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Posted Date: 5 December 2025

doi: 10.20944/preprints202512.0436.v1

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Pair of Associated η-Ricci–Bourguignon Almost Solitons with Vertical Torse-Forming Potential on Almost Contact Complex Riemannian Manifolds

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Abstract

Each of the studied manifolds has a pair of B-metrics, interrelated by an almost contact structure. The case where each of these metrics gives rise to an η-Ricci–Bourguignon almost soliton, where η is the contact form, is studied. In addition, the geometry-rich case where the soliton potential is torse-forming and is pointwise collinear on the Reeb vector field with respect to each of the two metrics is considered. Ricci tensors and scalar curvatures are expressed as functions of the parameters of the pair of almost solitons. Particular attention is paid to the special case when the manifold belongs to the only possible basic class of the corresponding classification. A necessary and sufficient condition has been found for these almost solitons to be η-Einstein for both metrics.

Keywords: Ricci–Bourguignon almost soliton; η-Ricci–Bourguignon almost soliton; almost contact B-metric manifold; almost contact complex Riemannian manifold; torse-forming vector field; vertical potential

MSC: 53C25; 53D15; 53C50; 53C44; 53D35; 70G45

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1. Introduction

The study of the Ricci–Bourguignon flow was proposed by Jean-Pierre Bourguignon in [1], developing some unpublished work of André Lichnerowicz in the sixties and a paper of Thierry Aubin [2]. Suppose that $g(t)$ is a time-dependent family of (pseudo-)Riemannian metrics on a smooth manifold $\mathcal{M}$. This family is said to evolve under the action of the Ricci–Bourguignon flow if $g(t)$, its Ricci tensor $\rho(t)$ and the scalar curvature $\tau(t)$ satisfy the following evolution equation for a real constant ℓ:

$$\frac{\partial}{\partial t}g = -2(\rho - \ell\tau g), \qquad g(0) = g_0. \tag{1}$$

Let us note that the Ricci–Bourguignon flow contains quite a few other geometric flows for special values of the constant ℓ in (1). The famous Ricci flow [3] occurs at $\ell = 0$, the Schouten flow [4] at $\ell = \frac{1}{2(m-1)}$, the traceless Ricci flow [5] at $\ell = \frac{1}{m}$ and the Einstein flow [6] at $\ell = \frac{1}{2}$, where m is the dimension of $\mathcal{M}$ [7,8]. For $m = 2$, the last three tensors are zero, therefore the flow is static, and in a higher dimension, the values of ℓ are strictly ordered as above, in ascending order.

Furthermore, this family can be considered as an interpolation between the Ricci flow and the Yamabe flow [9,10] obtained as a limit when $\ell \to -\infty$.

Ricci–Bourguignon flow also interpolates between the Ricci flow and the Yamabe flow when ℓ is non-positive. The short time existence and uniqueness for solution to the Ricci–Bourguignon flow (1) as a system of partial differential equations have been established in [11] for sufficiently small t and $\ell < \frac{1}{2(m-1)}$.

The Ricci–Bourguignon soliton (briefly RB soliton) is determined by the following equation [11,12]

$$\rho + \frac{1}{2}\mathcal{L}_{\vartheta}g + (\lambda + \ell\tau)g = 0, \tag{2}$$

where $\mathcal{L}_{\vartheta}g$ stands for the Lie derivative of g along the vector field ϑ called the soliton potential, and λ is the soliton constant. The solution is called a RB almost soliton if λ is a differential function on $\mathcal{M}$ [12].

A slightly more general notion of an RB (almost) soliton is obtained by perturbing (2) using a multiple of a (0,2)-tensor field $\eta \otimes \eta$ for a certain 1-form η on the manifold. Namely, this is an η-Ricci–Bourguignon (almost) soliton (e.g. [13]).

Torse-forming vector fields are defined by a certain recurrent condition for their covariant derivative with respect to the Levi-Civita connection of the basic metric [14]. Such a vector field is a geometric generalization of some important types of vector fields: recurrent, conformal, and parallel; as well as it causes specific directional deformation properties of the geometry of the manifold. This natural choice of soliton potential on manifolds with different structures has been studied by several authors (e.g. [15–19]).

Furthermore, the almost contact structure contains the Reeb vector field ξ, and therefore it is natural to choose the soliton potential vector field to be in the ξ-direction, determined the so-called vertical distribution. According to [20], the only basic class of almost contact B-metric manifolds that allows a torse-forming Reeb vector field is $\mathcal{F}_5$ in the Ganchev–Mihova–Gribachev classification [21]. This class is the counterpart of the class of β-Kenmotsu manifolds among the classes of almost contact metric manifolds.

2. Almost Contact Complex Riemannian Manifolds

In the present work we study an almost contact complex Riemannian manifold (abbreviated accR manifold), also known as an almost contact manifold with B-metric, which is introduced in [21]. This means that we have a $(2n+1)$-dimensional smooth manifold $\mathcal{M}$ equipped with an almost contact structure (φ, ξ, η) and a pseudo-Riemannian metric g with signature $(n+1, n)$, such that [21]

$$\begin{gathered}\varphi\xi = 0, \qquad \varphi^2 = -\iota + \eta \otimes \xi, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1, \\ g(\varphi x, \varphi y) = -g(x,y) + \eta(x)\eta(y),\end{gathered} \tag{3}$$

where ι is the identity in $\mathfrak{X}(\mathcal{M})$, the Lie algebra of tangent vector fields on $\mathcal{M}$ [21]. In the last equality and further on, by x, y, z, w we denote arbitrary elements of $\mathfrak{X}(\mathcal{M})$ or vectors in the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at an arbitrary point p of $\mathcal{M}$.

The remarkable thing about any accR manifold is that every B-metric has its twin, another B-metric. That is, on $(M, \varphi, \xi, \eta, g)$ there exists a B-metric $\widetilde{g}$ associated with the given B-metric g by means of the almost contact structure and is defined by

$$\widetilde{g}(x,y) = g(x, \varphi y) + \eta(x)\eta(y). \tag{4}$$

The studied manifolds are divided into eleven basic classes $\mathcal{F}_i$, $i \in \{1, 2, \dots, 11\}$ in the Ganchev–Mihova–Gribachev classification [21]. It is made with respect to the conditions for the $(0,3)$-tensor F, defined by

$$F(x,y,z) = g\big((\nabla_x\varphi)y, z\big)$$

and having the following basic properties:

$$\begin{gathered}F(x,y,z) = F(x,z,y) = F(x,\varphi y, \varphi z) + \eta(y)F(x,\xi,z) + \eta(z)F(x,y,\xi), \\ F(x,\varphi y, \xi) = (\nabla_x\eta)(y) = g(\nabla_x\xi, y).\end{gathered} \tag{5}$$

 doi:10.20944/preprints202512.0436.v1

The intersection of the basic classes is the special class $\mathcal{F}_0$, defined by the condition $F = 0$, and is known as the class of cosymplectic B-metric manifolds or cosymplectic accR manifolds.

The aforementioned classification uses the following 1-forms associated with F, also known as Lee forms of the considered manifold:

$$\theta = g^{ij}F(e_i, e_j, \cdot), \quad \theta^* = g^{ij}F(e_i, \varphi e_j, \cdot), \quad \omega = F(\xi, \xi, \cdot),$$

where $\left(g^{ij}\right)$ denotes the inverse of the matrix $\left(g_{ij}\right)$ of g with respect to a basis $\{e_i; \xi\}$ $(i = 1, 2, \dots, 2n)$ of $T_p\mathcal{M}$ at $p \in \mathcal{M}$.

The research in the present work mainly concerns one of the basic classes, namely $\mathcal{F}_5$, which is the counterpart of the class of well-known β-Kenmotsu manifolds among almost contact metric manifolds. The condition for defining $\mathcal{F}_5$-manifolds is the following

$$F(x, y, z) = -s\{g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y)\},$$

where for brevity of notation we denote the following function

$$s = \frac{\theta^*(\xi)}{2n}. \tag{6}$$

It is known that the Lee forms on every $\mathcal{F}_5$-manifold have the properties

$$\theta = 0, \quad \theta^* = 2n\, s\, \eta, \quad \omega = 0. \tag{7}$$

A subclass of the basic class $\mathcal{F}_5$ of accR manifolds with remarkable curvature properties is $\mathcal{F}_5^0$, defined by the additional condition that its only non-zero Lee form θ^* must be closed, i.e., $\mathrm{d}\theta^* = 0$ holds [22].

Using the second equality in (7), we find that the closedness of θ^* is equivalent to the following

$$\mathrm{d}s(x)\eta(y) = \mathrm{d}s(y)\eta(x),$$

which for an $\mathcal{F}_5$-manifold can be written in the form

$$\mathrm{d}s = \mathrm{d}s(\xi)\eta. \tag{8}$$

In [22], for each $\mathcal{F}_5^0$-manifold, a relation is given between the curvature tensors R and $\widetilde{R}$ generated by the Levi-Civita connections ∇ and $\widetilde{\nabla}$, corresponding to the B-metrics g and $\widetilde{g}$, respectively. This relation can be written, using (6), in the following form:

$$\begin{aligned}\widetilde{R}(x, y)z = R(x, y)z &- \{g(y, \varphi z) + g(\varphi y, \varphi z)\}\left\{s^2 x + \mathrm{d}s(\xi)\eta(x)\xi\right\}\\ &+ \{g(x, \varphi z) + g(\varphi x, \varphi z)\}\left\{s^2 y + \mathrm{d}s(\xi)\eta(y)\xi\right\}.\end{aligned} \tag{9}$$

Obviously, the following property holds:

$$\widetilde{R}(x, y)\xi = R(x, y)\xi.$$

By virtue of (4) and (9), we obtain

$$\begin{aligned}\widetilde{R}(x, y, z, w) = R(x, y, z, \varphi w) &+ \eta(R(x, y)z)\eta(w)\\ &- \{g(y, \varphi z) + g(\varphi y, \varphi z)\}\left\{s^2 g(x, \varphi w) + \left[s^2 + \mathrm{d}s(\xi)\right]\eta(x)\eta(w)\right\}\\ &+ \{g(x, \varphi z) + g(\varphi x, \varphi z)\}\left\{s^2 g(y, \varphi w) + \left[s^2 + \mathrm{d}s(\xi)\right]\eta(y)\eta(w)\right\}.\end{aligned} \tag{10}$$

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Taking the trace of (10) for $x = e_i$ and $w = e_j$ by the following consequence of (4)

$$\widetilde{g}^{ij} = -\varphi^j_m g^{im} + \xi^i \xi^j, \tag{11}$$

we get

$$\widetilde{\rho}(y,z) = \rho(y,z) - \left\{2n\, s^2 + \mathrm{d}s(\xi)\right\}\{g(y,\varphi z) + g(\varphi y, \varphi z)\}. \tag{12}$$

As a corollary we have the property $\xi \lrcorner \widetilde{\rho} = \xi \lrcorner \rho$. Moreover, $\widetilde{\rho} = \rho$ is true if and only if $\mathrm{d}s(\xi) = -2n\, s^2$ is fulfilled. The last differential equation has, for example, the solution $s = (2n\, t)^{-1}$, where $\mathrm{d}t$ is the coordinate 1-form on $\mathcal{H}^\perp = \mathrm{span}(\xi)$.

Applying (11) to (12) with $y = e_i$ and $z = e_j$, we express the scalar curvature for $\widetilde{g}$ of an arbitrary $\mathcal{F}^0_5$-manifold. The expression uses the associated quantity τ^* of τ with respect to φ, which is defined by $\tau^* = g^{ij}\rho(e_i, \varphi e_j)$. The resulting relation has the following form

$$\widetilde{\tau} = -\tau^* - 2n\left\{2\,\mathrm{d}s(\xi) + (2n+1)s^2\right\}. \tag{13}$$

Analogously, we can consider the associated quantity $\widetilde{\tau}^*$ of $\widetilde{\tau}$ with respect to φ, which is defined by $\widetilde{\tau}^* = \widetilde{g}^{ij}\widetilde{\rho}(e_i, \varphi e_j)$. Then, (11) and (12) imply the following relation on an arbitrary $\mathcal{F}^0_5$-manifold:

$$\widetilde{\tau}^* = \tau + 2n\left\{2\,\mathrm{d}s(\xi) + (2n+1)s^2\right\}. \tag{14}$$

3. Pair of Associated η-Ricci–Bourguignon Almost Solitons on an accR Manifold

We consider the so-called η-Ricci–Bourguignon almost soliton (in short, η-RB almost soliton) induced by the metric g and generalizing the RB soliton (2) in the following way

$$\rho + \frac{1}{2}\mathcal{L}_\vartheta g + (\lambda + \ell\tau)g + \mu\,\eta \otimes \eta = 0, \tag{15}$$

where μ is also a function on $\mathcal{M}$ [13]. We denote this almost soliton by $(g; \vartheta; \lambda, \mu, \ell)$.

Analogously to (15), we also have an η-RB almost soliton induced by the other B-metric $\widetilde{g}$ and defined for the corresponding Ricci tensor $\widetilde{\rho}$ and scalar curvature $\widetilde{\tau}$ as follows

$$\widetilde{\rho} + \frac{1}{2}\mathcal{L}_\vartheta \widetilde{g} + (\widetilde{\lambda} + \widetilde{\ell}\widetilde{\tau})\widetilde{g} + \widetilde{\mu}\,\eta \otimes \eta = 0, \tag{16}$$

where $\widetilde{\lambda}$ and $\widetilde{\mu}$ are differentiable functions on $\mathcal{M}$ and $\widetilde{\ell}$ is a real constant. We denote this almost soliton by $(\widetilde{g}; \vartheta; \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\ell})$.

The following concept was introduced in [23]. An accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g, \widetilde{g})$ is said to be equipped with a pair of associated η-RB almost solitons with potential vector field ϑ if the corresponding Ricci tensors ρ and $\widetilde{\rho}$, as well as their scalar curvatures τ and $\widetilde{\tau}$, satisfy (15) and (16), respectively.

The concept of an almost η-Einstein manifold equipped in the presence of some 1-form η, for example the contact form, is well known. The condition is that its Ricci tensor be expressed as a sum of the corresponding metric and $\eta \otimes \eta$, both multiplied by functions as coefficients, i.e.,

$$\rho = a\,g + b\,\eta \otimes \eta, \tag{17}$$

where (a, b) is some pair of functions on the manifold. This notion generalize almost Einstein manifolds for $b = 0$, as well as η-Einstein manifolds and Einstein manifolds for constants a, b and a constant a and $b = 0$, respectively.

In our case of an accR manifold, we also have the possibility of an almost η-Einstein manifold with respect to $\widetilde{\rho}$ for $\widetilde{g}$, i.e.,

$$\widetilde{\rho} = \widetilde{a}\,\widetilde{g} + \widetilde{b}\,\eta \otimes \eta \tag{18}$$

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for a pair of functions $(\widetilde{a}, \widetilde{b})$ on the manifold.

3.1. The Soliton Potential Is Double Torse-Forming

In the present study, we pay special attention to the case when the soliton potential ϑ is torse-forming with respect to the Levi-Civita connection ∇ of g. The definition of this type of vector field is as follows

$$\nabla_x \vartheta = f\, x + \gamma(x)\vartheta, \tag{19}$$

where f is a differentiable function on $\mathcal{M}$ (called the conformal scalar of ϑ) and γ is a 1-form on $\mathcal{M}$ (called the generating form of ϑ) [14,24].

Remark 1. *Some special types of torse-forming vector fields have been studied by various authors. Namely, a vector field ϑ determined by (19) is said to be of the following type if the corresponding specializing condition is satisfied: torqued, if $\gamma(\vartheta) = 0$ [25]; concircular, if $\gamma = 0$ [26]; concurrent, if $f - 1 = \gamma = 0$ [27]; recurrent, if $f = 0$ [28]; parallel, if $f = \gamma = 0$ (e.g. [29]).*

In (19), the Levi-Civita connection ∇ of the basic B-metric g is used. For a similar purpose, we can use the twin B-metric $\widetilde{g}$ and its Levi-Civita connection $\widetilde{\nabla}$ on the studied accR manifold. Furthermore, we require that the same vector field ϑ be torse-forming with respect to $\widetilde{\nabla}$, i.e., the following condition be satisfied:

$$\widetilde{\nabla}_x \vartheta = \widetilde{f}\, x + \widetilde{\gamma}(x)\vartheta, \tag{20}$$

where $\widetilde{f}$ and $\widetilde{\gamma}$ are also a differentiable function and a 1-form on $\mathcal{M}$, respectively. Then $\widetilde{f}$ and $\widetilde{\gamma}$ are called the conformal scalar and the generating form of ϑ with respect to $\widetilde{\nabla}$, respectively.

Further, when a vector field ϑ is torse-forming with respect to ∇ and $\widetilde{\nabla}$, we briefly say that ϑ is double torse-forming.

Due to (19) and (20), we obtain the following expressions for the Lie derivatives of g and $\widetilde{g}$ along ϑ, which is a double torse-forming vector field:

$$(\mathcal{L}_\vartheta g)(x, y) = g(\nabla_x \vartheta, y) + g(x, \nabla_y \vartheta) = 2f g(x, y) + \gamma(x) g(\vartheta, y) + \gamma(y) g(\vartheta, x), \tag{21}$$

$$(\mathcal{L}_\vartheta \widetilde{g})(x, y) = \widetilde{g}(\widetilde{\nabla}_x \vartheta, y) + \widetilde{g}(x, \widetilde{\nabla}_y \vartheta) = 2\widetilde{f}\widetilde{g}(x, y) + \widetilde{\gamma}(x)\widetilde{g}(\vartheta, y) + \widetilde{\gamma}(y)\widetilde{g}(\vartheta, x). \tag{22}$$

Substituting (21) and (22) into (15) and (16), respectively, we obtain the following expressions:

$$\rho(x, y) = -\{f + \lambda + \ell\tau\} g(x, y) - \mu\, \eta(x)\eta(y) - \frac{1}{2}\{\gamma(x) g(\vartheta, y) + \gamma(y) g(\vartheta, x)\}, \tag{23}$$

$$\widetilde{\rho}(x, y) = -\left\{\widetilde{f} + \widetilde{\lambda} + \widetilde{\ell}\widetilde{\tau}\right\}\widetilde{g}(x, y) - \widetilde{\mu}\, \eta(x)\eta(y) - \frac{1}{2}\{\widetilde{\gamma}(x)\widetilde{g}(\vartheta, y) + \widetilde{\gamma}(y)\widetilde{g}(\vartheta, x)\}. \tag{24}$$

Theorem 1. *Let an accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g, \widetilde{g})$ be equipped with a pair of associated η-RB almost solitons $(g; \vartheta; \lambda, \mu, \ell)$ and $(\widetilde{g}; \vartheta; \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\ell})$, where ϑ is double torse-forming. Then we have:*

(i) For $\ell \neq -\frac{1}{2n+1}$, *the Ricci tensor and the scalar curvature with respect to g of this manifold have the following form:*

$$\begin{aligned} \rho(x, y) = &-\frac{f + \lambda - \ell[\mu + \gamma(\vartheta)]}{1 + (2n+1)\ell} g(x, y) - \mu\, \eta(x)\eta(y) \\ &- \frac{1}{2}\{\gamma(x) g(\vartheta, y) + \gamma(y) g(\vartheta, x)\}, \end{aligned} \tag{25}$$

$$\tau = -\frac{(2n+1)(f + \lambda) + \mu + \gamma(\vartheta)}{1 + (2n+1)\ell}. \tag{26}$$

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(ii) *For $\ell = -\frac{1}{2n+1}$, the scalar curvature with respect to g cannot be expressed explicitly and we have the following relation between the Ricci tensor and the scalar curvature with respect to g:*

$$\begin{aligned}\rho(x,y) = &-\left\{f+\lambda-\frac{1}{2n+1}\tau\right\}g(x,y)-\mu\,\eta(x)\eta(y)\\ &-\frac{1}{2}\{\gamma(x)g(\vartheta,y)+\gamma(y)g(\vartheta,x)\}.\end{aligned} \tag{27}$$

In addition, the following property is valid

$$\gamma(\vartheta) = -(2n+1)(f+\lambda)-\mu, \tag{28}$$

as well as ϑ is geodesic with respect to ∇ if and only if $f = -\frac{1}{2n}\{(2n+1)\lambda+\mu\}$ holds.

(iii) *For $\widetilde{\ell} \neq -\frac{1}{2n+1}$, the Ricci tensor and the scalar curvature with respect to $\widetilde{g}$ of the considered manifold have the following form:*

$$\begin{aligned}\widetilde{\rho}(x,y) = &-\frac{\widetilde{f}+\widetilde{\lambda}-\widetilde{\ell}[\widetilde{\mu}+\widetilde{\gamma}(\vartheta)]}{1+(2n+1)\widetilde{\ell}}\widetilde{g}(x,y)-\widetilde{\mu}\,\eta(x)\eta(y)\\ &-\frac{1}{2}\{\widetilde{\gamma}(x)\widetilde{g}(\vartheta,y)+\widetilde{\gamma}(y)\widetilde{g}(\vartheta,x)\},\end{aligned} \tag{29}$$

$$\widetilde{\tau} = -\frac{(2n+1)(\widetilde{f}+\widetilde{\lambda})+\widetilde{\mu}+\widetilde{\gamma}(\vartheta)}{1+(2n+1)\widetilde{\ell}}. \tag{30}$$

(iv) *For $\widetilde{\ell} = -\frac{1}{2n+1}$, we have the following relation between $\widetilde{\rho}$ and $\widetilde{\tau}$, from which the scalar curvature cannot be expressed explicitly:*

$$\begin{aligned}\widetilde{\rho}(x,y) = &-\left\{\widetilde{f}+\widetilde{\lambda}-\frac{1}{2n+1}\widetilde{\tau}\right\}\widetilde{g}(x,y)-\widetilde{\mu}\,\eta(x)\eta(y)\\ &-\frac{1}{2}\{\widetilde{\gamma}(x)\widetilde{g}(\vartheta,y)+\widetilde{\gamma}(y)\widetilde{g}(\vartheta,x)\}.\end{aligned} \tag{31}$$

Furthermore, the following property is valid

$$\widetilde{\gamma}(\vartheta) = -(2n+1)(\widetilde{f}+\widetilde{\lambda})-\widetilde{\mu}, \tag{32}$$

as well as ϑ is geodesic with respect to $\widetilde{\nabla}$ if and only if $\widetilde{f} = -\frac{1}{2n}\{(2n+1)\widetilde{\lambda}+\widetilde{\mu}\}$ holds.

Proof. Contracting (23), we obtain the expression in (26) for the corresponding scalar curvature for g in the case $\ell \neq -\frac{1}{2n+1}$. Substituting (26) into (23) gives us the result in (25), thus proving case (i).

In the case of $\ell = -\frac{1}{2n+1}$, the equality in (23) implies (27) and then τ cannot be expressed explicitly, but instead we get (28). As a consequence of the latter and (19), we obtain the property $\nabla_\vartheta\vartheta = -\{2n\,f+(2n+1)\lambda+\mu\}\vartheta$, from which the last part of the statement in case (ii) follows.

In a similar way, the contraction of (24) using $\widetilde{g}$ gives the following equality

$$\left\{1+(2n+1)\widetilde{\ell}\right\}\widetilde{\tau} = -(2n+1)(\widetilde{f}+\widetilde{\lambda})-\widetilde{\mu}-\widetilde{\gamma}(\vartheta).$$

First, under the condition of case (iii), we obtain the expression in (30) for the scalar curvature with respect to $\widetilde{g}$ and substituting it into (24), we obtain (29).

Otherwise, i.e., if $\widetilde{\ell}$ has the value $-\frac{1}{2n+1}$, then (32) is true and (24) takes the form in (31). The proof of case (iv) completes by establishing the truth of the fact at the end of this case, using (20) and (32). □

3.2. The Double Torse-Forming Potential Is Vertical

In addition to the condition that the soliton potential ϑ be torse-forming with respect to the pair of Levi-Civita connections, we require that it be vertical, i.e., the following holds

$$\vartheta = k\,\xi, \tag{33}$$

where k is a nowhere-vanishing function on $\mathcal{M}$ and obviously $k = \eta(\vartheta)$ is true. This means that we exclude from consideration the trivial case in which ϑ is a zero soliton potential or equivalently, ϑ is a torqued vector field according to Remark 1.

Using the expression of ρ in (23), we calculate τ^* and obtain $\tau^* = -\gamma(\varphi\vartheta)$, which due to (33) implies the following

$$\tau^* = 0. \tag{34}$$

Similarly, by (24) for $\widetilde{\rho}$ we calculate $\widetilde{\tau}^*$ and get $\widetilde{\tau}^* = -\widetilde{\gamma}(\varphi\vartheta)$, which for vertical ϑ means that

$$\widetilde{\tau}^* = 0. \tag{35}$$

Taking into account (19), (20) and (33), we get

$$\mathrm{d}k(x) = f\,\eta(x) + k\,\gamma(x), \tag{36}$$

$$\nabla_x\xi = -\frac{f}{k}\,\varphi^2 x. \tag{37}$$

Note that from the expression in (37) it follows that ξ is geodesic with respect to ∇ and η is closed.

Due to (5) and (37), we obtain

$$F(x,\varphi y,\xi) = (\nabla_x\eta)(y) = -s\,g(\varphi x,\varphi y),$$
$$s = \frac{f}{k} = \frac{\theta^*(\xi)}{2n}. \tag{38}$$

Similarly, we have

$$\mathrm{d}k(x) = \widetilde{f}\,\eta(x) + k\,\widetilde{\gamma}(x), \tag{39}$$

$$\widetilde{\nabla}_x\xi = -\widetilde{s}\,\varphi^2 x, \qquad \widetilde{s} = \frac{\widetilde{f}}{k}. \tag{40}$$

It is clear, (40) shows that ξ is also geodesic with respect to $\widetilde{\nabla}$ in the considered case.

Let us recall a fact about the construction under consideration.

Lemma 1 ([30]). *Let us consider an accR manifold $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$ and a vertical vector field ϑ on it. If ϑ is torse-forming with respect to both ∇ and $\widetilde{\nabla}$ with conformal scalars f, $\widetilde{f}$ and generating forms γ, $\widetilde{\gamma}$, respectively, then we have the following*

$$\widetilde{f} = f, \qquad \widetilde{\gamma} = \gamma, \qquad \widetilde{\nabla}\vartheta = \nabla\vartheta. \tag{41}$$

Combining (19), (33), and (36), we get

$$\nabla_x\vartheta = -f\,\varphi^2 x + \mathrm{d}k(x)\xi$$

and due to (21) we have

$$(\mathcal{L}_\vartheta g)(x,y) = -2fg(\varphi x,\varphi y) + h(x,y), \tag{42}$$

where h denotes the symmetric (0,2)-tensor determined by

$$h(x,y) = \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x). \tag{43}$$

Similarly, (20), (33), (39), (41), and (22) imply

$$(\mathcal{L}_{\vartheta}\widetilde{g})(x,y) = 2fg(x,\varphi y) + h(x,y). \tag{44}$$

Using (37), (38) and (40), we get the following for the curvature tensors R and $\widetilde{R}$ of g and $\widetilde{g}$, respectively:

$$R(x,y)\xi = -\{\mathrm{d}s(x) + s^2\eta(x)\}\varphi^2 y + \{\mathrm{d}s(y) + s^2\eta(y)\}\varphi^2 x, \tag{45}$$

$$\widetilde{R}(x,y)\xi = -\{\mathrm{d}\widetilde{s}(x) + \widetilde{s}^2\eta(x)\}\varphi^2 y + \{\mathrm{d}\widetilde{s}(y) + \widetilde{s}^2\eta(y)\}\varphi^2 x. \tag{46}$$

Taking the trace of the maps $x \to R(x,\xi)\xi$ and $x \to \widetilde{R}(x,\xi)\xi$, based on (45) and (46), we obtain the following

$$\rho(\xi,\xi) = -2n\{\mathrm{d}s(\xi) + s^2\}, \tag{47}$$

$$\widetilde{\rho}(\xi,\xi) = -2n\{\mathrm{d}\widetilde{s}(\xi) + \widetilde{s}^2\}. \tag{48}$$

The expression in (47), due to (38), is equivalent to the following form

$$\rho(\xi,\xi) = -\frac{2n}{k^2}\Big\{k\mathrm{d}f(\xi) - f\mathrm{d}k(\xi) + f^2\Big\}. \tag{49}$$

Similarly, the expression in (48), using the second equality of (40), can be written in the following form

$$\widetilde{\rho}(\xi,\xi) = -\frac{2n}{k^2}\Big\{k\mathrm{d}\widetilde{f}(\xi) - \widetilde{f}\mathrm{d}k(\xi) + \widetilde{f}^2\Big\}.$$

From Theorem 1, assuming that ϑ is vertical, we have

Theorem 2. *Let an accR manifold $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$ be equipped with a pair of associated η-RB almost solitons $(g;\vartheta;\lambda,\mu,\ell)$ and $(\widetilde{g};\vartheta;\widetilde{\lambda},\widetilde{\mu},\widetilde{\ell})$, where ϑ is double torse-forming and vertical. Then we have:*

(i) *For $\ell \neq -\frac{1}{2n+1}$, the following expressions of ρ and τ are true:*

$$\rho = -\frac{(1+\ell)f + \lambda - \ell[\mu + \mathrm{d}k(\xi)]}{1+(2n+1)\ell}g + (f-\mu)\eta\otimes\eta - \frac{1}{2}h, \tag{50}$$

$$\tau = -\frac{2n f + (2n+1)\lambda + \mu + \mathrm{d}k(\xi)}{1+(2n+1)\ell}. \tag{51}$$

Furthermore, the following property for the functions used is valid

$$\begin{aligned}&\Big\{(1+2n\,\ell)k^2 + 2n[1+(2n+1)\ell]f\Big\}\mathrm{d}k(\xi)\\ &-2n[1+(2n+1)\ell]\Big\{k\,\mathrm{d}f(\xi) + f^2\Big\} - \{2n\,\ell\, f - \lambda - (1+2n\,\ell)\mu\}k^2 = 0.\end{aligned} \tag{52}$$

(ii) *For $\ell = -\frac{1}{2n+1}$, the following expressions of ρ and τ are true:*

$$\begin{aligned}\rho = &-\frac{1}{k^2}\Big\{2n[k\,\mathrm{d}f(\xi) + f\mu] + (2n+1)(\lambda+f)\big(k^2 + 2n\,f\big)\Big\}g\\ &+ (f-\mu)\eta\otimes\eta - \frac{1}{2}h,\end{aligned}$$

$$\tau = -\frac{2n(2n+1)}{k^2}\Big\{k\,\mathrm{d}f(\xi) + f\mu + \Big[(2n+1)f + k^2\Big](\lambda+f)\Big\}.$$

Furthermore, the following property for the functions used is valid

$$\mathrm{d}k(\xi) + 2n\,f + (2n+1)\lambda + \mu = 0 \tag{53}$$

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and k is a vertical constant if and only if $f = -\frac{1}{2n}\{(2n+1)\lambda + \mu\}$ *holds.*

(iii) *For* $\widetilde{\ell} \neq -\frac{1}{2n+1}$*, the following expressions of* $\widetilde{\rho}$ *and* $\widetilde{\tau}$ *are true:*

$$\widetilde{\rho} = -\frac{\left(1+\widetilde{\ell}\right)\widetilde{f} + \widetilde{\lambda} - \widetilde{\ell}\left[\widetilde{\mu} + \mathrm{d}k(\xi)\right]}{1+(2n+1)\widetilde{\ell}}\widetilde{g} + \left(\widetilde{f} - \widetilde{\mu}\right)\eta\otimes\eta - \frac{1}{2}h, \tag{54}$$

$$\widetilde{\tau} = -\frac{2n\,\widetilde{f} + (2n+1)\widetilde{\lambda} + \widetilde{\mu} + \mathrm{d}k(\xi)}{1+(2n+1)\widetilde{\ell}}. \tag{55}$$

Furthermore, the following property for the functions used is valid

$$\begin{aligned}&\left\{(1+2n\,\widetilde{\ell})k^2 + 2n\left[1+(2n+1)\widetilde{\ell}\right]\widetilde{f}\right\}\mathrm{d}k(\xi)\\&- 2n\left[1+(2n+1)\widetilde{\ell}\right]\left\{k\,\mathrm{d}\widetilde{f}(\xi) + \widetilde{f}^2\right\} - \left\{2n\,\widetilde{\ell}\,\widetilde{f} - \widetilde{\lambda} - (1+2n\,\widetilde{\ell})\widetilde{\mu}\right\}k^2 = 0.\end{aligned}$$

(iv) *For* $\widetilde{\ell} = -\frac{1}{2n+1}$*, the following expressions of* $\widetilde{\rho}$ *and* $\widetilde{\tau}$ *are true:*

$$\begin{aligned}\widetilde{\rho} = &-\frac{1}{k^2}\left\{2n\left[k\,\mathrm{d}\widetilde{f}(\xi) + \widetilde{f}\widetilde{\mu}\right] + (2n+1)\left(\widetilde{\lambda}+\widetilde{f}\right)\left(k^2 + 2n\,\widetilde{f}\right)\right\}\widetilde{g}\\&+\left(\widetilde{f} - \widetilde{\mu}\right)\eta\otimes\eta - \frac{1}{2}h,\\ \widetilde{\tau} = &-\frac{2n(2n+1)}{k^2}\left\{k\,\mathrm{d}\widetilde{f}(\xi) + \widetilde{f}\widetilde{\mu} + \left[(2n+1)\widetilde{f} + k^2\right]\left(\widetilde{\lambda}+\widetilde{f}\right)\right\}.\end{aligned}$$

Furthermore, the following property for the functions used is valid

$$\mathrm{d}k(\xi) + 2n\,\widetilde{f} + (2n+1)\widetilde{\lambda} + \widetilde{\mu} = 0, \tag{56}$$

and k is a horizontal constant if and only if $\widetilde{f} = -\frac{1}{2n}\{(2n+1)\widetilde{\lambda} + \widetilde{\mu}\}$ *holds.*

Proof. Considering (33) and (36), we obtain the following expression

$$\gamma(\vartheta) = \mathrm{d}k(\xi) - f. \tag{57}$$

Similarly, (33) and (39) imply

$$\widetilde{\gamma}(\vartheta) = \mathrm{d}k(\xi) - \widetilde{f}. \tag{58}$$

We use (33), (36) and (57) first in (25) and (26) to obtain (50) and (51) of case (i); and second in (27) and (28) to deduce the following relation between ρ and τ for case (ii):

$$\rho = -\left\{f + \lambda - \frac{1}{2n+1}\tau\right\}g + (f-\mu)\eta\otimes\eta - \frac{1}{2}h, \tag{59}$$

which does not lead to an explicit expression of τ, but the contraction of (59) implies (53) and through it the assertion in the last line of (ii).

The formula (50) in case (i) implies for $x = y = \xi$ the following equality

$$\rho(\xi,\xi) = \frac{2n\,\ell\,f - \lambda - (1+2n\,\ell)[\mu + \mathrm{d}k(\xi)]}{1+(2n+1)\ell},$$

which together with (49) gives the condition in (52).

Taking into account (53) for case (ii), we obtain that (59) implies

$$\rho(\xi,\xi) = \frac{1}{2n+1}\tau + 2n(f+\lambda). \tag{60}$$

We compare (60) with (47) and obtain an expression of τ in this case as follows:

$$\tau = -2n(2n+1)\Big\{\mathrm{d}s(\xi) + s^2 + f + \lambda\Big\}. \tag{61}$$

Based on (38) and (53), the formula in (61) can be written in terms of the functions λ, μ, f and k, as given in ((ii)).

Similarly for cases (iii) and (iv), we substitute (33), (39) and (58) first into (29) and (30) to arrive at (54) and (55); and second into (31) and (32) to get (56) and the following relation between $\widetilde{\rho}$ and $\widetilde{\tau}$:

$$\widetilde{\rho} = -\Big\{\widetilde{f} + \widetilde{\lambda} - \frac{1}{2n+1}\widetilde{\tau}\Big\}\widetilde{g} + \Big(\widetilde{f} - \widetilde{\mu}\Big)\eta\otimes\eta - \frac{1}{2}h,$$

which does not directly imply an explicit expression of $\widetilde{\tau}$.

The rest of the proof for cases (iii) and (iv) is completed with similar reasoning as in the first two cases. □

3.3. The Double Torse-Forming Vertical Potential Is on an $\mathcal{F}_5^0$-Manifold

In this subsection, we recall two more well-known facts regarding the position of the considered manifolds in the most popular classification of accR manifolds from [21] and continue the reasoning in this direction in order to study the curvature properties of these manifolds.

Proposition 1 ([30])**.** *Let $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$ be an accR manifold with a vertical vector field ϑ that is torse-forming with respect to both ∇ and $\widetilde{\nabla}$. Then the manifold belongs to $\mathcal{F}_5$ or to a direct sum of $\mathcal{F}_5$ with $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_6$ and $\mathcal{F}_{10}$. Furthermore, ϑ is recurrent if and only if the component of F relating $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$ to $\mathcal{F}_5$ vanishes, i.e., $\mathcal{F}_5$ is restricted to $\mathcal{F}_0$.*

Proposition 2 ([30])**.** *Let an accR manifold $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$ belong to $\mathcal{F}_5$, a vector field ϑ on $\mathcal{M}$ be vertical, and ϑ be torse-forming with respect to both ∇ and $\widetilde{\nabla}$. Then $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$ belongs to $\mathcal{F}_5^0$.*

Proposition 3. *Let an $\mathcal{F}_5^0$-manifold $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$ be equipped with a pair of associated η-RB almost solitons $(g;\vartheta;\lambda,\mu,\ell)$ and $(\widetilde{g};\vartheta;\widetilde{\lambda},\widetilde{\mu},\widetilde{\ell})$, where ϑ is vertical and double torse-forming. Then the scalar curvatures for g and $\widetilde{g}$ are equal and have the following expression:*

$$\tau = \widetilde{\tau} = -2n\Big\{2\,\mathrm{d}s(\xi) + (2n+1)s^2\Big\}. \tag{62}$$

Proof. Since the soliton potential ϑ is vertical (i.e., (33) is valid), then we have $\varphi\vartheta = 0$, and the vanishing of τ^* and $\widetilde{\tau}^*$ holds because of (34) and (35). Thereafter, due to (13) and (14) for an $\mathcal{F}_5^0$-manifold, we obtain that τ and $\widetilde{\tau}$ are equal and (62) is valid. □

We continue by considering of the following symmetric $(0,2)$-tensors

$$\alpha = \rho + \Big\{\mathrm{d}s(\xi) + 2n\,s^2\Big\}g, \qquad \widetilde{\alpha} = \widetilde{\rho} + \Big\{\mathrm{d}\widetilde{s}(\xi) + 2n\,\widetilde{s}^2\Big\}\widetilde{g}. \tag{63}$$

Then we have the following

Proposition 4. *Let an $\mathcal{F}_5^0$-manifold $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$ be equipped with a pair of associated η-RB almost solitons with the same potential ϑ, which is vertical and double torse-forming. Then α is invariant under the change of ∇ by $\widetilde{\nabla}$ and vice versa, i.e., $\alpha = \widetilde{\alpha}$ holds.*

Proof. Considering (38), the second equality of (40) and the first equality of (41), we conclude that $\widetilde{s} = s$. Then, (47) and (48) imply the following equality

$$\widetilde{\rho}(\xi,\xi) = \rho(\xi,\xi). \tag{64}$$

Moreover, taking into account (3) and (4), we obtain from (12) that

$$\widetilde{\rho}+\left\{\mathrm{d}\widetilde{s}(\xi)+2n\,\widetilde{s}^2\right\}\widetilde{g}=\rho+\left\{\mathrm{d}s(\xi)+2n\,s^2\right\}g.$$

The last equality means that on every $\mathcal{F}_5^0$-manifold with the conditions in the statement, the symmetric (0,2)-tensor α, defined by (63), is invariant under the replacement of g by $\widetilde{g}$ and vice versa. □

Corollary 1. *Let the requirements of Proposition 4 be fulfilled for an $\mathcal{F}_5^0$-manifold $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$. Then the invariant tensor α has the following expression:*

$$\alpha=\rho-\frac{1}{2n}\{\tau-\rho(\xi,\xi)\}g. \tag{65}$$

Moreover, α vanishes if and only if the manifold is Einstein with negative scalar curvatures with respect to each B-metric of the pair $(g,\widetilde{g})$. Then the Lee form θ^* is a constant multiple of the contact form η.

Proof. Using (47) and (62), we obtain the equality in (65). Taking into account it and $\alpha=0$, we get

$$\tau=(2n+1)\rho(\xi,\xi) \tag{66}$$

and therefore $\rho=\frac{\tau}{2n+1}g$ is valid, i.e., the manifold is Einstein. Then, applying (47) and (62) to (66), we obtain that $\mathrm{d}s(\xi)=0$ is valid, which is equivalent to concluding that $\theta^*(\xi)$ is constant in the vertical direction. Then from (8) we have that $\theta^*(\xi)$ (as well as s) is a constant in general and due to (7) we obtain the last part of the statement. The proof is complete once we notice that (66) implies $\tau=\widetilde{\tau}=-2n(2n+1)s^2<0$, where s is non-zero for every $\mathcal{F}_5$-manifold that is not in $\mathcal{F}_0$. □

Let us now consider $(g;\vartheta;\lambda,\mu,\ell)$ and $(\widetilde{g};\vartheta;\widetilde{\lambda},\widetilde{\mu},\widetilde{\ell})$ without restrictions on the value of the constants ℓ and $\widetilde{\ell}$.

Proposition 5. *Let an $\mathcal{F}_5^0$-manifold $(\mathcal{M},\varphi,\xi,\eta,g,\widetilde{g})$ be equipped with a pair of associated η-RB almost solitons $(g;\vartheta;\lambda,\mu,\ell)$ and $(\widetilde{g};\vartheta;\widetilde{\lambda},\widetilde{\mu},\widetilde{\ell})$, where ϑ is double torse-forming and vertical, determined by the functions f, k and the 1-form γ. Then all the parameters that determine the pair of almost solitons and their Ricci tensors are expressed in terms of f and k as follows*

$$\lambda=-f+\frac{1}{k^2}\left\{(1+4n\,\ell)[k\,\mathrm{d}f(\xi)-f\,\mathrm{d}k(\xi)]+2n[1+(2n+1)\ell]f^2\right\}. \tag{67}$$

$$\widetilde{\lambda}=-f+\frac{1}{k^2}\left\{(1+4n\,\widetilde{\ell})[k\,\mathrm{d}f(\xi)-f\,\mathrm{d}k(\xi)]+2n\left[1+(2n+1)\widetilde{\ell}\right]f^2\right\}. \tag{68}$$

$$\mu=\widetilde{\mu}=f+(2n-1)\frac{1}{k}\mathrm{d}f(\xi)-\left[1+(2n-1)\frac{f}{k^2}\right]\mathrm{d}k(\xi). \tag{69}$$

$$\rho=-\left\{\mathrm{d}s(\xi)+2n\,s^2\right\}g+\left\{\mathrm{d}k(\xi)-(2n-1)\mathrm{d}s(\xi)\right\}\eta\otimes\eta-\frac{1}{2}h. \tag{70}$$

$$\widetilde{\rho}=-\left\{\mathrm{d}s(\xi)+2n\,s^2\right\}\widetilde{g}+\left\{\mathrm{d}k(\xi)-(2n-1)\mathrm{d}s(\xi)\right\}\eta\otimes\eta-\frac{1}{2}h. \tag{71}$$

Moreover, the considered manifold is η-Einstein with respect to both B-metrics g and $\widetilde{g}$ if and only if k is a horizontal constant, i.e., $\mathrm{d}k\circ\varphi=0$ is valid.

Proof. According to (3) and (42), the expression (15) takes the following form

$$\rho=-(f+\lambda+\ell\tau)g+(f-\mu)\eta\otimes\eta-\frac{1}{2}h. \tag{72}$$

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Therefore, from (72) and (43) we have

$$\rho(\xi,\xi) = -\{\lambda + \ell\tau + \mu + \mathrm{d}k(\xi)\}. \tag{73}$$

According to (62) and (47) for the considered $\mathcal{F}_5^0$-manifold, (73) implies the following differential equation for the functions used in the construction of $(g;\vartheta;\lambda,\mu,\ell)$ without restrictions on the value of the constant ℓ:

$$\lambda + \mu + \mathrm{d}k(\xi) = 2n\Big\{(1+2\ell)\mathrm{d}s(\xi) + [1+(2n+1)\ell]s^2\Big\}. \tag{74}$$

By virtue of (38), the relation in (74) takes the following form:

$$\begin{aligned}\lambda + \mu = -\frac{1}{k^2}\Big\{&\Big[k^2 + 2n(1+2\ell)f\Big]\mathrm{d}k(\xi)\\ &-2n\Big[(1+2\ell)k\,\mathrm{d}f(\xi) + [1+(2n+1)\ell]f^2\Big]\Big\}.\end{aligned}$$

Contracting (72) and using (62) and (74), we get the following expression

$$\tau = -2n\Big\{f + \lambda + (1-4n\ell)\mathrm{d}s(\xi) + [1-2n(2n+1)\ell]s^2\Big\}. \tag{75}$$

Then, from (75) and (62), we obtain the following expression of the sum of f and λ as a function of s, which is the quotient of f and k:

$$f + \lambda = (1+4n\ell)\mathrm{d}s(\xi) + 2n[1+(2n+1)\ell]s^2. \tag{76}$$

Taking into account (38) in (76), we can express for λ in terms of f and k as in (67).

Excluding λ from (74) and (76), we express the function μ as follows

$$\mu = f - \mathrm{d}k(\xi) + (2n-1)\mathrm{d}s(\xi). \tag{77}$$

By virtue of (38), we can rewrite (77) as an expression for μ in terms of f and k, given in (69).

Substituting (76), (62) and (77) into (72), we obtain (70).

Taking into account (4), (44) and the first equality of (41), the expression (16) takes the following form

$$\widetilde{\rho} = -\Big(f + \widetilde{\lambda} + \widetilde{\ell}\widetilde{\tau}\Big)\widetilde{g} + (f - \widetilde{\mu})\eta\otimes\eta - \frac{1}{2}h. \tag{78}$$

Applying the last equality to the arguments (ξ,ξ), we obtain from (78) and (43) that

$$\widetilde{\rho}(\xi,\xi) = -\Big\{\widetilde{\lambda} + \widetilde{\ell}\widetilde{\tau} + \widetilde{\mu} + \mathrm{d}k(\xi)\Big\}. \tag{79}$$

According to (62) and (64) for the considered $\mathcal{F}_5^0$-manifold, (79) implies the following differential equation for the functions used in constructing $(\widetilde{g};\vartheta;\widetilde{\lambda},\widetilde{\mu},\widetilde{\ell})$:

$$\widetilde{\lambda} + \widetilde{\mu} + \mathrm{d}k(\xi) = 2n\Big\{\Big(1+2\widetilde{\ell}\Big)\mathrm{d}s(\xi) + \Big[1+(2n+1)\widetilde{\ell}\Big]s^2\Big\}. \tag{80}$$

We substitute (38) into (80) and the last equality takes the following form

$$\begin{aligned}\widetilde{\lambda} + \widetilde{\mu} = -\frac{1}{k^2}\Big\{&\Big[k^2 + 2n\Big(1+2\widetilde{\ell}\Big)f\Big]\mathrm{d}k(\xi)\\ &-2n\Big[\Big(1+2\widetilde{\ell}\Big)k\,\mathrm{d}f(\xi) + \Big[1+(2n+1)\widetilde{\ell}\Big]f^2\Big]\Big\}.\end{aligned}$$

Taking the trace of (78) by $\widetilde{g}$ and then using (62) and (80), we get the following expression for $\widetilde{\tau}$

$$\widetilde{\tau} = -2n\Big\{f + \widetilde{\lambda} + \Big(1-4n\widetilde{\ell}\Big)\mathrm{d}s(\xi) + \Big[1-2n(2n+1)\widetilde{\ell}\Big]s^2\Big\}. \tag{81}$$

 doi:10.20944/preprints202512.0436.v1

Then, comparing (81) and (62), we obtain the following condition for the sum of f and $\widetilde{\lambda}$ in terms of s:

$$f + \widetilde{\lambda} = \left(1 + 4n\widetilde{\ell}\right)\mathrm{d}s(\xi) + 2n\left[1 + (2n+1)\widetilde{\ell}\right]s^2 \tag{82}$$

or replacing s by (38), we rewrite (82) in terms of f and k as in (68).

Excluding $\widetilde{\lambda}$ from (80) and (82), we get that $\widetilde{\mu} = \mu$ is true, since (77) holds, i.e., we have the expression for $\widetilde{\mu}$ in (69) .

Subtracting (76) and (82), we get $\widetilde{\lambda} = \lambda + 2n\left(\widetilde{\ell} - \ell\right)\left\{2\mathrm{d}s(\xi) + (2n+1)s^2\right\}$. Therefore, the equality $\widetilde{\lambda} = \lambda$ is true if and only if either $\widetilde{\ell} = \ell$ or $\mathrm{d}s(\xi) = -\frac{1}{2}(2n+1)s^2$ is valid. One solution to the last ODE is, for example, $s = \frac{2}{2n+1}t^{-1}$, where t is the coordinate along ξ.

Using the analogue of (77) for $\widetilde{\mu}$, (82) and (62) in (78), we obtain (71).

Due to (17), (18), (43), (72), and (78), the last statement in the proposition is true precisely when the condition $\mathrm{d}k = \mathrm{d}k(\xi)\eta$ holds, which is equivalent to the horizontal constancy of k. $\square$

Funding: & & & DA OSTANE ZA 2026 & & & The research is partially supported by project FP25-FMI-010 of the Scientific Research Fund, University of Plovdiv Paisii Hilendarski.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results. The author declares no conflict of interest.

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