Article

Not peer-reviewed version

Geometry of CR-Slant Warped Products in Nearly Kaehler Manifolds

Siraj Uddin , Bang-Yen Chen * , Rawan Bossly

Posted Date: 15 May 2023

doi: 10.20944/preprints202305.1002.v1

Keywords: CR-product; CR-warped product; CR-slant warped product; Chen's inequality; nearly Kaehler manifolds

Article

Geometry of CR-Slant Warped Products in Nearly Kaehler Manifolds

Siraj Uddin [1], Bang-Yen Chen [2,*] and Rawan Mohammed Bossly [1,3]

1 Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia; sshehabaldeen@kau.edu.sa

2 Department of Mathematics, Michigan State University, East Lansing, Michigan, 48824–1027, U.S.A.; chenb@msu.edu

3 Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia; rboslly@stu.kau.edu.sa

* Correspondence: chenb@msu.edu

Abstract: Recently, we studied CR-slant warped products $B_1 \times_f M_\perp$, where $B_1 = M_T \times M_\theta$ is the Riemannian product of holomorphic and proper slant submanifolds and $M_\perp$ is a totally real submanifold in a nearly Kaehler manifold. In the continuation, in this paper, we study $B_2 \times_f M_\theta$, where $B_2 = M_T \times M_\perp$ is a CR-product of a nearly Kaehler manifold and establish Chen's inequality for the squared norm of the second fundamental form. Some special cases of Chen's inequality are given.

Keywords: CR-product; CR-warped product; CR-slant warped product; Chen's inequality; nearly Kaehler manifolds

MSC: 53B05; 53B20; 53C25; 53C40

---

1. Introduction

The study of CR-warped products was initiated by the second author in [6,7]. In [20], we studied CR-slant warped product submanifolds of the form $B_1 \times_f M_\perp$, where $B_1 = M_T \times M_\theta$ is the Riemannian product of holomorphic and proper slant submanifolds and $M_\perp$ is a totally real submanifold of a nearly Kaehler manifold $\tilde{M}$. In fact, we established the following Chen's inequality:

Theorem 1. *[20] Let $M = B_1 \times_f M_\perp$ be a CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$ such that M is $\mathfrak{D}^\perp \oplus \mathfrak{D}^\theta$-mixed totally geodesic in $\tilde{M}$, where $B_1 = M_T \times M_\theta$ is the Riemannian product of complex and proper slant submanifolds of $\tilde{M}$. Then:*

(i) The second fundamental form h satisfies

$$\|h\|^2 \geq 2s\|\vec{\nabla}^T(\ln f)\|^2 + s\cot^2\theta\|\vec{\nabla}^\theta(\ln f)\|^2 \tag{1}$$

where $s = \dim M_\perp$ and $\vec{\nabla}^T(\ln f)$ and $\vec{\nabla}^\theta(\ln f)$ denote the gradient components of $\ln f$ along M_T and M_θ, respectively.

(ii) If the equality sign in (1) holds identically, then M_T and M_θ are totally geodesic, B_1 is mixed totally geodesic in $\tilde{M}$ and $M_\perp$ is totally umbilical in $\tilde{M}$.

In the sequel, in this paper, we study CR-slant warped product submanifold $M = B_2 \times_f M_\theta^{n_3}$, where $B_2 = M_T^{n_1} \times M_\perp^{n_2}$ is the CR-product and $M_\theta^{n_3}$ is an n_3-dimensional proper θ-slant submanifold in a nearly Kaehler manifold $\tilde{M}^{2m}$. We prove that the second fundamental form h of M satisfies the following inequality:

$$\|h\|^2 \geq \frac{1}{9}n_3\cos^2\theta\|\vec{\nabla}^\perp(\ln f)\|^2 + 2n_3\left(1 + \frac{10}{9}\cot^2\theta\right)\|\vec{\nabla}^T(\ln f)\|^2$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 May 2023 doi:10.20944/preprints202305.1002.v1

where $\vec{\nabla}^{\perp}(\ln f)$ and $\vec{\nabla}^{T}(\ln f)$ are the gradients of $\ln f$ along $M_{\perp}$ and M_T, respectively. The equality case is discussed and some special cases of the inequality are given.

2. Basic definitions and formulas

Let $\tilde{M}^{2m}$ be an almost Hermitian manifold endowed with an almost complex structure J and a Riemannian metric $\tilde{g}$ such that

$$J^2(X) = -X, \quad \tilde{g}(JX, JY) = \tilde{g}(X, Y) \tag{2}$$

for any $X, Y \in \Gamma(T\tilde{M}^{2m})$, where $\Gamma(T\tilde{M}^{2m})$ denotes the Lie algebra of vector fields on $\tilde{M}^{2m}$. In addition, an almost Hermitian manifold is called *Kaehler manifold* if

$$(\tilde{\nabla}_X)Y = 0, \ \forall\, X, Y \in \Gamma(T\tilde{M}^{2m}),$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}^{2m}$. Furthermore, an almost Hermitian manifold $\tilde{M}^{2m}$ is *nearly Kaehler* if $(\tilde{\nabla}_X)X = 0, \ \forall\, X \in \Gamma(T\tilde{M}^{2m})$, equivalently

$$(\tilde{\nabla}_X)Y + (\tilde{\nabla}_Y)X = 0, \ \forall\, X, Y \in \Gamma(T\tilde{M}^{2m}). \tag{3}$$

Clearly, every Kaehler manifold is nearly Kaehler but the converse is not true in general. The best known example of a nearly Kaehler non-Kaehlerian manifold is 6-dimensional sphere $\mathbb{S}^6$.

Let M^n be a Riemannian manifold isometrically immersed in an almost Hermitian manifold $\tilde{M}^{2m}$, $(n < 2m)$ and from now on we denote the metric $\tilde{g}$ and the induced metric g on M by the same symbol g. Then the Gauss and Weingarten formulas are respectively given by (see, for instance, [6,10])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4}$$
$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi, \tag{5}$$

for vector fields $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^{\perp}M)$, where $\Gamma(T^{\perp}M)$ is the set of all vector fields normal to M and ∇ and $\nabla^{\perp}$ denote the induced connections on the tangent and normal bundles of M, respectively, and h is the second fundamental form, A is the shape operator of M and they are related by

$$g(A_\xi X, Y) = g(h(X, Y), \xi), \ \forall\, X, Y \in \Gamma(TM), \ \xi \in \Gamma(T^{\perp}M). \tag{6}$$

For each vector field X tangent to M^n, we write

$$JX = PX + FX. \tag{7}$$

where PX and FX are the tangential and normal components of JX. Complex and totally real submanifolds are defined on the behaviour of almost complex structure J on M^n. A submanifold M^n of an almost Hermitian manifold $\tilde{M}$ is complex if its tangent space remains the same under the action of almost complex structure J. On contrary, M is a totally real submanifold of $\tilde{M}$ if $JX \in \Gamma(T^{\perp}M)$ for any $X \in \Gamma(TM)$.

A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called *CR-submanifold if there exists on M a differentiable holomorphic distribution $\mathfrak{D} : p \to \mathfrak{D}_p \subset T_pM$ whose orthogonal complementary distribution $\mathfrak{D}^{\perp}$ is totally real. A CR-submanifold M of an almost Hermitian manifold $\tilde{M}$ is called a* CR-product if it is a Riemannian product of a holomorphic submanifold M_T and a totally real submanifold $M_{\perp}$ of $\tilde{M}$. The second author introduced the notion of CR-product of Kaehler manifolds in [3].

In [4], the second author introduced another important class of submanifolds and he called them slant submanifolds those are the generalization of complex and totally real submanifolds. He defined slant submanifolds as:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 May 2023 doi:10.20944/preprints202305.1002.v1

Definition 1. *A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called slant if for each $p \in M$, the Wirtinger angle $\theta(X)$ between JX and T_pM is constant on M, i.e., it does not depend on the choice of $X \in T_pM$ and $p \in M$ [4,5]. In this case, θ is called the slant angle of M.*

Complex (holomorphic) and totally real submanifolds are slant submanifolds with slant angles 0 and $\frac{\pi}{2}$, respectively. A slant submanifold is called *proper slant* if it is neither holomorphic nor totally real.

More generally, a distribution $\mathfrak{D}$ on M is called a *slant distribution* if the angle $\theta(X)$ between JX and $\mathfrak{D}_p$ is independent of the choice of $p \in M$ and of $0 \neq X \in \mathfrak{D}_p$.

He proved that a submanifold M of an almost Hermitian manifold $\tilde{M}$ is slant if and only if [4]

$$P^2X = -(\cos^2\theta)X, \quad X \in \Gamma(TM). \tag{8}$$

Clearly, from (7) and (8), we know that

$$g(PX, PY) = (\cos^2\theta)g(X,Y), \quad g(FX, FY) = (\sin^2\theta)g(X,Y), \tag{9}$$

for any vector fields X, Y tangent to M.

Definition 2. *A submanifold M of an almost Hermitian manifold $\tilde{M}$ is* CR-slant submanifold (skew CR-submanifold) if there exist orthogonal distributions $\mathfrak{D}$, $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ such that the tangent bundle TM is spanned by

$$TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta, \tag{10}$$

where $\mathfrak{D}$, $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ are complex, totally real and proper slant distributions.

The normal bundle of a CR-slant submanifold M is decomposed by

$$T^\perp M = J\mathfrak{D}^\perp \oplus F\mathfrak{D}^\theta \oplus \nu, \tag{11}$$

where ν is an invariant normal subbundle of $T^\perp M$.

A CR-slant product submanifold M is *semi-slant mixed totally geodesic* (resp., *hemi-slant mixed totally geodesic*) if its second fundamental satisfies

$$h(X_1, X_2) = 0 \quad \forall\, X_1 \in \Gamma(\mathfrak{D}), \ \forall\, X_2 \in \Gamma(\mathfrak{D}^\theta)$$
$$(resp., \ \ h(X_2, X_3) = 0 \quad \forall\, X_2 \in \Gamma(\mathfrak{D}^\theta), \ \forall\, X_3 \in \Gamma(\mathfrak{D}^\perp)).$$

3. CR-slant warped products $(M_T \times M_\perp) \times_f M_\theta$

In this section, first we recall the definition of warped product manifolds which are the generalizations of Riemannian products. In 1969, Bishop and O'Neill [2] introduced the notion of warped product manifolds as follows:

Definition 3. *A warped product $B \times_f F$ of two Riemannian manifolds (B, g_B) and (F, g_F) is the product manifold $M = B \times F$ equipped with the product structure*

$$g_M(X,Y) = g_B(\pi_{1*}X, \pi_{1*}Y) + (f \circ \pi_1)^2 g_F(\pi_{2*}X, \pi_{2*}Y)$$

where $f : B \to (0,\infty)$ and $\pi_1 : M \to B$, $\pi_2 : M \to F$ are projection maps given by $\pi_1(p,q) = p$ and $\pi_2(p,q) = q$ for any $(p,q) \in B \times F$ and $*$ denotes the symbol for tangent map.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 May 2023 doi:10.20944/preprints202305.1002.v1

The function f is called warping function, if f is constant, then M is simply a Riemannian product. It is known that, for any vector field X on B and a vector field Z on F, we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z \tag{12}$$

where ∇ is the Levi-Civita connection on M. Further, it is well known that the base manifold B is totally geodesic and the fiber F is totally umbilical in M.

Now, we define CR-slant warped products $(M_T \times M_\perp) \times_f M_\theta$.

Definition 4. *A submanifold M of an almost Hermitian manifold $\tilde{M}$ is said to be CR-slant warped product submanifold if it is a warped product of CR-product $M_T \times M_\perp$ and a proper θ-slant submanifold M_θ of $\tilde{M}$.*

In [20], we discussed CR-slant warped product submanifolds of the form $B_1 \times_f M_\perp$, where $B_1 = M_T \times M_\theta$. In this section we study CR-slant warped products of the form $B_2 \times_f M_\theta$, where $B_2 = M_T \times M_\perp$. For this, we use the following conventions: $X_1, Y_1, \ldots$ are vector fields on $\mathfrak{D}$ and $X_2, Y_2 \ldots$ are vector fields on $\mathfrak{D}^\theta$, while $X_3, Y_3, \ldots$ are vector fields on $\mathfrak{D}^\perp$.

First, we have the following preparatory lemmas.

Lemma 1. *On a CR-slant warped product submanifold $M = B_2 \times_f M_\theta$ of a nearly Kaehler manifold $\tilde{M}$, we have*

(i) $g(h(X_1, Y_1), FX_2) = 0$,
(ii) $2g(h(X_3, Y_3), FX_2) = g(h(X_2, X_3), JY_3) + g(h(X_2, Y_3), JX_3)$,

for any $X_1, Y_1 \in \Gamma(TM_T)$, $X_2 \in \Gamma(TM_\theta)$ and $X_3, Y_3 \in \Gamma(TM_\perp)$, where $B_2 = M_T \times M_\perp$ is the CR-product submanifold in $\tilde{M}$.

Proof. The first part is easy to prove by using (4), (3) and (12). For the second part, we have

$$g(h(X_3, Y_3), FX_2) = g(\tilde{\nabla}_{X_3} Y_3, JX_2) + g(\tilde{\nabla}_{X_3} PX_2, Y_3) - g(J\nabla_{X_3} Y_3, X_2) + g(\nabla_{X_3} Y_3, PX_2)$$

for any $X_2 \in \Gamma(TM_\theta)$ and $X_3, Y_3 \in \Gamma(TM_\perp)$. Since $\nabla_{X_3} Y_3 \in \Gamma(TM_\perp)$, then using orthogonality of vector fields and covariant derivative property of J with (12), we find

$$\begin{aligned} g(h(X_3, Y_3), FX_2) &= g((\tilde{\nabla}_{X_3} J)Y_3, X_2) - g(\tilde{\nabla}_{X_3} JY_3, X_2) + X_3(\ln f)g(PX_2, Y_3) \\ &= g((\tilde{\nabla}_{X_3} J)Y_3, X_2) + g(h(X_2, X_3), JY_3) \end{aligned} \tag{13}$$

Similarly, by interchanging X_3 with Y_3 in (13), we brain

$$g(h(X_3, Y_3), FX_2) = g((\tilde{\nabla}_{Y_3} J)X_3, X_2) + g(h(X_2, Y_3), JX_3). \tag{14}$$

Hence, the second part immediately follows from (13) and (14). □

Lemma 2. *Let $M = B_2 \times_f M_\theta$ be a CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$ such that $B_2 = M_T \times M_\perp$ is the CR-product submanifold in $\tilde{M}$. Then, we have*

$$g(h(X_1, X_3), FX_2) = \frac{1}{2} g(h(X_1, X_2), JX_3) \tag{15}$$

for any $X_1 \in \Gamma(TM_T)$, $X_2 \in \Gamma(TM_\theta)$ and $X_3 \in \Gamma(TM_\perp)$.

Proof. For any $X_1 \in \Gamma(TM_T)$, $X_2 \in \Gamma(TM_\theta)$ and $X_3 \in \Gamma(TM_\perp)$, we have

$$g(h(X_1, X_3), FX_2) = g((\tilde{\nabla}_{X_3} J)X_1, X_2) - g(\tilde{\nabla}_{X_3} JX_1, X_2) = g((\tilde{\nabla}_{X_3} J)X_1, X_2). \tag{16}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 May 2023 doi:10.20944/preprints202305.1002.v1

On the other hand, we know that

$$g(h(X_1, X_3), FX_2) = g((\tilde{\nabla}_{X_1} J)X_3, X_2) - g(\tilde{\nabla}_{X_1} JX_3, X_2) + g(X_3, \tilde{\nabla}_{X_1} PX_2). \tag{17}$$

Then, the lemma follows from (16) and (17) with the help of (3) and (12). □

Lemma 3. *For a proper CR-slant warped product $M = B_2 \times_f M_\theta$ such that $B_2 = M_T \times M_\perp$ in a nearly Kaehler manifold $\tilde{M}$, we have*

$$g(h(JX_1, X_2), FY_2) = X_1(\ln f)g(X_2, Y_2) + \frac{1}{3}JX_1(\ln f)g(X_2, PY_2) \tag{18}$$

for any $X_1 \in \Gamma(TM_T)$, $X_2, Y_2 \in \Gamma(TM_\theta)$.

Proof. From (4) and (12), we have

$$g(h(X_1, X_2), FY_2) = g((\tilde{\nabla}_{X_2} J)X_1, Y_2) - JX_1(\ln f)g(X_2, Y_2), \tag{19}$$

for any orthogonal vector fields $X_1 \in \Gamma(TM_T)$, $X_2, Y_2 \in \Gamma(TM_\theta)$. On the other hand, we derive

$$g(h(X_1, X_2), FY_2) = g((\tilde{\nabla}_{X_1} J)X_2, Y_2) - X_1(\ln f)g(PX_2, Y_2) + g(h(X_1, Y_2), FX_2). \tag{20}$$

Then, from (19) and (20), we find

$$2g(h(X_1, X_2), FY_2) = X_1(\ln f)g(X_2, PY_2) - JX_1(\ln f)g(X_2, Y_2) + g(h(X_1, Y_2), FX_2). \tag{21}$$

Interchanging X_2 by Y_2, we obtain

$$2g(h(X_1, Y_2), FX_2) = X_1(\ln f)g(PX_2, Y_2) - JX_1(\ln f)g(X_2, Y_2) + g(h(X_1, X_2), FY_2). \tag{22}$$

Then, from (21) and (22), we derive

$$g(h(X_1, X_2), FY_2) = -JX_1(\ln f)g(X_2, Y_2) + \frac{1}{3}X_1(\ln f)g(X_2, PY_2). \tag{23}$$

Hence, (18) follows immediately by interchanging X_1 with JX_1 in (23), which proves the lemma completely. □

The following relations are immediate consequences of (18).

$$g(h(JX_1, PX_2), FY_2) = X_1(\ln f)g(PX_2, Y_2) + \frac{1}{3}\cos^2\theta JX_1(\ln f)g(X_2, Y_2), \tag{24}$$

$$g(h(JX_1, PX_2), FPY_2) = \cos^2\theta X_1(\ln f)g(X_2, Y_2) + \frac{1}{3}\cos^2\theta JX_1(\ln f)g(X_2, PY_2), \tag{25}$$

$$g(h(JX_1, X_2), FPY_2) = X_1(\ln f)g(X_2, PY_2) - \frac{1}{3}\cos^2\theta JX_1(\ln f)g(X_2, Y_2). \tag{26}$$

Lemma 4. *Let $M = B_2 \times_f M_\theta$ be a CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$ such that $B_2 = M_T \times M_\perp$ is the CR-product submanifold in $\tilde{M}$. Then, we have*

$$g(h(X_2, Y_2), JX_3) = g(h(X_2, X_3), FY_2) + \frac{1}{3}X_3(\ln f)g(X_2, PY_2) \tag{27}$$

for any $X_2, Y_2 \in \Gamma(TM_\theta)$ and $X_3 \in \Gamma(TM_\perp)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 May 2023 doi:10.20944/preprints202305.1002.v1

Proof. From the definition of covariant derivative with (4) and (7), we have

$$g(h(X_2,X_3),FY_2) = g((\tilde{\nabla}_{X_3}J)X_2,Y_2) - g(\tilde{\nabla}_{X_3}PX_2,Y_2) - g(\tilde{\nabla}_{X_3}FX_2,Y_2) - g(\tilde{\nabla}_{X_3}X_2,PY_2).$$

Again using (4), (5) and (12), we find

$$g(h(X_2,X_3),FY_2) = g((\tilde{\nabla}_{X_3}J)X_2,Y_2) + g(h(Y_2,X_3),FX_2). \tag{28}$$

On the other hand, we derive

$$\begin{aligned} g(h(X_2,X_3),FY_2) &= g((\tilde{\nabla}_{X_2}J)X_3,Y_2) - g(\tilde{\nabla}_{X_2}JX_3,Y_2) - g(\tilde{\nabla}_{X_2}X_3,PY_2) \\ &= g((\tilde{\nabla}_{X_2}J)X_3,Y_2) + g(h(X_2,Y_2),JX_3) - X_3(\ln f)g(X_2,PY_2). \end{aligned} \tag{29}$$

Then, from (28) and (29), we get

$$2g(h(X_2,X_3),FY_2) = g(h(X_2,Y_2),JX_3) + g(h(Y_2,X_3),FX_2) - X_3(\ln f)g(X_2,PY_2). \tag{30}$$

Interchanging X_2 by Y_2, we obtain

$$2g(h(Y_2,X_3),FX_2) = g(h(X_2,Y_2),JX_3) + g(h(X_2,X_3),FY_2) + X_3(\ln f)g(X_2,PY_2). \tag{31}$$

Then, from (30) and (31), we get (27); which proves the Lemma completely. □

4. Chen's inequality and its consequences

In this section first we prove the following main result by using Lemma 3.

Theorem 2. *Let $M = B_2 \times_f M_\theta$ be a proper CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$. Then, M is simply Riemannian product if and only if either M is semi-slant mixed totally geodesic i.e., $h(X_1,X_2) = 0,\ \forall\, X_1 \in \Gamma(\mathfrak{D}),\ X_2 \in \Gamma(\mathfrak{D}^\theta)$ or $h(\mathfrak{D},\mathfrak{D}^\theta)$ is orthogonal to $F\mathfrak{D}^\theta$.*

Proof. From Lemma 3, we find

$$g(h(JX_1,X_2),FY_2) = \frac{1}{3}JX_1(\ln f)g(X_2,PY_2) + X_1(\ln f)g(X_2,Y_2), \tag{32}$$

for any $X_1 \in \Gamma(\mathfrak{D}),\ X_2, Y_2 \in \Gamma(\mathfrak{D}^\theta)$. Then, from (26) and (32), we derive

$$g(h(JX_1,X_2),FY_2) + \frac{1}{3}g(h(X_1,X_2),FPY_2) = \left(1 - \frac{1}{9}\cos^2\theta\right) X_1(\ln f)g(X_2,Y_2). \tag{33}$$

If M is semi-slant mixed totally geodesic or $h(\mathfrak{D},\mathfrak{D}^\theta)$ is orthogonal to $F\mathfrak{D}^\theta$ then from (33), we find

$$\left(1 - \frac{1}{9}\cos^2\theta\right) X_1(\ln f)g(X_2,Y_2) = 0$$

Since g is a Riemannian metric and $-1 \leq \cos\theta \leq 1$, then from above equation we get $X_1(\ln f) = 0$, i.e., f is constant along M_T.

Conversely, if f is constant then again from (33), we get

$$g(h(JX_1,X_2),FY_2) + \frac{1}{3}g(h(X_1,X_2),FPY_2) = 0. \tag{34}$$

Interchanging X_1 by JX_1 and Y_2 by PY_2 in (34), we derive

$$g(h(X_1, X_2), FPY_2) + \frac{1}{3}\cos^2\theta g(h(JX_1, X_2), FY_2) = 0. \tag{35}$$

Then, from (34) and (35), we obtain

$$\left(1 - \frac{1}{9}\cos^2\theta\right) g(h(JX_1, X_2), FY_2) = 0. \tag{36}$$

Since $-1 \leq \cos\theta \leq 1$ for any value of $\theta \in \mathbb{R}$, thus we find either $h(\mathfrak{D}, \mathfrak{D}^\theta) = \{0\}$ or $h(\mathfrak{D}, \mathfrak{D}^\theta)$ is orthogonal to $F\mathfrak{D}^\theta$, which completes the proof. □

Next, we derive the Chen's inequality for CR-slant wanted products $M = B_2 \times_f M_\theta$, where $B_2 = M_T \times M_\perp$ is a CR-product in a nearly Kaehler manifold.

Theorem 3. *Let $M = \left(M_T^{n_1} \times M_\perp^{n_2}\right) \times_f M_\theta^{n_3}$ be a CR-slant warped product submanifold of a nearly Kaehler manifold $\tilde{M}$ such that M is hemi-slant mixed totally geodesic. Then, the squared norm of the second fundamental form satisfies*

$$\|h\|^2 \geq \frac{1}{9} n_3 \cos^2\theta \|\vec{\nabla}^\perp(\ln f)\|^2 + 2n_3\left(1 + \frac{10}{9}\cot^2\theta\right)\|\vec{\nabla}^T(\ln f)\|^2 \tag{37}$$

where $\vec{\nabla}^T(\ln f)$ and $\vec{\nabla}^\perp(\ln f)$ denote the gradient components of $\ln f$ along M_T and $M_\perp$, respectively.

Furthermore, if the equality holds in (37), then $M_T \times M_\perp$ is totally geodesic and M_θ is totally umbilical in $\tilde{M}$. Moreover, M is not a semi-slant mixed totally geodesic submanifold of $\tilde{M}$.

Proof. If we denote the tangent bundles of M_T, $M_\perp$ and M_θ by $\mathfrak{D}$, $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, respectively; then we use the following frame fields for the CR-slant warped product

$$\begin{aligned}
&\mathfrak{D} = \mathrm{Span}\{e_1, \cdots, e_p, e_{p+1} = Je_1, \cdots, e_{n_1} = e_{2p} = Je_p\},\\
&\mathfrak{D}^\perp = \mathrm{Span}\{e_{n_1+1} = \hat{e}_1, \cdots, e_{n_1+n_2} = \hat{e}_{n_2}\},\\
&\mathfrak{D}^\theta = \mathrm{Span}\{e_{n_1+n_2+1} = e_1^*, \cdots, e_{n_1+n_2+q} = e_q^*, e_{n_1+n_2+q+1} = \sec\theta Pe_1^*, \cdots,\\
&\qquad e_n = e_{2q}^* = \sec\theta Pe_q^*\}.
\end{aligned}$$

And the normal bundle frame will be

$$\begin{aligned}
&J\mathfrak{D}^\perp = \mathrm{Span}\{e_{n+1} = \tilde{e}_1 = J\hat{e}_1, \cdots, e_{n+n_2} = \tilde{e}_{n_2} = J\hat{e}_{n_2}\},\\
&F\mathfrak{D}^\theta = \mathrm{Span}\{e_{n+n_2+1} = \tilde{e}_{n_2+1} = E_1^* = \csc\theta Fe_1^*, \cdots, e_{n+n_2+q} = \tilde{e}_{n_2+q} = E_q^* = \csc\theta Fe_q^*,\\
&\qquad e_{n+n_2+q+1} = \tilde{e}_{n_2+q+1} = E_{q+1}^* = \csc\theta\sec\theta FPe_1^*, \cdots,\\
&e_{n+n_2+n_3} = \tilde{e}_{n_2+n_3} = E_{n_3}^* = \csc\theta\sec\theta FPe_q^*\},\\
&\nu = \mathrm{Span}\{e_{n+n_2+n_3+1} = \tilde{e}_{n_2+n_3+1}, \cdots, e_{2m} = \tilde{e}_{2m-n-n_2-n_3}\}.
\end{aligned}$$

From the definition of h, we find

$$\begin{aligned}
\|h\|^2 &= \|h(\mathfrak{D}, \mathfrak{D})\|^2 + \|h(\mathfrak{D}^\perp, \mathfrak{D}^\perp)\|^2 + \|h(\mathfrak{D}^\theta, \mathfrak{D}^\theta)\|^2\\
&\quad + 2\left(\|h(\mathfrak{D}, \mathfrak{D}^\perp)\|^2 + \|h(\mathfrak{D}, \mathfrak{D}^\theta)\|^2 + \|h(\mathfrak{D}^\perp, \mathfrak{D}^\theta)\|^2\right).
\end{aligned} \tag{38}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 May 2023 doi:10.20944/preprints202305.1002.v1

Using the frame fields and preparatory lemmas, we solve each term of (38) as follows:

$$\|h(\mathfrak{D},\mathfrak{D})\|^2 = \sum_{k=1}^{n_2}\sum_{i,j=1}^{n_1}\left(g(h(e_i,e_j),J\hat{e}_k)\right)^2 + \sum_{k=1}^{n_3}\sum_{i,j=1}^{n_1}\left(g(h(e_i,e_j),E_k^*)\right)^2 + \sum_{k=1}^{2m-n-n_2-n_3}\sum_{i,j=1}^{n_1}\left(g(h(e_i,e_j),\tilde{e}_k)\right)^2.$$

Leaving the ν-components terms and the is no warped product relation for the first term, then from Lemma 1 (i), we get

$$\|h(\mathfrak{D},\mathfrak{D})\|^2 \geq 0. \tag{39}$$

Similarly, for the second term of (38), we derive

$$\|h(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\|^2 = \sum_{k=1}^{n_2}\sum_{i,j=1}^{n_2}\left(g(h(\hat{e}_i,\hat{e}_j),J\hat{e}_k)\right)^2 + \sum_{k=1}^{n_3}\sum_{i,j=1}^{n_2}\left(g(h(\hat{e}_i,\hat{e}_j),E_k^*)\right)^2 + \sum_{k=1}^{2m-n-n_2-n_3}\sum_{i,j=1}^{n_2}\left(g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_k)\right)^2.$$

Using Lemma 1 (ii) with the given hemi-slant totally geodesic condition and leaving the first and last positive terms, we find

$$\|h(\mathfrak{D}^\perp,\mathfrak{D}^\perp)\|^2 \geq 0. \tag{40}$$

For the third term of (38), we find

$$\|h(\mathfrak{D}^\theta,\mathfrak{D}^\theta)\|^2 = \sum_{k=1}^{n_2}\sum_{i,j=1}^{n_3}\Big(g(h(e_i^*,e_j^*),J\hat{e}_k)\Big)^2 + \sum_{k=1}^{n_3}\sum_{i,j=1}^{n_3}\Big(g(h(e_i^*,e_j^*),E_k^*)\Big)^2 + \sum_{k=1}^{2n-n_2-n_3}\sum_{i,j=1}^{n_3}\Big(g(h(e_i^*,e_j^*),\tilde{e}_k)\Big)^2$$

Leaving the last two positive terms and using Lemma 4 with mixed totally geodesic condition, we get

$$\|h(\mathfrak{D}^\theta,\mathfrak{D}^\theta)\|^2 \geq \frac{2q}{9}\cos^2\theta\sum_{k=1}^{n_2}(e_k(\ln f))^2 = \frac{1}{9}n_3\cos^2\theta\|\vec{\nabla}^\perp(\ln f)\|^2. \tag{41}$$

Similarly, we derive the other terms of (38) as follows

$$\|h(\mathfrak{D},\mathfrak{D}^\perp)\|^2 = \sum_{k,j=1}^{n_2}\sum_{i=1}^{n_1}\left(g(h(e_i,\hat{e}_j),J\hat{e}_k)\right)^2 + \sum_{k=1}^{n_3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\left(g(h(e_i,\hat{e}_j),E_k^*)\right)^2 + \sum_{k=1}^{2m-n-n_2-n_3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\left(g(h(e_i,\hat{e}_j),\tilde{e}_k)\right)^2$$

There is no relation for the first positive term in terms of warped products and leaving the last ν-components term. Then, using Lemma 2, we derive

$$\|h(\mathfrak{D},\mathfrak{D}^{\perp})\|^2 \geq \frac{1}{4}\sum_{j=1}^{n_2}\sum_{i=1}^{n_1}\sum_{k=1}^{n_3}\left(g(h(e_i,e_k^*),J\hat{e}_j)\right)^2 \geq 0. \tag{42}$$

On the other hand, we also have

$$\|h(\mathfrak{D},\mathfrak{D}^{\theta})\|^2 = \sum_{k=1}^{n_2}\sum_{j=1}^{n_3}\sum_{i=1}^{n_1}\left(g(h(e_i,e_j^*),J\hat{e}_k)\right)^2 + \sum_{k,j=1}^{n_3}\sum_{i=1}^{n_1}\left(g(h(e_i,e_j^*),E_k^*)\right)^2$$
$$+ \sum_{k=1}^{2m-n-n_2-n_3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_3}\left(g(h(e_i,e_j^*),\tilde{e}_k)\right)^2$$

For the first term we use (42) and omit the ν-components terms and using frame fields of $\mathfrak{D}^{\theta}$ and $F\mathfrak{D}^{\theta}$, we derive

$$\|h(\mathfrak{D},\mathfrak{D}^{\theta})\|^2 \geq \csc^2\theta\sum_{k,j=1}^{q}\sum_{i=1}^{n_1}\left(g(h(e_i,e_j^*),Fe_k^*)\right)^2 + \csc^2\theta\sec^2\theta\sum_{k,j=1}^{q}\sum_{i=1}^{n_1}\left(g(h(e_i,Te_j^*),Fe_k^*)\right)^2$$
$$+ \csc^2\theta\sec^2\theta\sum_{k,j=1}^{q}\sum_{i=1}^{n_1}\left(g(h(e_i,e_j^*),FTe_k^*)\right)^2$$
$$+ \csc^2\theta\sec^4\theta\sum_{k,j=1}^{q}\sum_{i=1}^{n_1}\left(g(h(e_i,Te_j^*),FTe_k^*)\right)^2.$$

Using Lemma 3 with (23)-(26), we obtain

$$\|h(\mathfrak{D},\mathfrak{D}^{\theta})\|^2 \geq 2q\csc^2\theta\sum_{i=1}^{n_1}(e_i(\ln f))^2 + \frac{2q}{9}\cot^2\theta\sum_{i=1}^{n_1}(e_i(\ln f))^2$$
$$= n_3\left(\csc^2\theta + \frac{1}{9}n_3\cot^2\theta\right)\|\vec{\nabla}^T(\ln f)\|^2. \tag{43}$$

Last term of (38) is identically zero by the hemi-slant mixed totally geodesic condition. Then, for all values of h from (39)-(43), finally we get the required inequality (37).

For the equality case, since M is $\mathfrak{D}^{\perp}\oplus\mathfrak{D}^{\theta}$-mixed totally geodesic, i.e.,

$$h(\mathfrak{D}^{\perp},\mathfrak{D}^{\theta}) = \{0\}. \tag{44}$$

Form the leaving and vanishing terms, we also find

$$h(\mathfrak{D},\mathfrak{D}) = \{0\},\ \ h(\mathfrak{D}^{\perp},\mathfrak{D}^{\perp}) = \{0\},\ \ h(\mathfrak{D},\mathfrak{D}^{\perp}) = \{0\},$$
$$h(\mathfrak{D}^{\theta},\mathfrak{D}^{\theta}) \subseteq J\mathfrak{D}^{\perp},\ h(\mathfrak{D},\mathfrak{D}^{\theta}) \subseteq F\mathfrak{D}^{\theta}. \tag{45}$$

Then, $M_T \times M_{\perp}$ is totally geodesic and M_{θ} is totally umbilical in $\tilde{M}$ due to the fact that $M_T \times M_{\perp}$ is totally geodesic and M_{θ} is totally umbilical in M [2,6] with equality holding case of (45). Furthermore, due to Theorem 2 and Lemma 2, we observe that M is not a $\mathfrak{D}\oplus\mathfrak{D}^{\theta}$-mixed totally submanifold of $\tilde{M}$. Hence, the proof is complete. □

Now, we give the following consequences of Theorem 3.

A warped submanifold of the form $M = M_{\theta}\times_f M_{\perp}$ in a nearly Kaehler manifold $\tilde{M}$ is called *hemi-slant* if $M_{\perp}$ is a totally real submanifold and M_{θ} is a proper slant submanifold.

If $\dim M_T = 0$ in Theorem 3, then we have

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 May 2023 doi:10.20944/preprints202305.1002.v1

Theorem 4. *Let $M = M_{\perp}^{n_1} \times_f M_{\theta}^{n_2}$ be a mixed totally geodesic hemi-slant warped product submanifold in a nearly Kaehler manifold $\tilde{M}$. Then*

(i) The second fundamental form h of M satisfies

$$||h||^2 \geq \frac{1}{9} n_2 \cos^2\theta \, ||\vec{\nabla}^{\perp}(\ln f)||^2, \tag{46}$$

where $\vec{\nabla}^{\perp}(\ln f)$ is the gradient of $\ln f$ along $M_{\perp}$.

(ii) if the equality sign of (46) holds identically, then $M_{\perp}$ and M_{θ} are totally geodesic and totally umbilical submanifolds of $\tilde{M}$, respectively.

On the other hand, if $M_{\perp} = \{0\}$, we have the following special case of Theorem 3.

Theorem 5. [1] *Let $M = M_T^{n_1} \times_f M_{\theta}^{n_2}$ be a semi-slant warped product submanifold in a nearly Kaehler manifold $\tilde{M}$. Then, we have*

(i) The second fundamental form h and the warping function f satisfy

$$||h||^2 \geq 2n_2 \left(1 + \frac{10}{9}\cot^2\theta\right) ||\vec{\nabla}^{T}(\ln f)||^2. \tag{47}$$

where $\vec{\nabla}^T \ln f$ is gradient of $\ln f$ along M_T.

(ii) If the equality sign in (47) holds identically, then M_T is totally geodesic and M_{θ} is totally umbilical in $\tilde{M}$. Moreover, M is a minimal submanifold in $\tilde{M}$.

Also, if $\dim M_{\perp} = 0$ and $\theta = \frac{\pi}{2}$ in Theorem 3, then $M = M_T^{n_1} \times_f M_{\perp}^{n_2}$ is a CR-warped product submanifold of a nearly Kaehler manifold $\tilde{M}$ and they were studied in [17] and hence the main Theorem 4.2 of [17] is a special case of Theorem 3.

Author Contributions: Writing—original draft, S.U. and B.-Y. C.; Writing—review & editing, R. M. B. All authors have read and agreed to the published version of the manuscript.

Funding: This research work was funded by Institutional Fund Projects under grant no. (IFPIP: 1156-130-1443). The authors gratefully acknowledge the technical and financial support provided by the Ministry of Education and King Abdulaziz University, DSR, Jeddah, Saudi Arabia.

Data Availability Statement: Not applicable

Conflicts of Interest: The authors declare no conflict of interest.

References

1. F. R. Al-Solamy, V. A. Khan and S. Uddin, *Geometry of Warped Product Semi-Slant Submanifolds of Nearly Kaehler Manifolds*, Results. Math. **71** (2017), no. 3-4, 783–799.
2. R. L. Bishop and B. O'Neill, *Manifolds of Negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1-49.
3. B.-Y. Chen, *CR-submanifolds of a Kaehler manifold*, I. J. Differential Geometry **16** (1981), no. 2, 305–322.
4. B.-Y. Chen, *Slant immersions*, Bull. Austral. Math. Soc. **41** (1990), 135-147.
5. B.-Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, Belgium, 1990.
6. B.-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds*, Monatsh. Math. **133** (2001), 177–195.
7. B.-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds II*, Monatsh. Math. **134** (2001), no. 2, 103–119.
8. B.-Y. Chen, *Pseudo-Riemannian geometry, δ-invariants and applications*, World Scientific, Hackensack, NJ, 2011.
9. B.-Y. Chen, S. Uddin, *Slant geometry of warped products in Kaehler and nearly Kaehler manifolds*, in: Complex Geometry of Slant Submanifolds, 61–100, Springer, Singapore (2022).
10. B.-Y. Chen, *Differential geometry of warped product manifolds and submanifolds*, World Scientific, Hackensack, NJ, 2017.

 doi:10.20944/preprints202305.1002.v1

11. B.-Y. Chen and S. Uddin, *Warped product pointwise bi-slant submanifolds of Kaehler manifolds*, Publ. Math. Debrecen **92** (2018), no. 1-2, 183–199.
12. B.-Y. Chen, *CR-warped product submanifolds in Kaehler manifolds*, in: Geometry of Cauchy-Riemann submanifolds, 1–25, Springer, Singapore, 2016.
13. A. Gray, *Nearly Kähler manifolds*, J. Differential Geom. **4** (1970), 283–309.
14. A. Gray, *The structure of nearly Kähler manifolds*, Math. Ann. **223** (1976), no. 3, 233–248.
15. P.-A. Nagy, *Nearly Kähler geometry and Riemannian foliations*, Asian J. Math. **6** (2002), no. 3, 481–504.
16. P.-A. Nagy, *On nearly Kähler geometry*, Ann. Global Anal. Geom. **22** (2002), no. 2, 167–178.
17. B. Sahin, CR-Warped product submanifolds of nearly Kaehler manifolds, Beitr. Algebra Geom. **49** (2009), no. 2, 383–397.
18. S. Uddin, F. R. Al-Solamy, K.A. Khan *Geometry of warped product pseudo-slant submanifolds in Kaehler manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **62** (2016), no. 2, vol. 3, 927–938.
19. S. Uddin, B.-Y. Chen and F. R. Al-Solamy, *Warped product bi-slant immersions in Kaehler manifolds*, Mediterr. J. Math. **14** (2017), no. 2, Art. 95, 11 pp.
20. S. Uddin, L.S. Alqahtani, A. A. Alkhaldi and F. Y. Mofarreh, *CR-slant warped product submanifolds in nearly Kaehler manifolds*, Int. J. Geom. Methods Mod. Phys. **17** (2020), no. 1, Art.2050003, 11 pp. DOI: 10.1142/S0219887820500036