceil - 1)$, but the absolute value of $\log_2(\lceil 4r^2/\varepsilon \rceil - 1) - \log_2(\lceil 4j^4/\varepsilon \rceil - 1)$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $j \in \mathbb{N}$ where:

$$\log_2(\lceil 4j^4/\varepsilon \rceil - 1) \leq \log_2(\lceil 4r^2/\varepsilon \rceil - 1) \quad (75)$$

$$4j^4 \leq 4r^2 \quad (76)$$

$$j^4 \leq r^2 \quad (77)$$

$$j \leq \sqrt{r} \quad (78)$$

$$j = \lfloor \sqrt{r} \rfloor \quad (79)$$

$$4j^4/\varepsilon = 4\lfloor \sqrt{r} \rfloor^4/\varepsilon \quad (80)$$

$$\lceil 4j^4/\varepsilon \rceil = \lceil 4\lfloor \sqrt{r} \rfloor^4/\varepsilon \rceil \quad (81)$$

which gives:

$$\lceil 4\lfloor \sqrt{r} \rfloor^4/\varepsilon \rceil \leq \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)| \leq \lceil 4r^2/\varepsilon \rceil \quad (82)$$

where using note 6 crit. 1: $|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)| = \lceil \text{Area}(F_r^*)/\varepsilon \rceil = \lceil 4r^2/\varepsilon \rceil$, we wish to prove Section 5.4.2 crit. 33a:

$$\begin{aligned} 1 &= \limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \lceil 4\lfloor \sqrt{r} \rfloor^4/\varepsilon \rceil / \lceil 4r^2/\varepsilon \rceil \leq \limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{a}(\varepsilon, r, \omega, \psi) = \\ &= \limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \frac{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|}{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)|} \leq \\ &= \limsup_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \lceil 4r^2/\varepsilon \rceil / \lceil 4r^2/\varepsilon \rceil = 1 \end{aligned} \quad (83)$$

Hence Section 5.4.2 crit. 33a is true and then we prove Section 5.4.2 crit. 33b:

Note, since F_j^{**} is a square, using note 6 crit. 2 and note 7 crit. 2, it's simple to see:

$$\mathbb{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi))) = \log_2(\lceil \text{Area}(F_j^{**})/\varepsilon \rceil - 1) = \log_2(\lceil 4j^4/\varepsilon \rceil - 1)$$

Thus, using Section 5.4.2 0a and note 6 crit. 2, swap $r \in \mathbb{N}$ with $j \in \mathbb{N}$ and F_r^* with F_j^* , such that:

$$\begin{aligned} \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^*, \omega), \psi)|} &= \\ \sup \left\{ |\mathcal{S}(\mathbf{C}(\varepsilon, F_j^*, \omega'), \psi')| : r \in \mathbb{N}, \omega' \in \Omega_{\varepsilon,r}, \psi' \in \Psi_{\varepsilon,r,\omega'}, \mathbb{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^*, \omega'), \psi')))) \leq \mathbb{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi))) \right\} &= \\ \sup \left\{ |\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi')| : r \in \mathbb{N}, \omega' \in \Omega_{\varepsilon,r}, \psi' \in \Psi_{\varepsilon,r,\omega'}, \mathbb{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega'), \psi')))) \leq \log_2(\lceil 4j^4/\varepsilon \rceil - 1) \right\} &= \end{aligned} \quad (84)$$

Note, since F_r^* is also a square, for all $\omega' \in \Omega_{\varepsilon,r}$ and $\psi' \in \Psi_{\varepsilon,r,\omega'}$

$$0 < \mathbb{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^{**}, \omega'), \psi')))) \leq \log_2(\lceil \text{Area}(F_r^*)/\varepsilon \rceil - 1) = \log_2(\lceil 4r^2/\varepsilon \rceil - 1) \quad (85)$$

Therefore, using Equations (84) and (85):

1. For every $j \in \mathbb{N}$, we find a $r \in \mathbb{N}$, where $\log_2(\lceil 4r^2/\varepsilon \rceil - 1) \leq \log_2(\lceil 4j^4/\varepsilon \rceil - 1)$, but the absolute value of $\log_2(\lceil 4j^4/\varepsilon \rceil - 1) - \log_2(\lceil 4r^2/\varepsilon \rceil - 1)$ is minimized. In other words, for every $j \in \mathbb{N}$, we want a $r \in \mathbb{N}$, where:

$$\log_2(\lceil 4r^2/\varepsilon \rceil - 1) \leq \log_2(\lceil 4j^4/\varepsilon \rceil - 1) \quad (86)$$

$$4r^2 \leq 4j^4 \quad (87)$$

$$r^2 \leq j^4 \quad (88)$$

$$r \leq j^2 \quad (89)$$

$$r = \lfloor j^2 \rfloor = j^2 \quad (90)$$

$$4r^2/\varepsilon = 4(j^2)^2/\varepsilon = \quad (91)$$

$$\lceil 4r^2/\varepsilon \rceil = \lceil 4j^4/\varepsilon \rceil \quad (92)$$

Hence:

$$\left| \mathcal{S}(\mathbf{C}(\varepsilon, F_j^*, \omega), \psi) \right| = \lceil 4j^4/\varepsilon \rceil \quad (93)$$

and, using note 6 crit. 1 and note 6 crit. 1 with equation 92: $|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*, \omega), \psi)| = \lceil \text{Area}(F_r^*)/\varepsilon \rceil = \lceil 4r^2/\varepsilon \rceil = \lceil 4j^4/\varepsilon \rceil$. Thus, we wish to prove Section 5.4.2 crit. 33a using Equation(93):

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon, j}} \sup_{\psi \in \Psi_{\varepsilon, j, \omega}} \bar{\alpha}(\varepsilon, j, \omega, \psi) = \quad (94)$$

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow \infty} \sup_{\omega \in \Omega_{\varepsilon, j}} \sup_{\psi \in \Psi_{\varepsilon, j, \omega}} \left| \mathcal{S}(\mathbf{C}(\varepsilon, F_j^*, \omega), \psi) \right| / \left| \mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}, \omega), \psi) \right| = \quad (95)$$

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow \infty} \lceil 4j^4/\varepsilon \rceil / \lceil 4j^4/\varepsilon \rceil = 1 \quad (96)$$

where Section 5.4.2 crit. 33b is true.

Therefore, since Section 5.4.2 crit. 3 is correct in this case, “the measure” of (F_r^*) increases at a rate linear to that of $(F_j^{**})_{j \in \mathbb{N}}$

5.5. Defining The Actual Rate of Expansion of Sequence of Bounded Sets From C

In the next section, we will define the actual rate of expansion of sequence of bounded sets from C and give an example.

5.5.1. Definition of Actual Rate of Expansion of Sequence of Bounded Sets

Suppose $(F_r)_{r \in \mathbb{N}}$ is a bounded sequence of subsets of $\mathbb{R}^n$, and $d(Q, R)$ is the Euclidean distance between points $Q, R \in \mathbb{R}^n$. Therefore, using the “chosen” center point $C \in \mathbb{R}^n$, when:

$$\mathcal{G}(C, F_r) = \sup\{d(C, y) : y \in F_r\}$$

the **actual rate of expansion** is:

$$\mathcal{E}(C, F_r) = \mathcal{G}(C, F_{r+1}) - \mathcal{G}(C, F_r)$$

Note, there are cases of $(F_r)_{r \in \mathbb{N}}$ when $\mathcal{E}$ isn’t fixed and $\mathcal{E} \neq E$ (i.e., **the fixed, expected rate of expansion**).

5.5.2. Example

Suppose $A = \mathbb{R}^2$, $(F_r)_{r \in \mathbb{N}} = ([-r, r] \times [-r, r])_{r \in \mathbb{N}}$, and $C = (1, 1)$. One can clearly see $(-r, -r)$ is the point on F_r that is farthest from $(1, 1)$. Hence,

$$\mathcal{G}(C, F_r) = \sup\{d(C, y) : y \in F_r\} = \quad (97)$$

$$d((-r, -r), (1, 1)) = \quad (98)$$

$$\sqrt{(-r-1)^2 + (-r-1)^2} = \quad (99)$$

$$\sqrt{-(r+1)^2 + -(r+1)^2} = \quad (100)$$

$$\sqrt{(r+1)^2 + (r+1)^2} = \quad (101)$$

$$\sqrt{2(r+1)^2} = \quad (102)$$

$$\sqrt{2}(r+1) \quad (103)$$

and the actual rate of expansion is:

$$\mathcal{E}(C, F_r) = \mathcal{G}(C, F_{r+1}) - \mathcal{G}(C, F_r) = \quad (104)$$

$$\sqrt{2}((r+1)+1) - \sqrt{2}(r+1) = \quad (105)$$

$$\sqrt{2}(r+2) - \sqrt{2}(r+1) = \quad (106)$$

$$\sqrt{2}r + 2\sqrt{2} - \sqrt{2}r - \sqrt{2} = \quad (107)$$

$$\sqrt{2} \quad (108)$$

5.6. Reminder

See if Section 3.1 is easier to understand?

6. My Attempt At Answering The Approach of Section 1.3

6.1. Choice Function

Suppose $\mathbf{B}(\mathbb{R}^n)$ is the set of all bounded Borel subsets of $\mathbb{R}^n$. We then define the following:

1. If $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$ is an arbitrary set, then for all $r \in \mathbb{N}$, F_r^* satisfies (1), (2), (3), (4), and (5) of the **leading question** in Section 3.1
2. For all $j \in \mathbb{N}$, $F_j^{**} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$

Further note, from Section 5.4.2 (0b), if we take:

$$\begin{aligned} |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*), \omega), \psi)| &= \quad (109) \\ \inf\left\{|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}), \omega'), \psi')| : j \in \mathbb{N}, \omega' \in \Omega_{\varepsilon, j}, \psi' \in \Psi_{\varepsilon, j, \omega}, \mathbb{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}), \omega'), \psi')) \geq \mathbb{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*), \omega), \psi))\right\} \end{aligned}$$

and from Section 5.4.2 (0b), we take:

$$\begin{aligned} |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*), \omega), \psi)| &= \quad (110) \\ \sup\left\{|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}), \omega'), \psi')| : j \in \mathbb{N}, \omega' \in \Omega_{\varepsilon, j}, \psi' \in \Psi_{\varepsilon, j, \omega}, \mathbb{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{**}), \omega'), \psi')) \leq \mathbb{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*), \omega), \psi))\right\} \end{aligned}$$

then, using Section 5.4.1 (2), Equations (109) and (110):

$$\sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*), \omega), \psi)| = |\mathcal{S}'(\varepsilon, F_r^*)| = |\mathcal{S}'| \quad (111)$$

$$\sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*), \omega), \psi)| = |\mathcal{S}'(\varepsilon, F_r^*)| = |\mathcal{S}'| \quad (112)$$

$$\sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^*), \omega), \psi)| = |\mathcal{S}'(\varepsilon, F_r^*)| = |\mathcal{S}'| \quad (113)$$

6.2. Approach

We manipulate the definitions of Section 5.4.2 (0a), (0b) and Section 5.4.2 (1) to solve Section 3.1 (1)-(5) of the *leading question*.

6.3. Potential Answer

6.3.1. Preliminaries (Definition of T)

Suppose:

- The average of F_r^* for every $r \in \mathbb{N}$ is:

$$\text{Avg}(F_r^*) = \frac{1}{\mathcal{H}^{\dim_{\mathcal{H}}(F_r^*)}(F_r^*)} \int_{F_r^*} (x_1, \dots, x_n) d\mathcal{H}^{\dim_{\mathcal{H}}(F_r^*)} \quad (114)$$

- $d(P, Q)$ is the n -d Euclidean distance between points $P, Q \in \mathbb{R}^n$
- The difference of point $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ is:

$$X - Y = (x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$$

then define an *explicit* injective $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}$, where $r, j \in \mathbb{N}$, such that:

- If $d(\text{Avg}(F_r^*), C) < d(\text{Avg}(F_j^{**}), C)$, then $\mathcal{F}(\text{Avg}(F_r^*) - C) < \mathcal{F}(\text{Avg}(F_j^{**}) - C)$
- If $d(\text{Avg}(F_r^*), C) > d(\text{Avg}(F_j^{**}), C)$, then $\mathcal{F}(\text{Avg}(F_r^*) - C) > \mathcal{F}(\text{Avg}(F_j^{**}) - C)$
- If $d(\text{Avg}(F_r^*), C) = d(\text{Avg}(F_j^{**}), C)$, then $\mathcal{F}(\text{Avg}(F_r^*) - C) \neq \mathcal{F}(\text{Avg}(F_j^{**}) - C)$

where:

$$T(C, F_r^*) = \mathcal{F}(\text{Avg}(F_r^*) - C) \quad (115)$$

6.3.2. Question

Does T exist? If so, how do we define it?

Thus, using $|S'|$, $\lfloor S' \rfloor$, $\lceil S' \rceil$, E , $\mathcal{E}(C, F_r^*)$ (Section 5.5), and $T(C, F_r^*)$ with the absolute value function $\|\cdot\|$, ceiling function $\lceil \cdot \rceil$, and nearest integer function $[\cdot]$, we define:

$$K(\varepsilon, F_r^*) = (1 + \|\mathcal{E}(C, F_r^*) - E\|) \left(\left\| \frac{|S'| \left(1 + \left\lceil \frac{|S'|(|S'|+2|S'|)}{(|S'|+|S'|)(|S'|+|S'|)} \right\rceil \right) (1 + \lfloor |S'|/|S'| \rfloor)}{(1 + \lfloor |S'|/|S'| \rfloor) (1 + \lceil |S'|/|S'| \rceil)} - |S'| \right\| + |S'| \right) - T(C, F_r^*) \quad (116)$$

the choice function, which answers the **leading question** in Section 3.1, could be the following, s.t. we explain the reason behind choosing the choice function in Section 6.4:

Theorem 3. If we define:

$$\begin{aligned} \mathcal{M}(\varepsilon, F_r^*) &= |S'(\varepsilon, F_r^*)| (K(\varepsilon, F_r^*) - |S'(\varepsilon, F_r^*)|) \\ \mathcal{M}(\varepsilon, F_j^{**}) &= |S'(\varepsilon, F_j^{**})| (K(\varepsilon, F_j^{**}) - |S'(\varepsilon, F_j^{**})|) \end{aligned}$$

where for $\mathcal{M}(\varepsilon, F_r^*)$, we define $\mathcal{M}(\varepsilon, F_r^*)$ to be equivalent to $\mathcal{M}(\varepsilon, F_j^{**})$ when swapping " $j \in \mathbb{N}$ " with " $r \in \mathbb{N}$ " (for Equations (109) and (110)) and sets F_r^* with F_j^{**} (for Equation (109)–(116)), then for constant $v > 0$ and variable $v^* > 0$, if:

$$\overline{\mathcal{S}}(\varepsilon, r, v^*, F_j^{**}) = \inf \left(\left\{ |S'(\varepsilon, F_j^{**})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{**}) \geq \mathcal{M}(\varepsilon, F_r^*) \geq v^* \right\} \cup \{v^*\} \right) + v \quad (117)$$

and:

$$\underline{\mathcal{S}}(\varepsilon, r, v^*, F_j^{**}) = \sup \left(\left\{ |S'(\varepsilon, F_j^{**})| : j \in \mathbb{N}, v^* \leq \mathcal{M}(\varepsilon, F_j^{**}) \leq \mathcal{M}(\varepsilon, F_r^*) \right\} \cup \{-v^*\} \right) + v \quad (118)$$

then for all $F_j^* \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$ (Section 6 crit. 2), if:

$$\inf \left\{ \|1 - c\| : \forall (\epsilon > 0) \exists (c > 0) \forall (r \in \mathbb{N}) \exists (j \in \mathbb{N}) \left(\left\| \frac{|S'(\epsilon, F_r^*)|}{|S'(\epsilon, F_j^{**})|} - c \right\| < \epsilon \right) \right\} \quad (119)$$

where $\lceil \cdot \rceil$ is the ceiling function, E is the fixed rate of expansion, Γ is the gamma function, n is the dimension of $\mathbb{R}^n$, $\dim_H(F_r^*)$ is the Hausdorff dimension of set $F_r^* \subseteq \mathbb{R}^n$, and $\mathbf{A}_r$ is area of the smallest n -dimensional box that contains F_r^* , then

$$V(\epsilon, F_r^*, n) = \left[\left(\mathbf{A}_r^{1-\text{sign}(E)} (E - \text{sign}(E) + 1) \left(\frac{\exp(n \ln(\pi)/2)}{\Gamma(n/2 + 1)} \right) \left(r^{(n-\dim_H(F_r^*))} \right) \left(r^{\text{sign}(E)(\dim_H(F_r^*)-\text{sign}(\dim_H(F_r^*))+1)} \right) + (1 - \text{sign}(\dim_H(F_r^*))) \right) / \epsilon \right] / |S'(\epsilon, F_r^*)| \quad (120)$$

the choice function is:

$$\limsup_{\epsilon \rightarrow 0} \lim_{v^* \rightarrow \infty} \limsup_{r \rightarrow \infty} \left(\frac{\text{sign}(\mathcal{M}(\epsilon, F_r^*)) \bar{\mathcal{S}}(\epsilon, r, v^*, F_j^{**})}{|S'(\epsilon, F_r^*)| + v} - c^{-V(\epsilon, F_r^*, n)} \right) \quad (121)$$

$$\liminf_{\epsilon \rightarrow 0} \lim_{v^* \rightarrow \infty} \liminf_{r \rightarrow \infty} \left(\frac{\text{sign}(\mathcal{M}(\epsilon, F_r^*)) \bar{\mathcal{S}}(\epsilon, r, v^*, F_j^{**})}{|S'(\epsilon, F_r^*)| + v} - c^{-V(\epsilon, F_r^*, n)} \right) = \left(\frac{\text{sign}(\mathcal{M}(\epsilon, F_r^*)) \underline{\mathcal{S}}(\epsilon, r, v^*, F_j^{**})}{|S'(\epsilon, F_r^*)| + v} - c^{-V(\epsilon, F_r^*, n)} \right) = 0 \quad (122)$$

such that $F_r^* \in \mathcal{B}$ (Section 6 crit. 2) must satisfy Equations (121) and (122). (Note, we want $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$, and $(F_r^*)_{r \in \mathbb{N}}$ to answer the leading question of Section 3.1) where the answer to problems 1 and 2 & the approach of Section 1.3 is $\text{Avg}(F_r^*)$ (when it exists).

6.4. Explaining The Choice Function and Evidence The Choice Function Is Credible

Notice, before reading the programming in code 8, without the “c”-terms in Equation 121 and Equation 122:

1. The choice function in Equation (121) and Equation (122) is zero, when *what I’m measuring* from $(F_r^*)_{r \in \mathbb{N}}$ (Section 5.4.2 criteria 1) increases at a rate superlinear to that of $(F_j^{**})_{j \in \mathbb{N}}$, where $\text{sign}(\mathcal{M}(\epsilon, F_r^*)) = 0$.
2. The choice function in Equation (121) and Equation (122) is zero, when for a given $(F_r^*)_{r \in \mathbb{N}}$ and $(F_j^{**})_{j \in \mathbb{N}}$ there doesn’t exist c where Equation 119 is satisfied or $c = 0$.
3. When c does exist, suppose:

$$\left\{ \mathcal{J}(r) : r \in \mathbb{N}, \frac{|S'(\epsilon, F_r^*)|}{|S'(\epsilon, F_{\mathcal{J}(r)}^{**})|} \approx c \right\} \quad (123)$$

- (a) When $|S'(\epsilon, F_r^*)| < |S'(\epsilon, F_{\mathcal{J}(r)}^{**})|$, then:

$$\limsup_{\epsilon \rightarrow 0} \lim_{v^* \rightarrow \infty} \limsup_{r \rightarrow \infty} \frac{\text{sign}(\mathcal{M}(\epsilon, F_r^*)) \bar{\mathcal{S}}(\epsilon, r, v^*, F_j^{**})}{|S'(\epsilon, F_r^*)| + v} = c \quad (124)$$

$$\liminf_{\epsilon \rightarrow 0} \lim_{v^* \rightarrow \infty} \liminf_{r \rightarrow \infty} \frac{\text{sign}(\mathcal{M}(\epsilon, F_r^*)) \underline{\mathcal{S}}(\epsilon, r, v^*, F_j^{**})}{|S'(\epsilon, F_r^*)| + v} = 0 \quad (125)$$

- (b) When $|S'(\epsilon, F_r^*)| > |S'(\epsilon, F_{\mathcal{J}(r)}^{**})|$, then:

$$\limsup_{\epsilon \rightarrow 0} \lim_{v^* \rightarrow \infty} \limsup_{r \rightarrow \infty} \frac{\text{sign}(\mathcal{M}(\epsilon, F_r^*)) \bar{\mathcal{S}}(\epsilon, r, v^*, F_j^{**})}{|S'(\epsilon, F_r^*)| + v} = +\infty \quad (126)$$

$$\liminf_{\epsilon \rightarrow 0} \lim_{v^* \rightarrow \infty} \liminf_{r \rightarrow \infty} \frac{\text{sign}(\mathcal{M}(\epsilon, F_r^*)) \underline{\mathcal{S}}(\epsilon, r, v^*, F_j^{**})}{|S'(\epsilon, F_r^*)| + v} = 1/c \quad (127)$$

Hence, for each sub-criteria under crit. (3), if we subtract one of their limits by their limit value, then Equation 121 and Equation (122) is zero. (We do this by using the “c”-term in Equations (121) and (122)). However, when the exponents of the “c”-terms aren’t equal to -1 , the limits of Equation 121 and 122 aren’t equal to zero. We, infact, want this when we swap $|S'(\epsilon, F_r^*)|$ with $|S'(\epsilon, F_j^{**})|$. Moreover, we define function $V(\epsilon, F_r^*, n)$ (i.e., Equation (120)), where:

- i. When $S'(\epsilon, F_r^*) \gg \text{Numerator}(V(\epsilon, F_r^*, n))$, then Equation 121 and 122 without the “c”-terms are zero. (The “c”-terms approach zero and still allow Equations (121) and (122) to equal zero.)
- ii. When $S'(\epsilon, F_r^*) \ll \text{Numerator}(V(\epsilon, F_r^*, n))$ then $\text{sign}(\mathcal{M}(\epsilon, F_r^*))$ is zero, which makes Equation 121 and 122 equal zero.

- iii. Here are some examples of the numerator of $V(\varepsilon, F_r^*, n)$ (Equation (120)):
- When $E = 0, n = 1$, and $\dim_H(A) = 0$, the numerator of $V(\varepsilon, F_r^*, n)$ is $(\lceil Ar! + 1 \rceil / \varepsilon)$
 - When $E = z, n = 1$, and $\dim_H(A) = 0$, the numerator of $V(\varepsilon, F_r^*, n)$ is $\lceil (2zr \cdot r! + 1) / \varepsilon \rceil$
 - When $E = 0, n = z_2$, and $\dim_H(A) = z_2$, the numerator of $V(\varepsilon, F_r^*, n)$ is ceiling of constant A times the volume of an n -dimensional ball with finite radius: i.e.,

$$\left\lceil \frac{Az_1 \exp(z_2 \ln(\pi)/2)}{\Gamma(z_2/2 + 1)} \right\rceil / \varepsilon$$

- When $E = z_1, n = z_2$, and $\dim_H(A) = z_2$, the numerator of $V(\varepsilon, F_r^*, n)$ is ceiling of the volume of the n -dimensional ball: i.e.,

$$\left\lceil \frac{z_1 \exp(z_2 \ln(\pi)/2)}{\Gamma(z_2/2 + 1)} r^{z_2} \right\rceil / \varepsilon$$

Notation 8. Now, consider the code for Equation (121) and Equation (122). (Note, the set theoretic limit F_r^* is set A . In this example, $A = \mathbb{R}$ and:

- $(F_r^*)_{r \in \mathbb{N}} = (\{x^2 + y^2 = 6r^2\})_{r \in \mathbb{N}}$
- $(F_j^{**})_{j \in \mathbb{N}} = (\{x^2/(9j^2) + y^2/(4j^2) = 1\})_{j \in \mathbb{N}}$

We leave it to mathematicians to figure LengthS1, LengthS2, Entropy1 and Entropy2 for different A , $(F_r^*)_{r \in \mathbb{N}}$, and $(F_j^{**})_{j \in \mathbb{N}}$.

Listing 8: Code for Equation 121 and 122 on note 8

```
Clear["Global`*"]

(*We're using Mathematica*)

(*'A' is an arbitrary subset of  $\mathbb{R}^n$ *)
A=Reals

eps=.001 (*Since 'A' doesn't have Hausdorff dimension zero, we set 'eps' or  $\varepsilon$  to a small value*)

(*'LengthS1' is  $|S'(\varepsilon, F_r^*)|$ *)
LengthS1[r_] := LengthS1[r] = Ceiling[(6 Pi r^2)/eps]

(*'Entropy1' is the approximation of  $\sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} E(\mathcal{L}(S(C(\varepsilon, F_r^*), \omega), \psi))$  using asymptotic analysis*)
Entropy1[r_] := Entropy1[r] = Log2[(6 Pi r^2)/eps] (*We don't use Ceiling due to the code below 'q': 'Ceiling' can't be used in 'Solve'*)

(*'LengthS2' is  $|S'(\varepsilon, F_j^{**})|$ *)
LengthS2[j_] := LengthS2[j] = Ceiling[(6 Pi r^2)/eps]

(*'Entropy2' is the approximation  $\sup_{\omega \in \Omega_{\varepsilon, j}} \sup_{\psi \in \Psi_{\varepsilon, j, \omega}} E(\mathcal{L}(S(C(\varepsilon, F_j^{**}), \omega), \psi))$  using asymptotic analysis *)
Entropy2[j_] := Entropy2[j] = Log2[(6 Pi j^2)/eps] -.014427

q = 35; (*We want 'q' as large as possible; however, this is limited by computation time.*)

(*Below is the process of solving 'TableLowAlphr' which is  $|S'(\varepsilon, F_r^*)|$ *)
LowAlphValuesr = Table[
  {sol1[r_] :=
    sol1[r] = Reduce[j > 0 && Entropy2[j] <= Entropy1[r], j, Integers],
    LowSampler = Max[j /. Solve[sol1[r], {j}], Integers]},
  {r, 3, q}];
TableLowAlphr = Table[LowAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}]

(*Below is the process of solving 'TableUpAlphr' which is  $\overline{|S'(\varepsilon, F_r^*)|}$ *)
UpAlphValuesr = Table[
  {sol11[r_] :=
    sol11[r] =
      Reduce[j < 5000 && Entropy2[j] >= Entropy1[r], j, Integers],
    UpSampler = Min[j /. Solve[sol11[r], {j}], Integers]},
  {r, 3, q}];
TableUpAlphr = Table[UpAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}]

(*Below is the process of solving 'TableLowAlphj' which is  $|S'(\varepsilon, F_j^{**})|$ *)
LowAlphValuesj = Table[
  {sol2[j_] :=
```

```

sol2[j] =
  Reduce[r > 0 && Entropy1[r] <= Entropy2[j], r, Integers],
  LowSamplej = Max[r /. Solve[sol2[j], {r}, Integers]],
  LowAlphj = N[LengthS1[LowSamplej]], {j, 3, q}];
TableLowAlphj = Table[LowAlphValuesj[[j - 3 + 1, 3]], {j, 3, q}]

(*Below is the process of solving 'TableUpAlphj' which is  $\overline{|S'(\varepsilon, F_j^*)|}$ *)
UpAlphValuesj = Table[
  {sol21[j_] :=
    sol21[j] =
      Reduce[r < 5000 && Entropy1[r] >= Entropy2[j], r, Integers],
    UpSamplej = Min[r /. Solve[sol21[j], {r}, Integers]],
    UpAlphj = N[LengthS1[UpSamplej]], {j, 3, q}];
TableUpAlphj = Table[UpAlphValuesj[[j - 3 + 1, 3]], {j, 3, q}]

a[r_] := a[r] = TableUpAlphr[[r - 3 + 1]] (*This is  $|S'(\varepsilon, F_r^*)|$ *)
b[r_] := b[r] = LengthS1[r] (*This is  $|S'(\varepsilon, F_r^*)|$ *)
c[r_] := c[r] = TableLowAlphr[[r - 3 + 1]] (*This is the same as  $\overline{|S'(\varepsilon, F_r^*)|}$ *)

(* 'K1' is  $K(\varepsilon, F_r^*)$  *)
K1[r_] :=
  K1[r] = N[
    RealAbs[(b[
      r] (1 + Ceiling[(b[
        r] (a[r] + 2 b[r]))/((a[r] + b[r]) (a[r] + b[r] +
          c[r])) (1 + Round[a[r]/b[r]))]/((1 +
            Round[b[r]/c[r]) (1 + Round[a[r]/c[r])) - b[r] + b[r])
      ])]

a1[j_] :=
  a1[j] = TableUpAlphj[[j - 3 + 1]] (*This is  $|S'(\varepsilon, F_j^{**})|$ *)
b1[j_] := b1[j] = LengthS2[j] (*This is  $|S'(\varepsilon, F_j^{**})|$ *)
c1[j_] := c1[j] = TableLowAlphj[[j - 3 + 1]] (*This is  $\overline{|S'(\varepsilon, F_j^{**})|}$ *)

(* 'K2' is  $K(\varepsilon, F_j^{**})$  *)
K2[j_] :=
  K2[j] = N[
    RealAbs[(b1[
      j] (1 + Ceiling[(b1[
        j] (a1[j] + 2 b1[j]))/((a1[j] + b1[j]) (a1[j] +
          b1[j] + c1[j])) (1 + Round[a1[j]/b1[j]))]/((1 +
            Round[b1[j]/c1[j]) (1 + Round[a1[j]/c1[j])) - b1[j] +
          b1[j])
      ])]

(* 'Mr' is  $M'(\varepsilon, F_r^*)$  *)
Mr = Table[N[LengthS1[r] (K1[r] - LengthS1[r]), {r, 3, q - 1}]

(* 'Mj' is  $M'(\varepsilon, F_j^{**})$  *)
Mj = Table[N[LengthS2[j] (K2[j] - LengthS2[j]), {j, 3, q - 1}]

(* 'DownS' is  $\underline{S}(\varepsilon, r, v^*, F_j^{**})$  *)
DownS = Table[
  LengthS2[Flatten[
    Position[Mj, Max[Select[Mj, # <= Mr[[r - 4 + 2]] &]]][[1]] +
    4 - 2], {r, 4, q - 3}]

(* 'UpS' is  $\overline{S}(\varepsilon, r, v^*, F_j^{**})$  *)
UpS = Table[
  LengthS2[Flatten[
    Position[Mj, Min[Select[Mj, # >= Mr[[r - 4 + 2]] &]]][[1]] +
    4 - 2], {r, 4, q - 3}]

E1 = 1 (*Constant rate of expansion*)
dimH = 2 (*Hausdorff Dimension of A*)

(* 'Ar' is the smallest 2-dimensional box or  $[-\sqrt{6}r, \sqrt{6}r] \times [-\sqrt{6}r, \sqrt{6}r]$ 
  covering  $F_r^*$  with an area of  $24r^2$  *)
Ar[r_] := Ar[r] = 24r^2

(* 'V' is  $V(\varepsilon, F_r^*)$  or eq. 120. Note, n is the dimension
  of n-Euclidean Plane for which A is a subset *)
V[r_, n_] :=
  V[r, n] =
    Ceiling[(((Ar[r])^(1 - Sign[E1])) (E1 + (1 - Sign[E1])) ((Pi^(n/2))/

```

```

Gamma[n/2 + 1]) (r!^(n -
  dimH)) (r^(Sign[E1] (dimH - Sign[dimH] + 1))) + (1 -
  Sign[dimH])
Simplify[V[r]/eps]/LengthS1[r]

(* We couldn't add v, v* or convert this to a limit due to
limitations of the programming *)
ChoiceFunction =
Table[N[(((Sign[Mr[[r - 5 + 2]]] UpS[[r - 5 + 2]])/(LengthS1[
  r]) - (LengthS1[r]/LengthS2[r])^(-V[r,2]))*((Sign[Mr[[r - 5 + 2]]]
  DownS[[r - 5 + 2]])/(LengthS1[r]) -
  (LengthS1[r]/LengthS2[r])^(-V[r,2]))], {r, 5, q - 3}]

```

7. Questions

1. Does Section 6 answer the leading question in Section 3.1
2. If $\|\cdot\|_d$ is the n -d Euclidean norm, using Section 1.1 and thm. 3, where $A = A'$, is $\text{Avg}((F_r^*), A')$ unique and $\|\text{Avg}((F_r^*), A')\|_d$ finite?
3. If $\|\cdot\|_d$ is the n -d Euclidean norm, using Section 1.2 and thm. 3, where $A = A''$, is $\text{Avg}((F_r^*), A'')$ unique and $\|\text{Avg}((F_r^*), A')\|_d$ finite?
4. If there's no time to check questions 1, 2, and 3, see Section 4.

8. Appendix of Section 5.4.1

8.1. Example of Section 5.4.1, step 1

Suppose

1. $A = \mathbb{R}^2$
2. $(F_r^*)_{r \in \mathbb{N}} = ([-r, r] \times [-r, r])_{r \in \mathbb{N}}$

Then one example of $C(13/6, F_2^*, 1)$, using Section 5.4.1 step 1, (where $F_2^* = [-2, 2] \times [-2, 2]$) is:

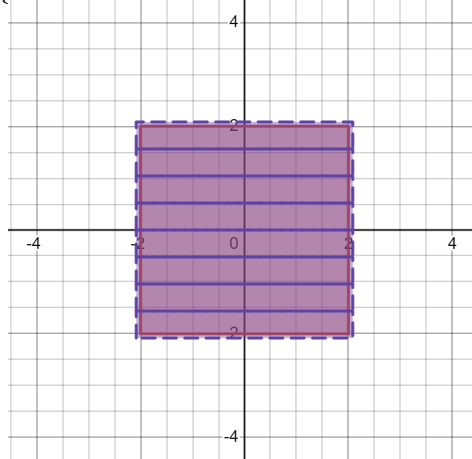
$$\begin{aligned}
 & \left\{ \left[-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}} \right] \times \left[-\sqrt{\frac{13}{3}}, -\frac{3}{4}\sqrt{\frac{13}{3}} \right], \left[-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}} \right] \times \left[-\frac{3}{4}\sqrt{\frac{13}{3}}, -\frac{1}{2}\sqrt{\frac{13}{3}} \right], \left[-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}} \right] \times \left[-\frac{1}{2}\sqrt{\frac{13}{3}}, -\frac{1}{4}\sqrt{\frac{13}{3}} \right], \right. \\
 & \left[-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}} \right] \times \left[-\frac{1}{4}\sqrt{\frac{13}{3}}, 0 \right], \left[-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}} \right] \times \left[0, \frac{1}{4}\sqrt{\frac{13}{3}} \right], \left[-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}} \right] \times \left[\frac{1}{4}\sqrt{\frac{13}{3}}, \frac{1}{2}\sqrt{\frac{13}{3}} \right], \\
 & \left. \left[-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}} \right] \times \left[\frac{1}{2}\sqrt{\frac{13}{3}}, \frac{3}{4}\sqrt{\frac{13}{3}} \right], \left[-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}} \right] \times \left[\frac{3}{4}\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}} \right] \right\}
 \end{aligned} \tag{128}$$

Note, the area of each of the rectangles is $13/6$, where the borders could be approximated as:

$$\begin{aligned}
 & \{[-2.082, 2.082] \times [-2.082, -1.561], [-2.082, 2.082] \times [-1.561, -1.041], [-2.082, 2.082] \times [-1.041, -.520], \\
 & [-2.082, 2.082] \times [-.520, 0], [-2.082, 2.082] \times [0, .520], [-2.082, 2.082] \times [.520, 1.041], \\
 & [-2.082, 2.082] \times [1.041, 1.561], [-2.082, 2.082] \times [1.561, 2.082]\}
 \end{aligned} \tag{129}$$

and we'll illustrate this as purple rectangles covering F_2^* (i.e., the red square).

Figure 1. Purple rectangles are the “covers” and the red square is F_2^* (Ignore the boundaries)



(Note, the purple rectangles in fig. 1, satisfy step (1) of Section 5.4.1, since the Hausdorff measures *in its dimension* of the rectangles is $13/6$ and there is a minimum 8 covers over-covering F_2^* : i.e.,

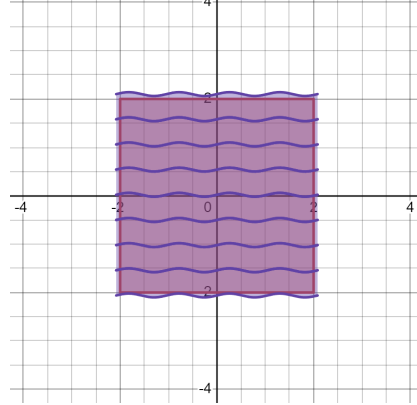
Definition 3 (Minimum Covers of Measure ε covering F_r^*). We can compute the minimum covers $\mathbf{C}(\varepsilon, F_r^*, \omega)$, using the formula:

$$\lceil \mathcal{H}^{\dim_H(F_r^*)}(F_r^*)/\varepsilon \rceil$$

where $\lceil \mathcal{H}^{\dim_H(F_2^*)}(F_2^*)/\varepsilon \rceil = \lceil \text{Area}([-2, 2]^2)/(13/6) \rceil = \lceil 16/(13/6) \rceil = \lceil 96/13 \rceil = \lceil 7 + (5/13) \rceil = 8$

Note the covers in $\mathbf{C}(13/6, F_r^*, \omega)$ need not be rectangles. In fact, they could be any set as long as the “area” of those sets is $13/6$, and F_2^* is over-covered by the smallest number of sets possible. Here is an example:

Figure 2. The eight purple sets are the “covers” and the red square is F_2^* . (Ignore the boundaries)



To define this cover, start off with:

$$B_1(x) = -2.062 + .04 \sin(6.037x) \quad (130)$$

$$B_2(x) = -1.541 + .04 \sin(6.037x) \quad (131)$$

$$B_3(x) = -1.021 + .04 \sin(6.037x) \quad (132)$$

$$B_4(x) = -0.500 + .04 \sin(6.037x) \quad (133)$$

$$B_5(x) = 0.020 + .04 \sin(6.037x) \quad (134)$$

$$B_6(x) = 0.540 + .04 \sin(6.037x) \quad (135)$$

$$B_7(x) = 1.061 + .04 \sin(6.037x) \quad (136)$$

$$B_8(x) = 1.581 + .04 \sin(6.037x) \quad (137)$$

$$B_9(x) = 2.102 + .04 \sin(6.037x) \quad (138)$$

$$(139)$$

Then, for each $i \in \mathbb{N}$ in $1 \leq i \leq 7$, we define:

$$\mathcal{B}_i = \{(x, y) : -2.082 \leq x \leq 2.082, B_i(x) \leq y < B_{i+1}(x)\}$$

except $i = 8$ where:

$$\mathcal{B}_8 = \{(x, y) : -2.082 \leq x \leq 2.082, B_8(x) \leq y \leq B_9(x)\}$$

such that an example of $\mathbf{C}(13/6, F_2^*, 2)$ is:

$$\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8\} \quad (140)$$

In the case of F_2^* , there are uncountably many $\mathbf{C}(13/6, F_2^*, \omega)$ of different shapes and sizes which we can use. However, these examples were the ones taken.

8.2. Example of Section 5.4.1, step 2

Suppose

1. $A = \mathbb{R}^2$
2. $(F_r^*)_{r \in \mathbb{N}} = ([-r, r] \times [-r, r])_{r \in \mathbb{N}}$
3. $F_2^* = [-2, 2] \times [-2, 2]$

4. $\mathbf{C}(13/6, F_2^*, 3)$, using Equation (129) and Figure 1, is

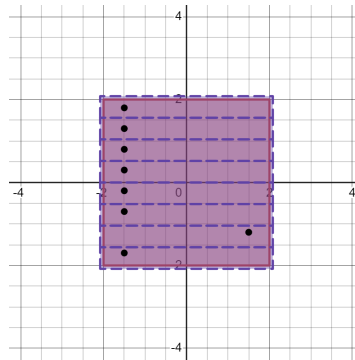
$$\begin{aligned} & \{[-2.082, 2.082] \times [-2.082, -1.561], [-2.082, 2.082] \times [-1.561, -1.041], [-2.082, 2.082] \times [-1.041, -.520], \\ & [-2.082, 2.082] \times [-.520, 0], [-2.082, 2.082] \times [0, .520], [-2.082, 2.082] \times [.520, 1.041], \\ & [-2.082, 2.082] \times [1.041, 1.561], [-2.082, 2.082] \times [1.561, 2.082]\} \end{aligned} \quad (141)$$

Then, an example of $\mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)$ is:

$$\{(-1.5, -1.7), (1.5, -1.2), (-1.5, -.7), (-1.5, -.2), (-1.5, .3), (-1.5, .8), (-1.5, 1.3), (-1.5, 1.8)\} \quad (142)$$

Below, is an illustration of the sample: i.e., the set of all black points in each purple rectangle of $\mathbf{C}(13/6, F_2^*, 3)$ covering $F_2^* = [-2, 2]^2$:

Figure 3. The black points are the “sample points”, the eight purple rectangles are the “covers”, and the red square is F_2^* . (Ignore the boundaries)



Note there are multiple samples we can take, as long as one sample point is taken from each cover in $\mathbf{C}(13/6, F_2^*, 1)$.

8.3. Example of Section 5.4.1, step 3

Suppose

1. $A = \mathbb{R}^2$
2. $(F_r^*)_{r \in \mathbb{N}} = ([-r, r] \times [-r, r])_{r \in \mathbb{N}}$
3. $F_2^* = [-2, 2] \times [-2, 2]$
4. $\mathbf{C}(13/6, F_2^*, 3)$, using Equation (129) and Figure 1, is

$$\begin{aligned} & \{[-2.082, 2.082] \times [-2.082, -1.561], [-2.082, 2.082] \times [-1.561, -1.041], [-2.082, 2.082] \times [-1.041, -.520], \\ & [-2.082, 2.082] \times [-.520, 0], [-2.082, 2.082] \times [0, .520], [-2.082, 2.082] \times [.520, 1.041], \\ & [-2.082, 2.082] \times [1.041, 1.561], [-2.082, 2.082] \times [1.561, 2.082]\} \end{aligned}$$

5. $\mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)$, using Equation 142, is:

$$\{(-1.5, -1.7), (1.5, -1.2), (-1.5, -.7), (-1.5, -.2), (-1.5, .3), (-1.5, .8), (-1.5, 1.3), (-1.5, 1.8)\} \quad (143)$$

Therefore, consider the following process:

8.3.1. Step 33a

If $\mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)$ is:

$$\{(-1.5, -1.7), (1.5, -1.2), (-1.5, -.7), (-1.5, -.2), (-1.5, .3), (-1.5, .8), (-1.5, 1.3), (-1.5, 1.8)\} \quad (144)$$

suppose $x_0 = (-1.5, -1.7)$. Note, the following:

1. $x_1 = (-1.5, -.7)$ is the next point in the “pathway” since it’s a point in $\mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)$ with the smallest 2-d Euclidean distance to x_0 instead of x_0 .
2. $x_2 = (-1.5, -.2)$ is the third point since it’s a point in $\mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)$ with the smallest 2-d Euclidean distance to x_1 instead of x_0 and x_1 .
3. $x_3 = (-1.5, .3)$ is the fourth point since it’s a point in $\mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)$ with the smallest 2-d Euclidean distance to x_2 instead of x_0, x_1 , and x_2 .
4. we continue this process, where the “pathway” of $\mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)$ is:

$$(-1.5, -1.7) \rightarrow (-1.5, -.7) \rightarrow (-1.5, -.2) \rightarrow (-1.5, .3) \rightarrow (-1.5, .8) \rightarrow (-1.5, 1.3) \rightarrow (-1.5, 1.8) \rightarrow (1.5, -1.2) \quad (145)$$

Notation 9. If more than one point has the minimum 2-d Euclidean distance from x_0, x_1, x_2 , etc. take all potential pathways: e.g., using the sample in Equation (144), if $x_0 = (-1.5, -.2)$, then since $(-1.5, -.7)$ and $(-1.5, .3)$ have the smallest Euclidean distance from $(-1.5, -.2)$, and for point $x_1 = (-1.5, -.7)$, since $(-1.5, -1.7)$ and $(-1.5, .3)$ have the smallest Euclidean distance from $(-1.5, -.7)$, we take *three* pathways:

$$\begin{aligned} &(-1.5, -.2) \rightarrow (-1.5, -.7) \rightarrow (-1.5, -1.7) \rightarrow (-1.5, .3) \rightarrow (-1.5, .8) \rightarrow (-1.5, 1.3) \rightarrow (-1.5, 1.8) \rightarrow (1.5, -1.2) \quad (\text{path 1}) \\ &(-1.5, -.2) \rightarrow (-1.5, -.7) \rightarrow (-1.5, .3) \rightarrow (-1.5, .8) \rightarrow (-1.5, 1.3) \rightarrow (-1.5, 1.8) \rightarrow (-1.5, -1.7) \rightarrow (1.5, -1.2) \quad (\text{path 2}) \\ &(-1.5, -.2) \rightarrow (-1.5, .3) \rightarrow (-1.5, .8) \rightarrow (-1.5, 1.3) \rightarrow (-1.5, 1.8) \rightarrow (-1.5, -.7) \rightarrow (-1.5, -1.7) \rightarrow (1.5, -1.2) \quad (\text{path 3}) \end{aligned}$$

8.3.2. Step 33b

Take the length of all line segments in the pathway. In other words, suppose $d(P, Q)$ is the n -th dim. Euclidean distance between points $P, Q \in \mathbb{R}^n$. Using the pathway of Equation (145), we want:

$$\{d((-1.5, -1.7), (-1.5, -.7)), d((-1.5, -.7), (-1.5, -.2)), d((-1.5, -.2), (-1.5, .3)), d((-1.5, .3), (-1.5, .8)), d((-1.5, .8), (-1.5, 1.3)), d((-1.5, 1.3), (-1.5, 1.8)), d((-1.5, 1.8), (1.5, -1.2))\} \quad (146)$$

Whose distances can be approximated as:

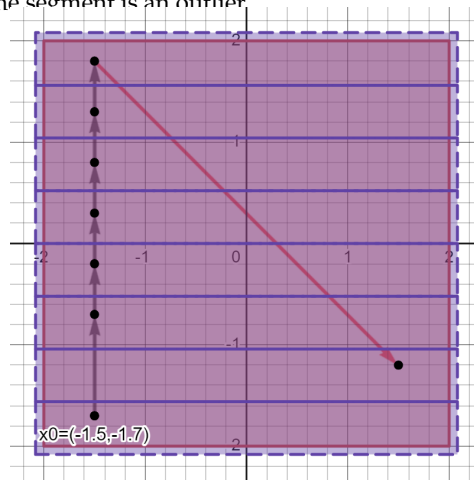
$$\{1, .5, .5, .5, .5, .5, 4.25\} \quad (147)$$

As we can see, the outlier is 4.25 (i.e., note these outliers become more prominent for $\varepsilon \ll 13/6$). Therefore, remove 4.25 from the set of distances:

$$\{1, .5, .5, .5, .5, .5\}$$

which we can illustrate with:

Figure 4. x_0 is the start point in the pathway. The black line segments in the “pathway” have lengths which aren’t outliers. The length of the red line segment is an outlier.



Hence, when $x_0 = (-1.5, -1.7)$, using Section 5.4.1 step 33b & Equation (144), we note:

$$\mathcal{L}((-1.5, -1.7), S(C(13/6, F_2^*, 3), 1)) = \{1, .5, .5, .5, .5, .5\} \quad (148)$$

8.3.3. Step 33c

To convert the set of distances in Equation (148) into a probability distribution, we take:

$$\sum_{x \in \{1, .5, .5, .5, .5, .5\}} x = 1 + .5 + .5 + .5 + .5 + .5 = 3.5 \quad (149)$$

Then divide each element in $\{1, .5, .5, .5, .5, .5\}$ by 3.5

$$\{1/(3.5), .5/(3.5), .5/(3.5), .5/(3.5), .5/(3.5), .5/(3.5)\}$$

which gives us the probability distribution:

$$\{2/7, 1/7, 1/7, 1/7, 1/7, 1/7\}$$

Hence,

$$\mathbb{P}(\mathcal{L}((-1.5, -1.7), \mathcal{S}(\mathbf{C}(\epsilon, F_2^*, 3), 1))) = \{2/7, 1/7, 1/7, 1/7, 1/7, 1/7\} \quad (150)$$

8.3.4. Step 33d

Take the shannon entropy of Equation (150):

$$\begin{aligned} E(\mathbb{P}(\mathcal{L}((-1.5, -1.7), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 1), 3)))) &= \\ \sum_{x \in \mathbb{P}(\mathcal{L}((-1.5, -1.7), \mathcal{S}(\mathbf{C}(13/6, F_2^*, \omega), \psi)))} -x \log_2 x &= \sum_{x \in \{2/7, 1/7, 1/7, 1/7, 1/7, 1/7\}} -x \log_2 x = \\ &= -(2/7) \log_2(2/7) - (1/7) \log_2(1/7) - (1/7) \log_2(1/7) - (1/7) \log_2(1/7) - (1/7) \log_2(1/7) - (1/7) \log_2(1/7) = \\ &= -(2/7) \log_2(2/7) - (5/7) \log_2(1/7) \approx 2.5216 \end{aligned}$$

We shorten $E(\mathbb{P}(\mathcal{L}((-1.5, -1.7), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))))$ to $E(\mathcal{L}((-1.5, -1.7), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)))$, giving us:

$$E(\mathcal{L}((-1.5, -1.7), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) \approx 2.5216 \quad (151)$$

8.3.5. Step 33e

Take the entropy w.r.t all pathways of:

$$\begin{aligned} \mathcal{S}(\mathbf{C}(13/6, F_2^*, 1), 3)) &= \\ \{(-1.5, -1.7), (1.5, -1.2), (-1.5, -0.7), (-1.5, -0.2), (-1.5, .3), (-1.5, .8), (-1.5, 1.3), (-1.5, 1.8)\} \end{aligned}$$

In other words, we'll compute:

$$E(\mathcal{L}(\mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) = \sup_{x_0 \in \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)} E(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1)))$$

We do this by repeating Section 8.3.1-Section 8.3.4 for different $x_0 \in \mathcal{S}(\mathbf{C}(13/6, F_2^*, 1), 3))$ (i.e., in the equations with multiple values, see note 9)

$$E(\mathcal{L}((-1.5, -1.7), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) \approx 2.52164 \quad (152)$$

$$E(\mathcal{L}((1.5, -1.2), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) \approx 2.34567 \quad (153)$$

$$E(\mathcal{L}((-1.5, -0.7), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) \approx 2.2136 \quad (154)$$

$$E(\mathcal{L}((-1.5, -0.2), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) \approx 2.32193, 2.11807, 2.3703 \quad (155)$$

$$E(\mathcal{L}((-1.5, 0.3), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) \approx 2.24244, 2.40051 \quad (156)$$

$$E(\mathcal{L}((-1.5, 0.8), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) \approx 2.22393, 2.41651 \quad (157)$$

$$E(\mathcal{L}((-1.5, 1.3), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) \approx 2.22208, 2.4076 \quad (158)$$

$$E(\mathcal{L}((-1.5, 1.8), \mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) \approx 2.52164 \quad (159)$$

Hence, since the largest value out of Equations (152)–(159) is 2.52164:

$$E(\mathcal{L}(\mathcal{S}(\mathbf{C}(13/6, F_2^*, 3), 1))) = \sup_{x_0 \in \mathcal{S}(\mathbf{C}(\epsilon, F_2^*, 3), 1)} E(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\epsilon, F_2^*, 3), 1))) \approx 2.52164$$

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