doi:10.20944/preprints202209.0089.v5

holomorphic functions of n, which makes those objects omnidimensional

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The study shows that the volumes and surfaces of the n-simplices, n-orthoplices, n-cubes, and n-balls are holomorphic functions of n, which makes those objects omnidimensional. Furthermore, the volume of an n-simplex is shown to be a bivalued function of n, and thus the surfaces of n-simplices and n-orthoplices are also bivalued functions of n. Applications of these formulas to the omnidimensional polytopes inscribed in and circumscribed about n-balls reveal previously unknown properties of these geometric objects in negative dimensions. In particular, for $0 < n < 1$, the volumes of the omnidimensional polytopes are larger than those of circumscribing n-balls, while their volumes and surfaces are smaller than the volumes of inscribed n-balls. Reflection relations around $n = 0$ for volumes and surfaces of these polytopes inscribed in and circumscribed about n-balls are disclosed. Specific products and quotients of volumes and surfaces of the omnidimensional polytopes and n-balls are shown to be independent of the gamma function.

Keywords: regular basic convex polytopes; circumscribed and inscribed polytopes; negative dimensions; fractal dimensions; complex dimensions; emergent dimensionality

I. INTRODUCTION

The notion of dimension n is intuitively defined as a natural number of coordinates of a point within Euclidean space $\mathbb{R}^n$. However, this is not the only possible definition [1, 2] of a dimension of a set. Negative dimensions [2–5], for example, can be defined by analytic continuations from positive dimensions [6]. Fractional (or fractal), including negative [7], dimensions are consistent with experimental results and enable the examination of transport parameters in multiphase fractal media [8, 9]. This renders dimension n real, or at least a rational number. Complex [2], including complex fractional [10], dimensions can also be considered. Complex geodesic paths, for example, emerge in the presence of black hole singularities [11], and when studying entropic dynamics on curved statistical manifolds [12]. Complex wavelengths were reported in Maxwell-Boltzmann, and Fermi-Dirac statistics on black hole event horizons [13]. Fractional derivatives of complex functions could be able to describe different physical phenomena [14]. For example, it was recently shown [15] that magnetic monopole motion in 3-simplicial spin ice crystal lattice is limited to a fractal cluster.

In n-dimensional space, n-dimensional objects have $(n-1)$-dimensional surfaces which have a dimension of volume in $(n-1)$-dimensional space. However, this sequence has a singularity at $n = -1$. A 0-dimensional point in 0-dimensional space has a vanishing (-1)-surface being a vanishing volume of the (-1)-dimensional void. But the surface of the (-1)-dimensional void is not (-2)-dimensional. It is undefined. This discontinuity, along with the recently discovered [16] reflection relations around $n = 0$ for volumes and surfaces of n-cubes inscribed in n-balls, hint that thinking about dimension in terms of a point on a number axis, with negative dimensions being simply analytic continuations from positive ones [6], may be misleading. Thinking of the dimension as a point on a number semiaxis, similar to a point on a radius, seems more appropriate. Thus n-dimension corresponds to $(-n)$-dimension. Considering the dimension of a set as the length exponent at which that set can be measured makes the negative dimensions refer to densities as positive ones refer to quantities [5]. Thus, the (-2)-dimensional pressure, for example, considered in terms of a density (e.g., in units of $\mathrm{N/m^2}$) corresponds to a 2-dimensional area (e.g., in units of $\mathrm{m^2}$) that it acts upon. Following the same logic, gravitational force $F = GMm/R^2$ acting towards an *inside* enclosed by a 2-dimensional equipotential surface is (-2)-dimensional, whereas centripetal force $F = mV^2/R^1$ acting towards an *outside* of a 1-dimensional curve is (-1)-dimensional.

The notion of distance intuitively defines how far apart two points are. Thus, intuition suggests that it is a non-negative quantity. However, intuition can be misleading[1], and the Euclidean distance (the Pythagorean theorem) admits not only the principal, non-negative, square root but also a negative one. This fact, taken plainly, violates the nonnegativity axiom of the metric. However, diffuse metrics [17], including the Łukaszyk-Karmowski metric [18], are known to violate the identity of indiscernibles axiom of the metric. This hints that axiomatizing distance as a non-negative quantity may also be misleading.

This study extends the prior research [16, 19] presenting novel recurrence relations for volumes and surfaces of n-balls, regular n-simplices, and n-orthoplices to complex dimensions. It was signaled in the prior research [16]

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[1] E.g., in aviation, where relying on a sense of orientation (aka intuition) can be fatal.

that these recurrence relations are continuous on their domains of definitions for $n \in \mathbb{R}$, whereas the starting points for fractional dimensions can be provided, e.g., using spline interpolation between two (or three, in the case of n-balls) subsequent integer dimensions. It was also conjectured in [16] that for $0 < n < 1$, volumes of n-cubes inscribed in n-balls are larger than volumes of those n-balls.

The study shows that the recurrence relations of the prior research [16, 19] allow to remove indefiniteness present in known formulas for volumes and surface of these three polytopes present $\forall n \in \mathbb{N}_0$ [20] and thus makes them holomorphic functions of a complex dimension n and omnidimensional (i.e., well defined $\forall n \in \mathbb{C}$), including inscribed in and circumscribed about n-balls. It is shown that for $0 < n < 1$, their volumes are larger than those of circumscribing n-balls, while their volumes and surfaces are smaller than volumes of inscribed n-balls.

The paper is structured as follows. Section II summarizes known formulas for volumes and surfaces of regular, convex polytopes employed in the paper's further sections. Section III shows that these recurrence relations can be extended to complex dimensions, yielding complex values, as shown in Section IV. V examines the properties of the omnidimensional, regular, convex polytopes inscribed in and circumscribed about n-balls. Section VI presents a conjecture concerning reflection relations presented in the preceding section and discusses some of their properties. Section VII presents metric-independent relations between volumes and surfaces of omnidimensional polytopes and n-balls. Section VIII hints at possible applications and concludes the findings of this paper.

II. KNOWN FORMULAS

It is known that the volume of an n-ball (B) is

$$V(n)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n, \tag{1}$$

where R denotes the n-ball radius, $\Gamma(\mathbb{C} \to \mathbb{C})$ is the Euler's gamma function, and $n \in \mathbb{C} \setminus \{n = -2k-2, k \in \mathbb{N}_0\}$. As the gamma function is meromorphic, volumes of n-balls are complex in complex dimensions. The volume of an n-ball can be expressed [21] in terms of the volume of an $(n-2)$-ball of the same radius as a recurrence relation

$$V(n)_B = \frac{2\pi R^2}{n} V(n-2)_B, \tag{2}$$

where $V(0)_B \doteq 1$ and $V_1(R)_B \doteq 2R$. The relation (2) can be extended [16] into negative dimensions as

$$V(n)_B = \frac{n+2}{2\pi R^2} V(n+2)_B, \tag{3}$$

solving (2) for $V(n-2)$ and assigning new n as the previous $n-2$. A radius recurrence relation[2] [16]

$$f(n) \doteq \frac{2}{n} f(n-2), \tag{4}$$

where $n \in \mathbb{C}$, $f(0) \doteq 1$ and $f(1) \doteq 2$, allows for expressing the volume n-ball as

$$V(n)_B \doteq f(n)\pi^{\lfloor n/2 \rfloor} R^n, \tag{5}$$

where "$\lfloor x \rfloor$" is the floor function that yields the greatest integer less than or equal to its argument x. The relation (4) can be analogously as formula (2) extended [16] into complex dimensions with a negative real part as

$$f(n) = \frac{n+2}{2} f(n+2), \tag{6}$$

which allows to define $f(-1) \doteq 1$, $f(0) \doteq 1$ to initiate (4) or (6). Known [21] surface of an n-ball[3] is

$$S(n)_B = \frac{n}{R} V(n)_B. \tag{7}$$

Known volume of n-cube (C) is

$$V(n)_C = A^n, \tag{8}$$

where $n \in \mathbb{C}$, A is the edge length (the metric factor) and A^n is the cubic factor. The known surface of n-cube is

$$V(n)_C = 2nA^{n-1}, \tag{9}$$

where $n \in \mathbb{C}$.

Known [22, 23] volume of a regular n-simplex (S) is

$$V(n)_S = \frac{\sqrt{n+1}}{n!\sqrt{2^n}} A^n, \tag{10}$$

where $n \in \mathbb{N}$. The formula (10) can be written [16] as a recurrence relation

$$V(n)_S = AV(n-1)_S \sqrt{\frac{n+1}{2n^3}}, \tag{11}$$

with $V(0)_S \doteq 1$, to remove the indefiniteness of the factorial for $n < 1$. Formula (11) can be solved for $V(n-1)$. Assigning new n as the previous $n-1$, yields [16]

$$V(n)_S = \frac{V(n+1)_S}{A} \sqrt{\frac{2(n+1)^3}{n+2}}, \tag{12}$$

[2] We choose the notation $f(n)$ over f_n, as $n \in \mathbb{C}$.

[3] Commonly, the surface $S(n)$ is defined to correspond to $(n+1)$-dimensional object. However, the author finds it confusing: $(n-1)$-dimensional surface remains the property of the n-dimensional object. Thus, in this study $S(n)$ denotes $(n-1)$-dimensional surface of the n-dimensional object, not n-dimensional surface of the $(n+1)$-dimensional object.

 doi:10.20944/preprints202209.0089.v5

which also removes the singularity for $n = 0$ present in known formula (10). n-simplex has $n + 1$ $(n-1)$-facets [21]. Therefore, its surface is

$$S(n)_S = (n+1)V(n-1)_S. \tag{13}$$

Known [21] volume of n-orthoplex (O) is

$$V(n)_O = \frac{\sqrt{2^n}}{n!}A^n, \tag{14}$$

where $n \in \mathbb{N}_0$.

Formula (14) can be written [16] as a recurrence relation

$$V(n)_O = AV(n-1)_O\frac{\sqrt{2}}{n}, \tag{15}$$

where $n \neq 0$, with $V(0)_O \doteq 1$, to remove the indefiniteness of the factorial for $n < 0$. Solving (15) for $V(n-1)$ and assigning new n as the previous $n-1$, yields [16]

$$V(n)_O = V(n+1)_O\frac{n+1}{A\sqrt{2}}, \tag{16}$$

which also removes singularity for $n = 0$ present in formula (14). Any n-orthoplex has 2^n facets [21], which are regular $(n-1)$-simplices. Therefore, its surface is

$$S(n)_O = 2^nV(n-1)_S. \tag{17}$$

III. HOLOMORPHIZING KNOWN FORMULAS

Known formulas presented in the preceding section can be extended to complex dimensions. We note that volumes and surfaces of n-cubes are already defined $\forall n \in \mathbb{C}$ and thus holomorphic.

Theorem 1. *The volume of an n-ball is a holomorphic function of a complex dimension n.*

Proof. First, we note that the recurrence relations (3) and (5) correspond to each other

$$\begin{aligned} V(n)_B &= \frac{n+2}{2\pi R^2}V(n+2)_B = \frac{n+2}{2}f(n+2)\pi^{\lfloor n/2\rfloor}R^n \\ V(n+2)_B &= \pi f(n+2)\pi^{\lfloor n/2\rfloor}R^{n+2}, \end{aligned} \tag{18}$$

which, after setting $m \doteq n+2 \rightarrow n = m-2$ in (18), yields

$$V(m)_B = f(m)\pi^{1+\lfloor (m-2)/2\rfloor}R^m = f(m)\pi^{\lfloor m/2\rfloor}R^m. \tag{19}$$

Comparing the non-recurrence general n-ball volume formula (1), which is valid within the domain of the gamma function, i.e. $\forall n \in \mathbb{C} \setminus \{n = -2k-2,\ k \in \mathbb{N}_0\}$, with the recurrence relation (3)

$$V(n)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)}R^n = \frac{n+2}{2\pi R^2}V(n+2)_B \tag{20}$$

yields

$$V(n+2)_B = \frac{2\pi^{n/2+1}}{(n+2)\Gamma(n/2+1)}R^{n+2}. \tag{21}$$

Setting $m \doteq n+2 \rightarrow n = m-2$ yields

$$\begin{aligned} V(m)_B &= \frac{2\pi^{m/2}}{m\Gamma(m/2)}R^m = \frac{\pi^{m/2}}{\frac{m}{2}\Gamma(m/2)}R^m = \\ &= \frac{\pi^{m/2}}{\Gamma(m/2+1)}R^m, \end{aligned} \tag{22}$$

which recovers (1), as $\frac{n}{2}\Gamma(n/2) = \Gamma(n/2+1)$, $\forall n \in \mathbb{C} \setminus \{n = -2k,\ k \in \mathbb{N}_0\}$. Thus we have proved that the recurrence relations (2), (3), and (5) correspond to the general n-ball volume formula (1) within this domain.

However, now we can use any of the *backward* recurrence relations (3) or (5) with (6) to determine the values of the n-ball volume outside this domain: $\forall n+2 \in \mathbb{C}$ we can find $V(n)_B$.

On the other hand

$$\lim_{n\rightarrow -2k-2, k\in\mathbb{N}_0}\frac{\pi^{n/2}R^n}{\Gamma(n/2+1)} = \frac{a}{\infty} = 0, \tag{23}$$

where $a \in \mathbb{R}$, so the poles of the meromorphic gamma function $\Gamma(n/2+1)$ present in (1), now defined in the sense of a limit of a function, vanish - which completes the proof. □

Corollary 1.1. *The surface of an n-ball is a holomorphic function of a complex dimension n.*

Proof. If the volume of an n-ball is a holomorphic function by Theorem 1, then, using formula (7), the surface of an n-ball is also a holomorphic function.

Using (1) and (7) the surface of an n-ball is given by

$$S(n)_B = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}R^{n-1}. \tag{24}$$

□

Theorem 2. *The volume of a regular n-simplex is a holomorphic, bivalued function of a complex dimension n.*

Proof. Expressing the factorial in the volume of a regular n-simplex formula (10) by the gamma function extends the domain of applicability of the formula (10) to complex dimensions

$$V(n)_S = \frac{\sqrt{n+1}}{n!\sqrt{2^n}}A^n = \frac{\sqrt{n+1}}{\Gamma(n+1)2^{n/2}}A^n. \tag{25}$$

On the other hand, comparing (25) with the recurrence relation (12)

$$\frac{\sqrt{n+1}}{\Gamma(n+1)2^{n/2}}A^n = \frac{V(n+1)_S}{A}\sqrt{\frac{2(n+1)^3}{n+2}} \tag{26}$$

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yields

$$V(n+1)_S = \frac{\sqrt{n+1}\sqrt{n+2}}{\Gamma(n+1)2^{(n+1)/2}\sqrt{(n+1)^3}}A^{n+1}, \quad (27)$$

which, after setting $m \doteq n+1$ in (27), yields

$$\begin{aligned} V(m)_S =& \frac{\sqrt{m}\sqrt{m+1}}{\Gamma(m)2^{m/2}\sqrt{m^3}}A^m \\ =& \frac{\sqrt{m+1}}{\Gamma(m+1)2^{m/2}}A^m \frac{m\sqrt{m}}{\sqrt{m^3}} \\ =& \pm \frac{\sqrt{m+1}}{\Gamma(m+1)2^{m/2}}A^m, \end{aligned} \quad (28)$$

and recovers (25) as $m\Gamma(m) = \Gamma(m+1)\ \forall m \in \mathbb{C} \setminus \{m = -k, k \in \mathbb{N}_0\}$ and $m\sqrt{m}/\sqrt{m^3} = \pm 1$ (cf. Appendix). Thus, we have proved that the recurrence relation (12) corresponds to the general n-simplex volume formula (25) within this domain.

However, now we can use the recurrence relation (12) to determine the values of the n-simplex volume outside this domain: $\forall n+1 \in \mathbb{C}$ we can find $V(n)_S$. For example, even though $m\sqrt{m}/\sqrt{m^3}$ is undefined for $m = 0$, we can determine that $V(0)_S = \pm 1$ using the recurrence relation (12) with $V(1)_S = \pm 1$ obtained from (28).

On the other hand

$$\lim_{n \to -k-1, k \in \mathbb{N}_0} \pm \frac{2^{-n/2}A^n\sqrt{n+1}}{\Gamma(n+1)} = \frac{a}{\infty} = 0, \quad (29)$$

where $a \in \mathbb{C}$, so the poles of the meromorphic gamma function $\Gamma(n+1)$ present in (28), now defined in the sense of a limit of a function, vanish - which completes the proof. □

For $n < -1 \in \mathbb{R} \setminus \{n \in \mathbb{Z}\}$, n-simplex volume formula (28) is imaginary.

Corollary 2.1. *The surface of a regular n-simplex is a holomorphic, bivalued function of a complex dimension n.*

Proof. If the volume of a regular n-simplex is a holomorphic, bivalued function by Theorem 2, then its surface, using formula (13), is also a holomorphic, bivalued function.

Using (13) and (28) the surface of a regular n-simplex is given by (cf. Appendix)

$$\begin{aligned} S(n)_S =& \frac{n^{3/2}(n+1)}{\Gamma(n+1)2^{(n-1)/2}}A^{n-1}\frac{(n-1)\sqrt{n-1}}{\sqrt{(n-1)^3}} = \\ =& \pm \frac{n^{3/2}(n+1)}{\Gamma(n+1)2^{(n-1)/2}}A^{n-1}. \end{aligned} \quad (30)$$

□

Again, even though $(n-1)\sqrt{n-1}/\sqrt{(n-1)^3}$ is undefined for $n = 1$, we can determine that $S(1)_S = \pm 2$ directly from formula (13) and knowing that $V(0)_S = \pm 1$.

For $n < 0 \in \mathbb{R} \setminus \{n \in \mathbb{Z}\}$, n-simplex surface formula (30) is imaginary.

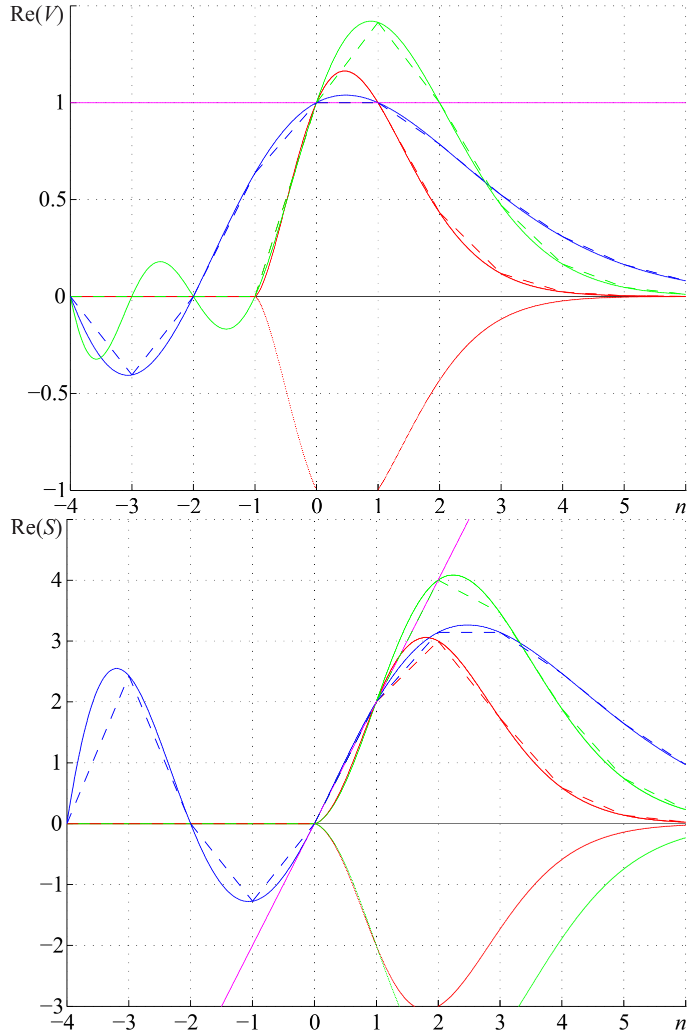

Figure 1. Graphs of volumes (V) and surfaces (S) of unit edge length regular n-simplices (red), n-orthoplices (green), n-cubes (pink), and unit diameter n-balls (blue), along with the integer recurrence relations (dashed lines) and the 2nd branches (dotted lines) for $n = [-4, 6]$.

Theorem 3. *The volume of an n-orthoplex is a holomorphic function of a complex dimension n.*

Proof. Expressing the factorial in the volume of a n-orthoplex formula (14) by the gamma function extends the domain of applicability of (14) to complex dimensions

$$V(n)_O = \frac{\sqrt{2^n}}{n!}A^n = \frac{\sqrt{2^n}}{\Gamma(n+1)}A^n. \quad (31)$$

On the other hand, comparing (31) with the recurrence relation (16)

$$\frac{\sqrt{2^n}}{\Gamma(n+1)}A^n = V(n+1)_O\frac{n+1}{A\sqrt{2}}, \quad (32)$$

yields

$$V(n+1)_O = \frac{2^{(n+1)/2}}{(n+1)\Gamma(n+1)}A^{n+1}, \quad (33)$$

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whereas setting $m = n + 1$ in (33) yields

$$V(m)_O = \frac{\sqrt{2^m}}{m\Gamma(m)} A^m = \frac{2^{m/2}}{\Gamma(m+1)} A^m, \qquad (34)$$

which recovers n-orthoplex volume (14), as $m\Gamma(m) = \Gamma(m+1)\ \forall m \in \mathbb{C} \setminus \{m = -k, k \in \mathbb{N}_0\}$. Thus, we have proved that the recurrence relation (16) corresponds to the general n-orthoplex volume formula (31) within this domain.

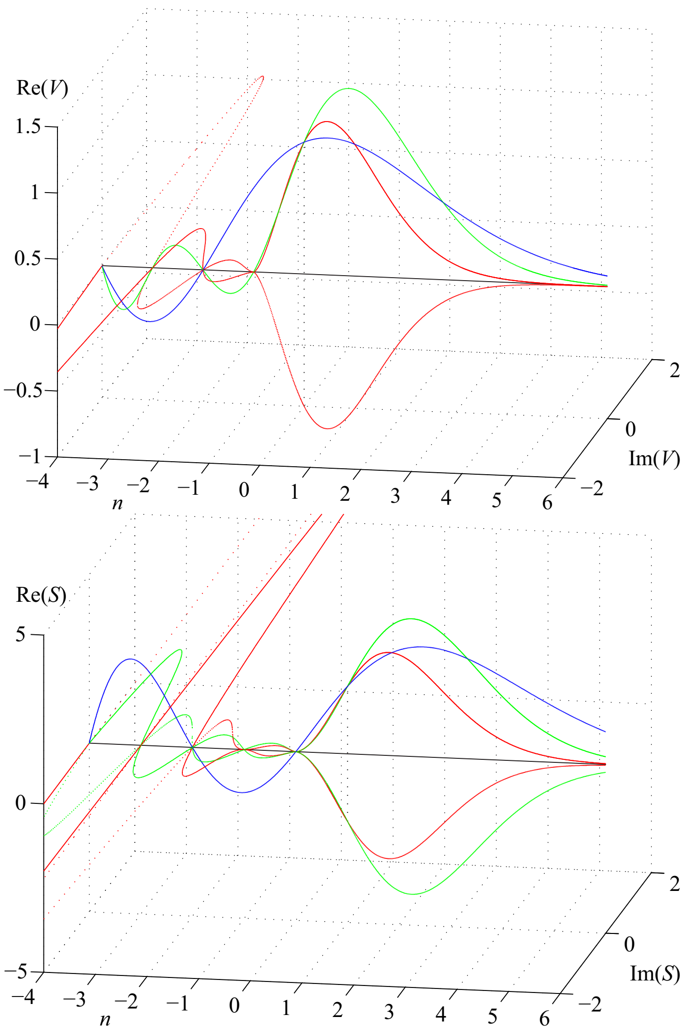

Figure 2. Graphs of volumes (V) and surfaces (S) of unit edge length regular n-simplices (red), n-orthoplices (green), and unit diameter n-balls (blue), along with the 2nd branches (dotted lines) for $n = [-4, 6]$.

However, now we can use the recurrence relation (16) to determine the values of the n-orthoplex volume outside this domain: $\forall n+1 \in \mathbb{C}$ we can find $V(n)_O$. On the other hand

$$\lim_{n \to -k-1, k \in \mathbb{N}_0} \frac{2^{n/2} A^n}{\Gamma(n+1)} = \frac{a}{\infty} = 0, \qquad (35)$$

where $a \in \mathbb{R}$, so the poles of the meromorphic gamma function $\Gamma(n+1)$ present in (34), now defined in the sense of a limit of a function, vanish - which completes the proof. □

Corollary 3.1. *The surface of an n-orthoplex is a holomorphic, bivalued function of a complex dimension n.*

Proof. If the volume of a regular n-simplex is a holomorphic function by Theorem 2, then, using formula (17), the surface of an n-orthoplex is a holomorphic, bivalued function.

Using (17) and (32) the surface of an n-orthoplex is given by

$$S(n)_O = \pm \frac{n^{3/2} 2^{(n+1)/2}}{\Gamma(n+1)} A^{n-1}. \qquad (36)$$

□

For $n < 0 \in \mathbb{R} \setminus \{n \in \mathbb{Z}\}$, n-orthoplex bivalued surface formula (36) is imaginary.

Holomorphic volumes (1), (28), (34), and surfaces (24), (30), (36) of n-balls, n-simplices, n-orthoplices, mean that these objects (along with n-cubes) are omnidimensional, i.e. well defined $\forall n \in \mathbb{C}$.

IV. VOLUMES AND SURFACES IN COMPLEX DIMENSIONS

The gamma function is defined for all complex numbers except the non-positive integers. Therefore, the volumes and surfaces of n-balls and omnidimensional polytopes containing the gamma function are also defined for all $n = a + ib \in \mathbb{C}$.

In the case of n-balls [24]

$$\pi^{n/2} = \pi^{(a+ib)/2} = \pi^{a/2} \left[\cos\left(\frac{b}{2}\ln\pi\right) + i\sin\left(\frac{b}{2}\ln\pi\right)\right], \qquad (37)$$

$$R^n = R^{a+ib} = R^a \left[\cos(b\ln R) + i\sin(b\ln R)\right], \qquad (38)$$

and the volume (1) and surface (24) become

$$V(n)_B = \frac{\pi^{a/2} R^a}{\Gamma\left(\frac{n}{2}+1\right)} \left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right] + i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}, \qquad (39)$$

$$S(n)_B = \frac{n\pi^{a/2} R^{a-1}}{\Gamma\left(\frac{n}{2}+1\right)} \left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right] + i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}, \qquad (40)$$

where we have used $\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a+b)$ and $\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a+b)$.

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In particular for $n = 3 + ib$, $b \in \mathbb{R}$ (spacetime dimensionality) equation (39) becomes

$$V(n)_B = \frac{\pi^{3/2} R^3}{\Gamma\left(\frac{3+ib}{2}+1\right)} \left\{\cos\left[b \ln\left(R\sqrt{\pi}\right)\right] + i \sin\left[b \ln\left(R\sqrt{\pi}\right)\right]\right\}, \quad (41)$$

which reduces to familiar $V_3(R)_B = 4\pi R^3/3$ for $n = 3 + 0i$, i.e. at the present moment. Note that the antisymmetry of the imaginary part of the volume (39), in a way, establishes the arrow of time and is independent on $\mathrm{Re}(n)$ for $\mathrm{Im}(n) = 0$.

In the case of n-cubes [24], the volume (8) is

$$V(n)_C = A^a \left\{\cos\left[b \ln(A)\right] + i \sin\left[b \ln(A)\right]\right\}. \quad (42)$$

We note in passing that the complex trigonometric parts of (39), (40), and (42) vanish not only for $b = 0$ but also for $R = 1/\sqrt{\pi}$ and $A = 1$. $R = \ell_P/\sqrt{\pi}$, where ℓ_P is the Planck length, is the radius of a 4-bit black hole [13], and one unit of a black hole entropy [25].

V. OMNIDIMENSIONAL POLYTOPES INSCRIBED IN AND CIRCUMSCRIBED ABOUT n-BALLS

Each regular, omnidimensional polytope discussed in Section III can be inscribed in and circumscribed about an n-ball, and this is considered in this section.

A. Regular n-Simplices Inscribed in n-Balls

The diameter D_{BCS} of an n-ball circumscribing a regular n-simplex (BCS) is known [23] to be

$$D_{BCS} = \frac{\sqrt{2n}}{\sqrt{n+1}} A, \quad (43)$$

where A is the edge length. Hence, the edge length A_{SIB} of a regular n-simplex inscribed (SIB) in an n-ball (B) with diameter D is

$$A_{SIB} = \frac{\sqrt{n+1}}{\sqrt{2n}} D, \quad (44)$$

so that the regular n-simplex volume (28) becomes

$$V(n)_{SIB} = \pm \frac{(1+n)^{(1+n)/2} n^{-n/2} 2^{-n}}{\Gamma(1+n)} D^n. \quad (45)$$

For $n < -1 \in \mathbb{R} \setminus \{n \in \mathbb{Z}\}$ the inscribed n-simplex volume (45) is imaginary and divergent with n approaching negative infinity. It is complex for $-1 < n < 0$, with the real part being equal to the imaginary part for $n = -1/2$. It is zero for $n = -k$, $k \in \mathbb{N}$, and for $0 < n < 1$ it is larger than the volume of the circumscribing n-ball.

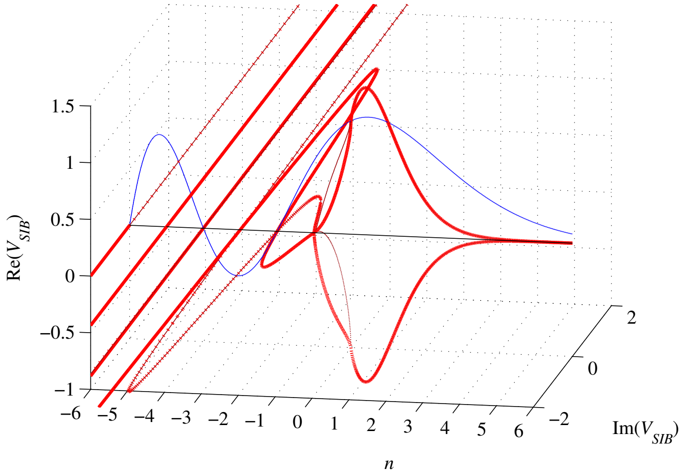

Figure 3. Graphs of volumes of regular n-simplices (red) and unit diameter n-balls (blue), they are inscribed in, along with reflection relations (dark red), for $n = [-6, 6]$.

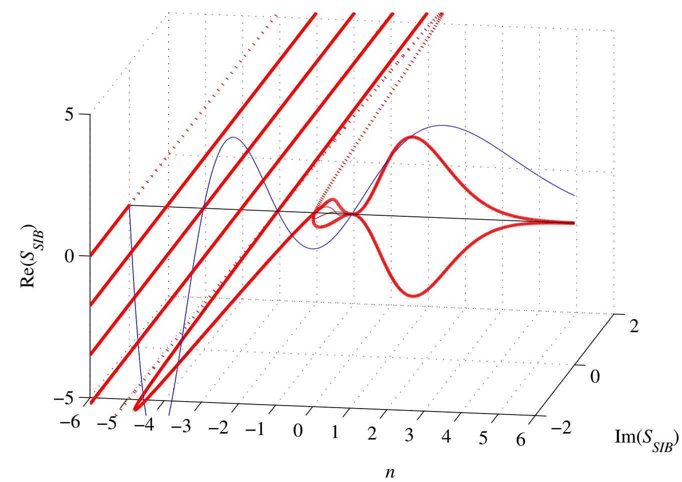

Figure 4. Graphs of surfaces of regular n-simplices (red) and unit diameter n-balls (blue), they are inscribed in, along with the reflection relations (dark red), for $n = [-6, 6]$.

Nontrivial reflection relation can be obtained by setting $m = -n$ in (45) to extract the imaginary unit. Switching back to $n = -m$ yields

$$V(n)_{SIBR} = \pm i \frac{(-n-1)^{(1+n)/2} (-n)^{-n/2} 2^{-n}}{\Gamma(1+n)} D^n. \quad (46)$$

Similarly, the surface (30) of a regular inscribed n-simplex with edge length A given by (44) is

$$S(n)_{SIB} = \pm \frac{n^{(4-n)/2} (1+n)^{(1+n)/2} 2^{1-n}}{\Gamma(1+n)} D^{n-1}, \quad (47)$$

as shown in Fig. 4. For $n < -1 \in \mathbb{R} \setminus \{n \in \mathbb{Z}\}$ the inscribed n-simplex surface (47) is imaginary and divergent with n approaching negative infinity. It is complex for $-1 < n < 0$, with the real part being equal to the imaginary part for $n = -1/2$. It is zero for $n = -k$, $k \in \mathbb{N}_0$.

The nontrivial reflection relation of the surface (47) is

$$S(n)_{SIBR} = \pm i \frac{(-n)^{(4-n)/2}(-n-1)^{(1+n)/2} 2^{-n+1}}{\Gamma(1+n)} D^{n-1}. \tag{48}$$

Volumes (45), (46) and surfaces (47), (48) of n-simplices inscribed in n-balls are shown in Figs. 3, 4.

B. Regular n-Simplices Circumscribed About n-Balls

The diameter D_{BIS} of an n-ball inscribed in a regular n-simplex (BIS) is known [23] to be

$$D_{BIS} = \frac{\sqrt{2}}{\sqrt{n}\sqrt{n+1}} A, \tag{49}$$

where A is the edge length. Hence, the edge length A_{SCB} of a regular n-simplex circumscribed (SCB) about an n-ball (B) with diameter D is

$$A_{SCB} = \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{2}} D, \tag{50}$$

so that its volume (28) becomes

$$V(n)_{SCB} = \pm \frac{n^{n/2}(1+n)^{(1+n)/2} 2^{-n}}{\Gamma(1+n)} D^n. \tag{51}$$

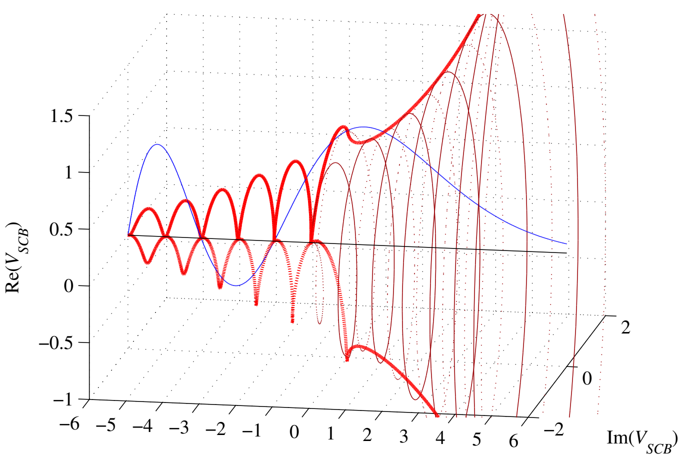

Figure 5. Graphs of volumes of regular n-simplices (red) and unit diameter n-balls (blue), they are circumscribed about, along with the reflection relations (dark red), for $n = [-6, 6]$.

For $n < 0$, the bivalued volume of the circumscribed n-simplex (51) is complex, whereas both branches are left-handed and convergent to zero with n approaching negative infinity. For $0 < n < 1$ it is smaller than the volume of the inscribed n-ball (cf. Table I). It is zero for $n = -k$, $k \in \mathbb{N}$ and real for $n = -(2k+1)/2$, $k \in \mathbb{N}$. For $D = 1$ it amounts

$$\begin{aligned} &V\left(\frac{-(2k+1)}{2}\right)_{SCB} = \\ &\pm \frac{i^{2k} 2^{(4k+1)/2} (2k-1)^{(1-2k)/4}}{\Gamma\left(\frac{1-2k}{2}\right)(2k+1)^{(1+2k)/4}} \approx \\ &\pm \{0.7, 0.5618, 0.4251, 0.3172, 0.2353, \ldots\}. \end{aligned} \tag{52}$$

Furthermore, for $n = -1/2$ and for $n = -(2k+3)/4$, $k \in \mathbb{N}$, the real part of the volume (51) is equal to the imaginary part up to a modulus. For $n = -1/2$ $V(-1/2)_{SCB} = \pm(1-i)/\sqrt{\pi} \approx \pm 0.5642(1-i)$. Otherwise, for $D = 1$ it amounts

$$\begin{aligned} &V\left(\frac{-(2k+3)}{4}\right)_{SCB} = \\ &\pm \frac{(1+i)(-i)^{k-3} 2^{(6k+3)/4} (2k-1)^{(1-2k)/8}}{\Gamma\left(\frac{1-2k}{4}\right)(2k+3)^{(2k+3)/8}} \approx \\ &\pm \{-0.3549, -0.3359, 0.2996, 0.2626, -0.2283, \ldots\} \\ &(1+i)(-i)^{k-3}. \end{aligned} \tag{53}$$

The volume (51) reflection relation is

$$V(n)_{SCBR} = \pm i^{1+2n} \frac{(-n-1)^{(1+n)/2}(-n)^{n/2} 2^{-n}}{\Gamma(1+n)} D^n. \tag{54}$$

Similarly, the surface (30) of a regular circumscribed n-simplex with edge length A_{SCB} (50) is

$$S(n)_{SCB} = \pm \frac{n^{(2+n)/2}(1+n)^{(1+n)/2} 2^{1-n}}{\Gamma(1+n)} D^{n-1}. \tag{55}$$

For real $0 < n < 1$ the surface (55) is smaller than the surface of the inscribed n-ball (cf. Table II). For real $n < 0$ it is complex with both branches being left-handed towards negative infinity or the branch point. It is zero for $n = -k$, $k \in \mathbb{N}_0$. I is real for $n = -(2k+1)/2$, $k \in \mathbb{N}$

$$\begin{aligned} &S\left(\frac{-(2k+1)}{2}\right)_{SCB} = \\ &\pm \frac{-i^{2k} 2^{(4k+1)/2} (2k-1)^{(1-2k)/4}}{\Gamma\left(\frac{1-2k}{2}\right)(2k+1)^{(2k-3)/4}} \approx \\ &\pm \{2.1, 2.809, \underline{2.976}, 2.854, 2.588, \ldots\}, \end{aligned} \tag{56}$$

(for $D = 1$), achieving maximum at $n \approx -7/2$. The real part of the surface (55) is equal to the imaginary part up to a modulus for $n = -1/2$ and for $n = -(2k+3)/4$,

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$k \in \mathbb{N}$

$$S\left(\frac{-(2k+3)}{4}\right)_{SCB} = \pm\frac{(1+i)(-i)^{k-1}\,2^{(6k-1)/4}\,(2k-1)^{(1-2k)/8}}{\Gamma\left(\frac{1-2k}{4}\right)(2k+3)^{(2k-5)/8}} \approx \pm\{0.8873, -1.1755, 1.3484, -1.4443, \underline{1.4842}, 1.4828, \ldots\}\,(1+i)(-i)^{k-1}, \tag{57}$$

(for $D = 1$). For $n = -1/2$ $S(-1/2)_{SCB} = \pm(-1+i)/\sqrt{\pi}$. The surface (55) is initially divergent to achieve a modulus maximum of about 2.9757 at $n \approx -3.4997$ (numerical) and a real maximum of about $n = -2.976$ at $n = -3.5$, and then becomes convergent to zero with n approaching negative infinity.

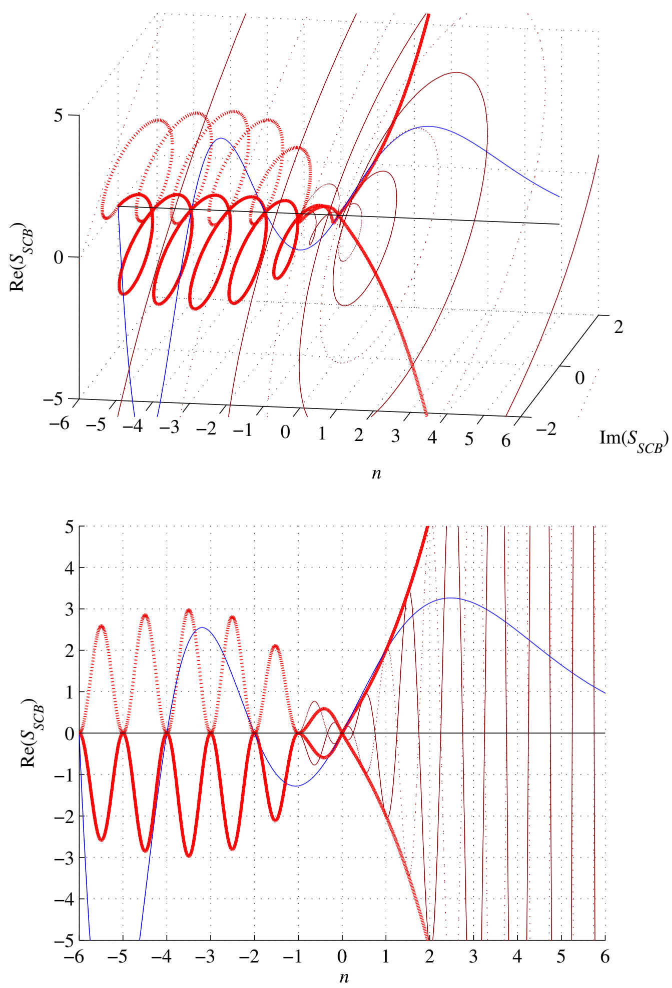

Figure 6. Graphs of surfaces of regular n-simplices (red) and unit diameter n-balls (blue), they are circumscribed about, along with reflection relations (dark red), for $n = [-6, 6]$.

The surface (55) reflection relation is

$$S(n)_{SCBR} = \pm\, i^{3+2n}\frac{(-n)^{(2+n)/2}(-n-1)^{(1+n)/2}\,2^{1-n}}{\Gamma(1+n)}D^{n-1}. \tag{58}$$

Volumes (51) and surfaces (55) of n-simplices circumscribed about n-balls are shown in Figs. 5 and 6.

C. n-Orthoplices Inscribed in n-Balls

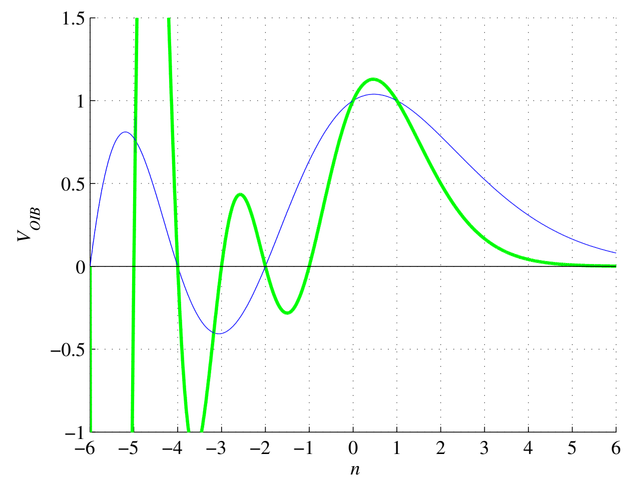

Figure 7. Graphs of real volumes of n-orthoplices (green) and unit diameter n-balls (blue), they are inscribed in, for $n = [-6, 6]$.

The diameter D_{BCO} of an n-ball circumscribing an n-orthoplex (BCO) is known [26] to be

$$D_{BCO} = \sqrt{2}A, \tag{59}$$

where A is the edge length. Hence, the edge length A_{OIB} of an n-orthoplex inscribed in an n-ball (OIB) with diameter D is

$$A_{OIB} = \frac{1}{\sqrt{2}}D, \tag{60}$$

so that its volume (34) becomes

$$V(n)_{OIB} = \frac{1}{\Gamma(n+1)}D^n. \tag{61}$$

The inscribed n-orthoplex volume (61) is real for $n \in \mathbb{R}$, vanishes for $n = -k$, $k \in \mathbb{N}$, and for $0 < n < 1$ it is larger than the volume of the circumscribing n-ball (cf. Table I).

Similarly, the surface (36) of the inscribed n-orthoplex with edge length A given by (60) becomes

$$S(n)_{OIB} = \pm\frac{2n^{3/2}}{\Gamma(n+1)}D^{n-1}. \tag{62}$$

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For $n < 0$, $n \notin \mathbb{Z}$ inscribed n-orthoplex surface (62) is imaginary and oscillatory divergent with n approaching negative infinity, and for $n = -k$, $k \in \mathbb{N}$ it vanishes.

The reflection relations for the volume (61) and surface (62) are trivial. In the second case $\pm n^{3/2} = \mp i(-n)^{3/2}$.

Volumes (61) and surfaces (62) of n-orthoplices inscribed in n-balls are shown in Figs. 7 and 8.

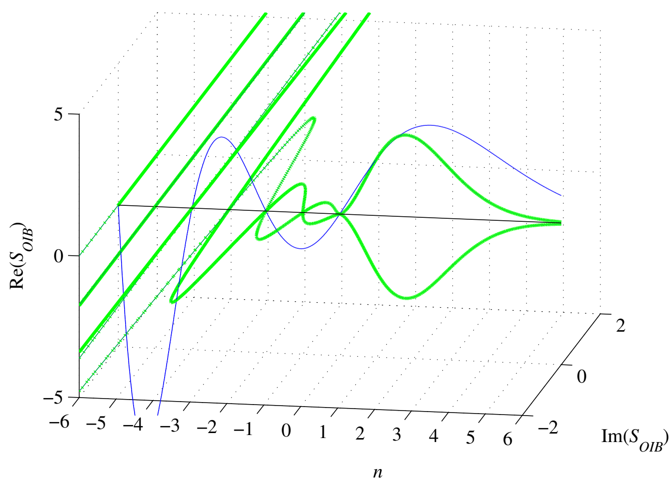

Figure 8. Graphs of surfaces of n-orthoplices (green) and unit diameter n-balls (blue), they are inscribed in, for $n = [-6, 6]$.

D. n-Orthoplices Circumscribed About n-Balls

The diameter D_{BIO} of an n-ball inscribed in an n-orthoplex (BIO) is known [26] to be

$$D_{BIO} = \sqrt{\frac{2}{n}}A, \quad (63)$$

where A is the edge length. Hence, the edge length A_{OCB} of an n-orthoplex circumscribed about an n-ball (OCB) with diameter D is

$$A_{OCB} = \sqrt{\frac{n}{2}}D, \quad (64)$$

so that its volume (34) becomes

$$V(n)_{OCB} = \frac{n^{n/2}}{\Gamma(n+1)}D^n, \quad (65)$$

as shown in Fig. 9.

Circumscribed n-orthoplex volume (65) is a single-valued function, is complex for $n < 0$, crossing the quadrants of the complex plane in the order $\{\mathrm{Re}(V_{OCB}) > 0, \mathrm{Im}(V_{OCB}) < 0\}, \{\mathrm{Re}(V_{OCB})\} > 0, \mathrm{Im}(V_{OCB}) > 0\}, \{\mathrm{Re}(V_{OCB}) < 0, \mathrm{Im}(V_{OCB})) > 0\}, \{\mathrm{Re}(V_{OCB}) < 0, \mathrm{Im}(V_{OCB}) < 0\}$. It oscillates and is initially convergent to achieve a modulus maximum of about 0.1181 at $n \approx -3.4976$ (numerical) and then becomes divergent with n approaching negative infinity. For $n = -k$, $k \in \mathbb{N}$ it vanishes. For $0 < n < 1$ it is smaller than the volume of the inscribed n-ball (cf. Table I). For $n = -(2k+1)/2$, $k \in \mathbb{N}_0$ the real part of the volume (65) equals the imaginary part up to a modulus, achieving maximum at $n \approx -3.5$ and for $D = 1$ it is

$$\begin{aligned} &V\left(\frac{-(2k+1)}{2}\right)_{OCB} = \\ &= \frac{(1+i)(-i)^{k-3}\,2^{(2k-1)/4}}{\Gamma\left(\frac{1-2k}{2}\right)(2k+1)^{(2k+1)/4}} \approx \\ &\{0.4744, -0.1472, 0.0952, -\underline{0.0835}, 0.0888, -0.1084, \ldots\} \\ &(1+i)(-i)^{k-3}. \end{aligned} \quad (66)$$

The volume (65) reflection relation is

$$V(n)_{OCBR} = i^n \frac{(-n)^{n/2}}{\Gamma(n+1)}D^n. \quad (67)$$

Similarly, the surface (36) of the circumscribed n-orthoplex with edge length A given by (64) becomes

$$S(n)_{OCB} = \pm\frac{2n^{n/2+1}}{\Gamma(n+1)}D^{n-1}, \quad (68)$$

as shown in Fig. 10.

Circumscribed n-orthoplex surface (68) is a bivalued function, is complex for $n < 0$, crossing the quadrants of the complex plane in the order $\{\mathrm{Re}(S_{OCB}) < 0, \mathrm{Im}(S_{OCB})) > 0\}, \mathrm{Re}(S_{OCB}) < 0, \mathrm{Im}(S_{OCB}) < 0\}, \{\mathrm{Re}(S_{OCB}) > 0, \{\mathrm{Im}(S_{OCB}) < 0\}$, and $\{\mathrm{Re}(S_{OCB}) > 0, \mathrm{Im}(S_{OCB}) > 0\}$. It oscillates and is initially convergent to achieve a modulus maximum of about 0.6244 at $n \approx -1.5$ (numerical) and then becomes divergent with n approaching negative infinity. For $n = -k$, $k \in \mathbb{N}$ it vanishes. For $0 < n < 1$ it is smaller than the surface of the inscribed n-ball (cf. Table II). Furthermore, its real part is equal to the imaginary part up to a modulus for $n = -(2k+1)/2$, $k \in \mathbb{N}_0$. It achieves maximum at $n = -3/2$ and for $D = 1$ amounts

$$\begin{aligned} &S\left(\frac{-(2k+1)}{2}\right)_{OCB} = \\ &\pm\frac{(1+i)(-i)^{k-1}\,2^{(2k-1)/4}\,(2k+1)^{(3-2k)/4}}{\Gamma\left(\frac{1-2k}{2}\right)} \approx \\ &\pm\{0.4744, -\underline{0.4415}, 0.4759, -0.5846, 0.7989, \ldots\} \\ &(1+i)(-i)^{k-1}. \end{aligned} \quad (69)$$

The surface (68) reflection relation is

$$S(n)_{OCBR} = \mp\frac{-2ni^n(-n)^{n/2}D^{n-1}}{\Gamma(n+1)}, \quad (70)$$

Volumes (65), (67) and surfaces (68), (70) of n-orthoplices inscribed in n-balls are shown in Figs. 9 and 10.

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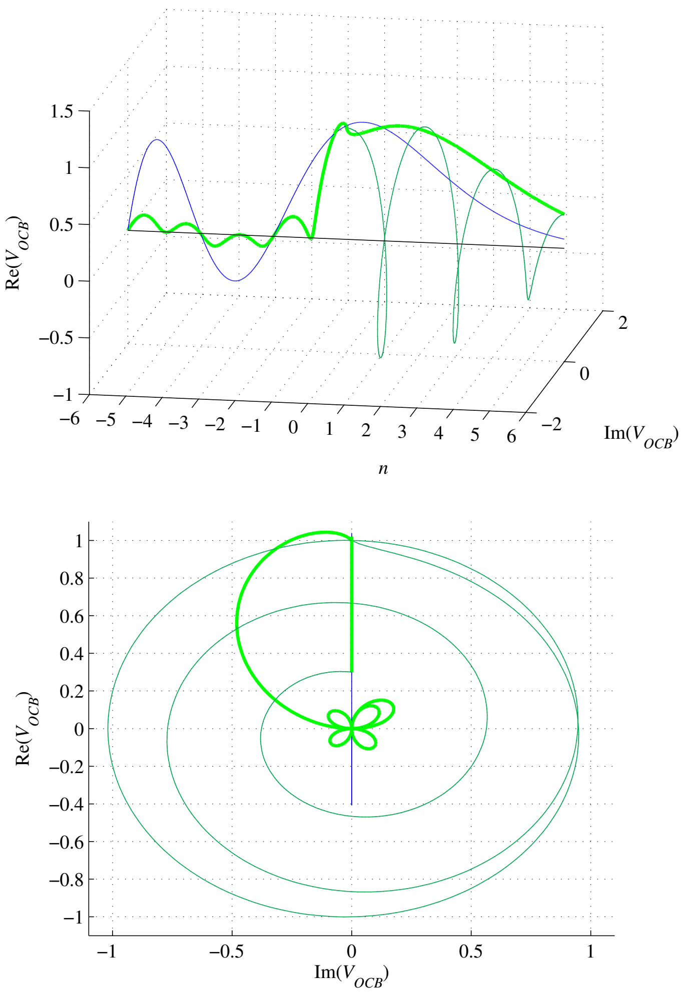

Figure 9. Graphs of volumes of n-orthoplices (green) and unit diameter n-balls (blue), they are circumscribed about, along with reflection relations (dark red), for $n = [-6, 6]$.

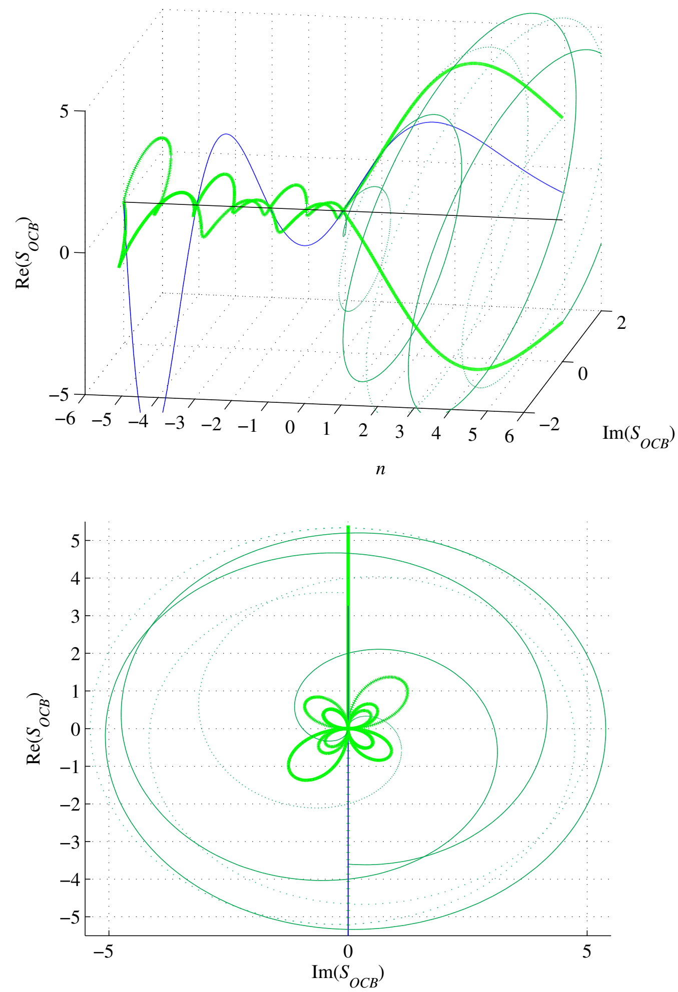

Figure 10. Graphs of surfaces of n-orthoplices (green) and unit diameter n-balls (blue), they are circumscribed about, along with reflection relations (dark red), for $n = [-6, 6]$.

E. n-Cubes Inscribed in and Circumscribed About n-Balls

The edge length $ACCB$ of an n-cube circumscribed about an n-ball (CCB) corresponds to the diameter D of this n-ball. Thus, the volume of this cube is simply

$$V(n)_{CCB} = D^n, \tag{71}$$

and the surface is

$$S(n)_{CCB} = 2nD^{n-1}. \tag{72}$$

However, the edge length A_{CIB} of an n-cube inscribed in an n-ball (CIB) of diameter D is

$$A(n)_{CIB} = D/\sqrt{n}, \tag{73}$$

which is singular for $n = 0$ and complex for $n < 0$, rendering [16] the following volume and the surface of an n-cube inscribed in an n-ball

$$V(n)_{CIB} = n^{-n/2}D^n, \tag{74}$$

$$S_{CIB} = 2n^{(3-n)/2}D^{n-1}. \tag{75}$$

The reflection relation can be obtained setting $m = -n$ in (74) and (75), yielding [16] the volume and the surface

$$V(n)_{CIBR} = i^{-n}(-n)^{-n/2}D^n, \tag{76}$$

$$S(n)_{CIBR} = -2i^{1-n}(-n)^{(3-n)/2}D^{n-1}, \tag{77}$$

which are complex for $m \in \mathbb{R}$. Volumes (74) and (76) correspond to each other [16] for $n \leq 0$, $n \in \mathbb{R}$ and for $n = 2k$, $k \in \mathbb{Z}$, as shown in Fig. 11. Surfaces (75) and (77) correspond to each other [16] for $n \in \mathbb{R}$, $n \leq 0$, and for $n = 2k - 1$, $k \in \mathbb{Z}$, as shown in Fig. 12.

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For $n \geq 0$ (by convention $0^0 \doteq 1$), the inscribed n-cube volume (74) is real, complex if $n < 0$, becoming real if n is negative and even and imaginary if n is negative and odd, and divergent with n approaching negative infinity. For $0 < n < 1$ it is larger than the volume of the circumscribing n-ball. For $n \geq 0$, the inscribed n-cube surface (75) is real, complex if $n < 0$, becoming real if n is negative and odd and imaginary if n is negative and even, and divergent with n approaching negative infinity.

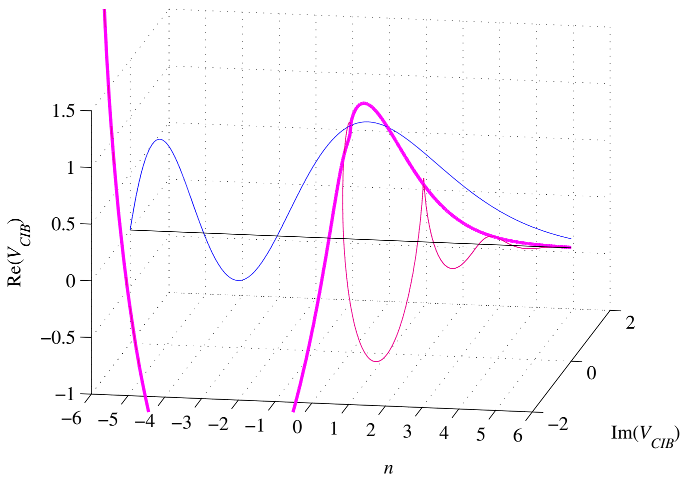

Figure 11. Graphs of volumes of n-cubes (pink) and unit diameter n-balls (blue), they are inscribed in, along with reflection relations (dark pink), for $n = [-6, 6]$.

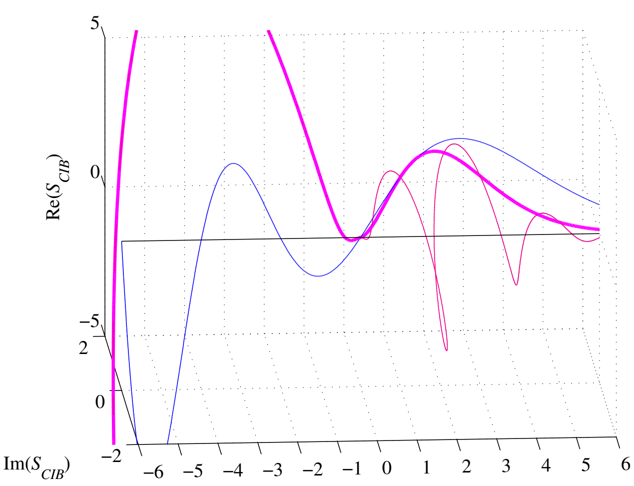

Figure 12. Graphs of surfaces of n-cubes (pink) and unit diameter n-balls (blue), they are inscribed in, along with reflection relations (dark pink), for $n = [-6, 6]$.

VI. REFLECTION RELATIONS

Conjecture 4. *If a function $F(\mathbb{C} \to \mathbb{C})$ contains at least one term $f(n)^{k/2}$, where $k \in \mathbb{C}$, and $f(n)$ is a linear function of $n \in \mathbb{C}$, then it has a nontrivial reflection relation $F(m)$, $m = -n$, containing the imaginary unit i.*

For one linear function of $n \in \mathbb{C}$ we have

$$[f(n)]^{k/2} = (a + bn)^{k/2}. \quad (78)$$

Setting $-m \doteq a + bn$ yields

$$\begin{aligned}(a+bn)^{k/2} &= [(-1)m]^{k/2} = i^k(-a-bn)^{k/2} = \\ &= i^k\left[-f(n)\right]^{k/2} = i^{2k}\left[f(n)\right]^{k/2}.\end{aligned} \quad (79)$$

Thus,

$$[f(n)]^{k/2} = \begin{cases} [f(n)]^{k/2}, & \text{if} \quad k \quad \text{is even} \\ -[f(n)]^{k/2}, & \text{if} \quad k \quad \text{is odd} \\ z \in \mathbb{C}, & \text{otherwise} \end{cases}. \quad (80)$$

Even k leads to an identity $(1 = 1)$, whereas odd k yields

$$2\,[f(n)]^{k/2} = 0 \to (a+bn)^{k/2} = 0 \to n = \frac{-a}{b}, \quad (81)$$

For two (or more) linear functions of $n \in \mathbb{C}$ we have

$$f_1(n)^{k_1/2} f_2(n)^{k_2/2} = (a_1 + b_1 n)^{k_1/2}(a_2 + b_2 n)^{k_2/2}. \quad (82)$$

Setting $-m_1 \doteq a_1 + b_1 n$ and $-m_2 \doteq a_2 + b_2 n$ yields

$$\begin{aligned}&(a_1+b_1n)^{k_1/2}(a_2+b_2n)^{k_2/2} = \\ &(-m_1)^{k_1/2}(-m_2)^{k_2/2} = \\ &= i^{k_1}(m_1)^{k_1/2} i^{k_2}(m_2)^{k_2/2} = \\ &= i^{k_1+k_2}(-a_1-b_1n)^{k_1/2}(-a_2-b_2n)^{k_2/2} = \\ &= i^{k_1+k_2}\left[-f_1(n)\right]^{k_1/2}\left[-f_2(n)\right]^{k_2/2} = \\ &= i^{2(k_1+k_2)}\left[f_1(n)\right]^{k_1/2}\left[f_2(n)\right]^{k_2/2}.\end{aligned} \quad (83)$$

Thus,

$$\begin{aligned}&[f_1(n)]^{k_1/2}\,[f_2(n)]^{k_2/2} = \\ &= \begin{cases} [f_1(n)]^{k_1/2}\,[f_2(n)]^{k_2/2}, & \text{if } k_1+k_2 \text{ is even} \\ -[f_1(n)]^{k_1/2}\,[f_2(n)]^{k_2/2}, & \text{if } k_1+k_2 \text{ is odd} \\ z \in \mathbb{C}, & \text{otherwise} \end{cases}.\end{aligned} \quad (84)$$

Volumes (1), (8) and surfaces (24), (9) of n-balls and n-cubes, and volume (34) of an n-orthoplex do not contain $f(n)^{k/2}$ terms. Thus, they have trivial reflection relations, symmetric around $n = 0$.

Bivalued volume (28) and surface (30) of an n-simplex and bivalued surfaces (36), (62) of n-orthoplices contain

Table I. Particular volumes of omnidimensional polytopes inscribed in and circumscribed about unit diameter n-balls.

n	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$
V_B	0.331	$2/\pi \approx 0.637$	0.867	1	1.039	1	0.908
V_{SIB}	$\frac{-i3^{3/4}}{\sqrt{\pi}} \approx -i1.286$	0	$\frac{1+i}{\sqrt{2\pi}} \approx 0.399\,(1+i)$	1	$\frac{3^{3/4}}{\sqrt{\pi}} \approx 1.286$	1	$\frac{5^{5/4}}{3^{7/4}\sqrt{\pi}} \approx 0.617$
V_{SCB}	$\frac{2^{3/2}}{3^{3/4}\sqrt{\pi}} \approx 0.7$	0	$\frac{1-i}{\sqrt{\pi}} \approx 0.564\,(1-i)$	1	$\frac{3^{3/4}}{\sqrt{2\pi}} \approx 0.909$	1	$\frac{5^{5/4}}{3^{7/4}\sqrt{\pi}} \approx 1.133$
V_{OIB}	$\frac{-1}{2\sqrt{\pi}} \approx -0.282$	0	$\frac{1}{\sqrt{\pi}} \approx 0.564$	1	$\frac{2}{\sqrt{\pi}} \approx 1.128$	1	$\frac{4}{3\sqrt{\pi}} \approx 0.752$
V_{OCB}	$\frac{1+i}{2^{3/4}3^{3/4}\sqrt{\pi}} \approx 0.147\,(1+i)$	0	$\frac{1-i}{2^{1/4}\sqrt{\pi}} \approx 0.474\,(1-i)$	1	$\frac{2^{3/4}}{\sqrt{\pi}} \approx 0.949$	1	$\frac{2^{5/4}}{2^{1/4}\sqrt{\pi}} \approx 1.02$
V_{CIB}	$\frac{(i-1)3^{3/4}}{2^{-5/4}} \approx 0.958\,(i-1)$	i	$\frac{(1+i)}{2^{3/4}} \approx 0.595\,(i+1)$	1	$2^{1/4} \approx 1.189$	1	$\frac{2^{3/4}}{3^{3/4}} \approx 0.738$
V_{CCB}	1	1	1	1	1	1	1

Table II. Particular surfaces of omnidimensional polytopes inscribed in and circumscribed about unit diameter n-balls.

n	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$
S_B	-0.992	$-4/\pi \approx -1.273$	-0.867	0	1.039	2	2.723
S_{SIB}	$\frac{-i3^{11/4}}{2\sqrt{\pi}} \approx -5.787i$	0	$\frac{1+i}{\sqrt{\pi}2^{3/2}} \approx 0.199\,(1+i)$	0	$\frac{3^{3/4}}{2\sqrt{\pi}} \approx 0.643$	2	$\frac{3^{1/4}5^{5/4}}{2\sqrt{\pi}} \approx 2.776$
S_{SCB}	$\frac{-2^{3/2}}{3^{1/4}\sqrt{\pi}} \approx -2.1$	0	$\frac{-1+i}{\sqrt{\pi}} \approx 0.564\,(-1+i)$	0	$\frac{3^{3/4}}{\sqrt{2\pi}} \approx 0.909$	2	$\frac{3^{3/4}5^{5/4}}{2^{3/2}\sqrt{\pi}} \approx 3.4$
S_{OIB}	$\frac{3^{3/2}i}{2^{3/2}\sqrt{\pi}} \approx 1.036i$	0	$\frac{-i}{\sqrt{2\pi}} \approx -0.399i$	0	$\frac{\sqrt{2}}{\sqrt{\pi}} \approx 0.798$	2	$\frac{2^{3/2}\sqrt{3}}{\sqrt{\pi}} \approx 2.764$
S_{OCB}	$\frac{-(1+i)3^{1/4}}{2^{1/4}\sqrt{\pi}} \approx -0.442\,(1+i)$	0	$\frac{(-1+i)}{2^{1/4}\sqrt{\pi}} \approx -0.474\,(-1+i)$	0	$\frac{2^{3/4}}{\sqrt{\pi}} \approx 0.949$	2	$\frac{2^{5/4}3^{3/4}}{\sqrt{\pi}} \approx 3.059$
S_{CIB}	$\frac{(1+i)3^{9/4}}{2^{7/4}} \approx 3.521\,(1+i)$	2	$\frac{1-i}{2^{5/4}} \approx 0.42\,(1-i)$	0	$2^{-1/4} \approx 0.841$	2	$3^{3/4}2^{1/4} \approx 2.711$
S_{CCB}	-3	-2	-1	0	1	2	3

just one $f(n)^{k/2}$ term with odd k, ($\pm(n+1)^{1/2}$ and $\pm n^{3/2}$), but in that case, the formula only changes its sign, and remains symmetric around $n = 0$.

Surface $S(n)_{SIB}$ (47) contains two $f(n)^{k/2}$ terms $n^{(4-n)/2}$ and $(1+n)^{(1+n)/2}$. So $k_1 + k_2 = 5$ is odd, $n = -a_1/b_1 = 0$, $n = -a_2/b_2 = -1$, and $n^{(4-n)/2}(1+n)^{(1+n)/2} = 0$ for $n = 0$. As shown in Fig. 4 (47) is complex for $-1 < n < 0$, similarly as volume $V(n)_{SIB}$ (45) shown in Fig. 3.

Formulas (45), (47), (51), and (55) contain two $f(n)^{k/2}$ terms with odd $k_1 + k_2$. Formulas (65), (68), (74), and (75), contain a single $f(n)^{k/2}$ term with k that can be even or odd.

VII. METRIC-INDEPENDENT RELATIONS

It was shown [16] that the following metric-independent relation holds between volumes (74), (76) of n-cubes inscribed in n-balls

$$V(n)_{CIB}V(-n)_{CIBR} = D^n n^{-n/2} i^n D^{-n} n^{n/2} = i^n. \tag{85}$$

Similar metric-independent relations can be derived for volumes of n-simplices (28)

$$V(n)_S V(-n)_S = \pm \frac{(1+n)^{1/2}(1-n)^{1/2}\sin[\pi(n+1)]}{\pi n}, \tag{86}$$

n-orthoplices (34)

$$V(n)_O V(-n)_O = -\frac{\sin[\pi(n+1)]}{\pi n}, \tag{87}$$

n-simplices inscribed in n-balls (45)

$$V(n)_{SIB}V(-n)_{SIB} = \pm \frac{(1+n)^{(1+n)/2}(1-n)^{(1-n)/2}(n)^{(-n)/2}(-n)^{(n)/2}\sin[\pi(n+1)]}{\pi n}, \tag{88}$$

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n-simplices circumscribed about n-balls (51)

$$V(n)_{SCB}V(-n)_{SCB} = \pm\frac{(1+n)^{(1+n)/2}(1-n)^{(1-n)/2}(n)^{(n)/2}(-n)^{(-n)/2}\sin\left[\pi(n+1)\right]}{\pi n}, \tag{89}$$

and n-orthoplices circumscribed about n-balls (65)

$$V(n)_{OCB}V(-n)_{OCB} = -\frac{(n)^{(n)/2}(-n)^{(-n)/2}\sin\left[\pi(n+1)\right]}{\pi n}, \tag{90}$$

where we used Euler's reflection formula, $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, with $m \doteq n+1$

$$\Gamma(n+1)\Gamma(-n+1) = \Gamma(m)\Gamma(2-m) = \Gamma(m)\Gamma(1-m)(1-m) = \frac{-\pi n}{\sin\left[\pi(n+1)\right]}, \tag{91}$$

and surfaces of n-simplices (30),

$$S(n)_S S(2-n)_S = \pm\frac{n^{1/2}(n+1)(2-n)^{1/2}(3-n)\sin\left[\pi(n+1)\right]}{\pi(1-n)}, \tag{92}$$

n-orthoplices (36)

$$S(n)_O S(2-n)_O = \pm\frac{4n^{1/2}(2-n)^{1/2}\sin\left[\pi(n+1)\right]}{\pi(1-n)}, \tag{93}$$

n-cubes inscribed in n-balls (75)

$$S(n)_{CIB}S(2-n)_{CIB} = 4n^{(3-n)/2}(2-n)^{(1+n)/2}, \tag{94}$$

n-simplices inscribed in n-balls (47)

$$S(n)_{SIB}S(2-n)_{SIB} = \pm\frac{n^{(2-n)/2}(n+1)^{(n+1)/2}(2-n)^{n/2}(3-n)^{(3-n)/2}\sin\left[\pi(n+1)\right]}{\pi(1-n)}, \tag{95}$$

n-simplices circumscribed about n-balls (55)

$$S(n)_{SCB}S(2-n)_{SCB} = \pm\frac{n^{n/2}(n+1)^{(n+1)/2}(2-n)^{(2-n)/2}(3-n)^{(3-n)/2}\sin\left[\pi(n+1)\right]}{\pi(1-n)}, \tag{96}$$

and n-orthoplices circumscribed about n-balls (68)

$$S(n)_{OCB}S(2-n)_{OCB} = \pm\frac{4n^{n/2}(2-n)^{(2-n)/2}\sin\left[\pi(n+1)\right]}{\pi(1-n)}, \tag{97}$$

where we also used $m = n+1$ and Euler's reflection formula

$$\Gamma(n+1)\Gamma(3-n) = \Gamma(m)\Gamma(1-m)(1-m)(2-m)(3-m) = \frac{-\pi n(1-n)(2-n)}{\sin\left[\pi(n+1)\right]}. \tag{98}$$

Notably, the relations (86)-(93) and (95)-(97) are also independent on the gamma function.

Furthermore, the following particular symmetries between $n = -1/2$ and $n = 1/2$ hold for (45), (47); (51), (55); (61), (62); (65), (68); (74), (75); and (71), (72)

$$V(-1/2)_{SIB} = \pm 2S(-1/2)_{SIB}D, \quad V(1/2)_{SIB} = \pm 2S(1/2)_{SIB}D, \tag{99}$$

$$V(-1/2)_{SCB} = \pm S(-1/2)_{SCB}D, \quad V(1/2)_{SCB} = \pm S(1/2)_{SCB}D, \tag{100}$$

$$V(-1/2)_{OIB} = i\sqrt{2}S(-1/2)_{OIB}D, \quad V(1/2)_{OIB} = \sqrt{2}S(1/2)_{OIB}D, \tag{101}$$

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$$V\left(-1/2\right)_{OCB}=\pm S\left(-1/2\right)_{OCB}D,\quad V\left(1/2\right)_{OCB}=\pm S\left(1/2\right)_{OCB}D, \tag{102}$$

$$V\left(-1/2\right)_{CIB}\sqrt{2}S\left(-1/2\right)^{*}_{CIB}D,\quad V\left(1/2\right)_{CIB}=\sqrt{2}S\left(1/2\right)_{CIB}D, \tag{103}$$

$$V\left(-1/2\right)_{CCB}=-S\left(-1/2\right)_{CCB}D,\quad V\left(1/2\right)_{CCB}=S\left(1/2\right)_{CCB}D, \tag{104}$$

$$V\left(-1/2\right)_{B}=-S\left(-1/2\right)_{B}D,\quad V\left(1/2\right)_{B}=S\left(1/2\right)_{B}D, \tag{105}$$

$$V\left(-1/2\right)_{S}=\pm S\left(-1/2\right)_{S}Ai2\sqrt{2},\quad V\left(1/2\right)_{S}=\pm S\left(1/2\right)_{S}A2\sqrt{\frac{2}{3}}, \tag{106}$$

where "*" denotes a complex conjugate. Furthermore, if $A=D$

$$S\left(3\right)_{OIB}=\pm S\left(3\right)_{S},\quad S\left(2\right)_{OIB}=\pm S\left(2\right)_{CIB}. \tag{107}$$

Knowing volumes (1) and surfaces (24) of n-balls in complex dimensions, we can extend the relations $S(n)_B S(2-n)_B = 4\,\mathrm{Re}(i^{n-1})$ (24), and $2\pi n V(n)_B V(-n)_B = 4\,\mathrm{Re}(i^{n-1})$ (27) between n-ball surfaces and volumes in integer dimensions, disclosed in the prior research [16]. Products of respectively (1) and (24) for n and $-n$ yield

$$\begin{aligned}V(n)_B V(-n)_B &= \frac{2\sin\left(\pi n/2\right)}{\pi n}\\ &= \frac{-2\sin\left[\pi\left(n/2+1\right)\right]}{\pi n},\end{aligned} \tag{108}$$

$$\begin{aligned}S(n)_B S(2-n)_B &= 4\sin\left(\pi n/2\right)\\ &= -4\sin\left[\pi\left(n/2+1\right)\right].\end{aligned} \tag{109}$$

Also, the following relations between volumes (71), (74) and surfaces (72), (75)

$$V(n)_{CCB}=n^{n/2}V(n)_{CIB}, \tag{110}$$

$$S(n)_{CCB}=n^{(n-1)/2}S(n)_{CIB}, \tag{111}$$

volumes (51), (45) and surfaces (55), (47)

$$V(n)_{SCB}=n^{n}V(n)_{SIB}, \tag{112}$$

$$S(n)_{SCB}=n^{n-1}S(n)_{SIB}, \tag{113}$$

and volumes (65), (61) and surfaces (68), (62)

$$V(n)_{OCB}=n^{n/2}V(n)_{OIB}, \tag{114}$$

$$S(n)_{OCB}=n^{(n-1)/2}S(n)_{OIB}, \tag{115}$$

can be easily obtained, as shown in Fig. 13. Notably, the ratio of the volume of n-cube circumscribed about n-ball and volume of n-cube inscribed in n-ball (110) is the same as the ratio of the volume of n-orthoplex circumscribed about n-ball and volume of n-orthoplex inscribed in n-ball (114), and the same holds true for the ratio of their surfaces (111), (115). This is not surprising: as n-cube is dual to n-orthoplex, these ratios remain invariant.

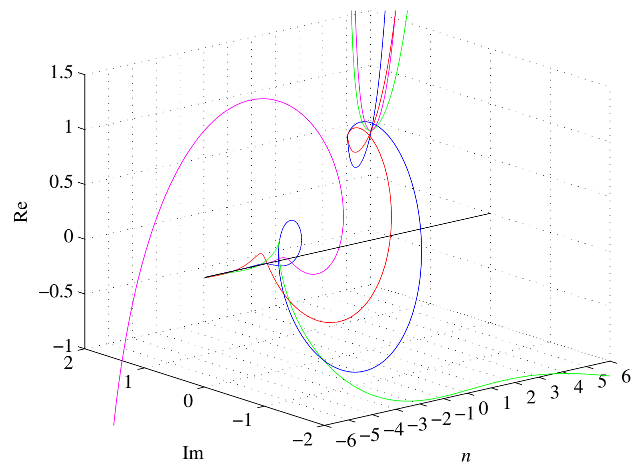

Figure 13. $n^{n/2}$ (red), n^{n} (green), $n^{(n-1)/2}$ (blue), n^{n-1} (pink), for $n=[-6,6]$.

VIII. CONCLUSIONS

The recurrence relations (2), (3), (5), (11), (12), (15), (16) defining volumes and surfaces of n-balls and the omnidimensional polytopes expressed by the gamma function (1), (24); (28), (30); (34), (36) show that volumes and surfaces of omnidimensional, convex polytopes and n-balls are holomorphic functions of a complex dimension n.

The volume of an n-simplex turns out to be a bivalued function of n. Thus, the surfaces of n-simplices and n-orthoplices are also bivalued functions of n.

Applications of these formulas to the omnidimensional polytopes inscribed in and circumscribed about n-balls revealed the properties of these geometric objects in complex dimensions. In particular, for $0 < n < 1$, volumes of the omnidimensional polytopes are larger than volumes of circumscribing n-balls, while their volumes and surfaces are smaller than volumes of inscribed n-balls.

It was shown that certain metric-independent products (86)-(93), (95)-(97) and quotients (110)-(114) of volumes and surfaces of these circumscribed and inscribed omnidimensional polytopes and n-balls are independent on the gamma function.

The results of this study could be applied in linguistic statistics, where the dimension in the distribution for frequency dictionaries is chosen to be negative [3], in fog computing, where n-simplex is related to a full mesh pattern, n-orthoplex is linked to a quasi-full mesh structure, and n-cube is referred to as a certain type of partial mesh layout [27], and in molecular physics and crystallography. Perhaps they are also related to the 2-dimensional quantum hall effect.

ACKNOWLEDGMENTS

I thank Tomek for his motivation.

Appendix: $n\sqrt{n}/\sqrt{n^3} = \pm 1 \quad \forall n \in \mathbb{C}$

It is known that complex number $n = |n|e^{i\theta} = |n|(\cos\theta + i\sin\theta) \neq 0$ has two square roots

$$\sqrt{n} = \sqrt{|n|}\left(\cos\frac{\theta+2k\pi}{2} + i\sin\frac{\theta+2k\pi}{2}\right), \tag{A.1}$$

for $k = 0, 1$, that is

$$\begin{aligned}\sqrt{n} &= \sqrt{|n|}\begin{cases}\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\\ \cos\frac{\theta+2\pi}{2} + i\sin\frac{\theta+2\pi}{2}\end{cases}\\ &= \sqrt{|n|}\begin{cases}\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\end{cases}\\ &= \sqrt{|n|}\begin{cases}\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\\ -\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\end{cases}\\ &= \pm\sqrt{|n|}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \pm\sqrt{|n|}e^{i\frac{\theta}{2}}.\end{aligned} \tag{A.2}$$

Thus,

$$\frac{n\sqrt{n}}{\sqrt{n^3}} = \frac{|n|e^{i\theta}\left(\pm\sqrt{|n|}e^{i\frac{\theta}{2}}\right)}{\pm\sqrt{|n|^3}e^{i\frac{3\theta}{2}}} = \pm 1. \tag{A.3}$$

The same holds true for $(n-1)\sqrt{n-1}/\sqrt{(n-1)^3} = \pm 1$, setting $m \doteq n - 1$.

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