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Article

The 4-Set Tree Connectivity of Hierarchical Folded Hypercube

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Abstract: The k-set tree connectivity, as a natural extension of classical connectivity, is an extremely significant index to measure the fault-tolerance of interconnection networks. Let $G = (V, E)$ be a connected graph and a subset $S \subseteq V$, an S-tree $T = (V', E')$ of graph G is a tree that contains the subset S. Any two S-trees T and T' are internally disjoint if and only if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. The cardinality of maximum internally disjoint S-trees is defined as $\kappa_G(S)$, and the k-set tree connectivity is denoted by $\kappa_k(G) = \min\{\kappa_G(S) | S \subseteq V(G) \text{ and } |S| = k\}$. In this note, we determine the k-set tree connectivity of hierarchical folded hypercube when $k = 4$, that is, $\kappa_4(HFQ_n) = n + 1$, where HFQ_n is hierarchical folded hypercube for $n \geq 7$.

Keywords: tree connectivity; hierarchical folded hypercube; internally disjoint trees; fault-tolerance

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1. Introduction

All graphs in this paper are finite simple and undirected. For more terminology and notation that are used but not described here, refer to reference [1]. A network is commonly molded by a graph $G = (V, E)$, where vertices correspond to processors and edges correspond communication links. For any vertex $x \in V$, let $N_G(x) = \{y \in V(G) \backslash x | xy \in E\}$ be the neighbour set of x in the graph G, and $N_G[x] = N_G(x) \cup \{x\}$. Let $d_G(x) = |N_G(x)|$ be the degree of x in G. Besides, we use $\delta(G)$ and $\Delta(G)$ to denote the minimum degree and maximum degree of G, respectively. If $\delta(G) = \Delta(G)$, then G is called a regular graph. For any $a, b \in V$, an (a, b)-path is a simple graph whose vertices begin with a and end with b. Any two (a, b)-paths A and B are internally disjoint if and only if $V(A) \cap V(B) = \{a, b\}$. For any vertex set $S \subseteq V$, let $G_{[S]}$ be the induced subgraph of G that contains all the vertices of S. For convenience sake, let $[n + 1] = \{1, 2, \ldots, n + 1\}$ and use "$w.l.o.g.$" to express"Without loss of generality". Let $x = x_1 x_2 \ldots x_n$ be n-bit binary string and x_i be the i-th bit of x. Denotes $H(x, y)$ by the Hamming distance of two distinct n-bit binary strings of x and y is that the number of positions where they differ.

With the rise and rapid development of high-performance parallel computer science, more and more attention is being paid to interconnection networks with excellent performance. An eminent topological structure can significantly improve the reliability of a network. Therefore, it is essential to consider the fault tolerance of the network while designing the topology of the network. This means that the interconnection network should be able to operate effectively even when certain nodes and edges fail, ensuring that it retains specific network properties. In fact, the topological structure of an interconnection network commonly modeled by an undirected simple graph $G = (V, E)$, where every vertex corresponds to a processor, and each edge corresponds to a communication link. Then many computer scientists and engineers use some parameters of graph theory to design and analyzing topological structures of interconnection networks, for instance the connectivity.

The *connectivity* $\kappa(G)$ of a connected graph G is the minimum cardinality of vertices whose removed to obtain a disconnected graph or only a vertex. A famous theorem that we know of Whitney[2] gives another equivalent definition of connectivity. For any vertex set $S = \{x, y\}$, let

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$\kappa_G(S)$ be the maximum cardinality of internally disjoint paths joining x and y in G. Then $\kappa(G) = \min\{\kappa_G(S)|S \subseteq V$ and $|S| = 2\}$. However, the connectivity has an evident drawback that in the practical application of interconnection networks, the probability of all neighbours adjacent to a vertex failing at the same time is extremely small. Therefore, it is not precise enough to measure the fault tolerance of a network by the parameter of connectivity.

In order to better evaluate the fault tolerance level of a network, Chartrand et al.[3] extended the connectivity and came up with the concept of the *k-set tree connectivity*. For any vertex set $S \subseteq V$, $G_{[S]}$ as the induced subgraph of G is actually a tree. An S-tree $T = (V', E')$ of a connected graph G is a tree that contains the subset S. Two S-trees T and T' are internally disjoint if and only if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. Let $\kappa_G(S)$ be the maximum number of internally disjoint S-trees in graph G. The *k-set tree connectivity* is denoted by $\kappa_k(G) = \min\{\kappa_G(S)|S \subseteq V$ and $|S| = k\}$.

The relation between the tree connectivity and the connectivity and the bounds of the tree connectivity have been extensively investigated [4–6]. Furthermore, the 3-set tree connectivity of some networks have been studied. Such as, Chartrand ea al. explored the 3-set tree connectivity of complete graphs[7]. Li ea al. derived the 3-set tree connectivity of card product graphs[8]. Li et al. determined product graphs[9]. Li et al. evaluated the 3-set tree connectivity of complete bipartite graph[10]. Zhao et al. investigated the 3-set tree connectivity of star graphs and alternating group graphs[11]. However, there are few results about 4-set tree connectivity. The results known about 4-set tree connectivity of networks so far, are hypercubes[12], exchanged hypercubes[13] and dual cube[14] and hierarchical cubic networks[15]. For more researches and results about the k-set tree connectivity, refer to [16–19,22,23].

In this paper, we study the 4-set tree connectivity of the hierarchical folded hypercube.

2. Preliminaries

An n-dimensional hierarchical folded hypercube HFQ_n can be divided into 2^n clusters, say C_1, $C_2, \ldots, C_{2^n}$ and every cluster is isomorphic to the n-dimensional folded hypercube FQ_n. Any vertex $a \in V(HFQ_n)$ can be denoted by a two-tuple address $a = (c(a), p(a))$, where both $c(a)$ and $p(a)$ are n-bit binary strings. The first binary string $c(a)$ identifies the cluster the vertex a belong to and the second binary string $p(a)$ identifies the vertex within the cluster. Two vertices $a = (c(a), p(a))$ and $b = (c(b), p(b))$ are adjacent in HFQ_n if and only if one of the following two prerequisites holds:

$$E_{cu}(HFQ_n) = \{ab|H(P(a), P(b)) = 1\},$$

$E_{cr}(HFQ_n) = \{ab|$ if $c(a) = P(a)$, then $c(b) = p(b) = \overline{c(a)}$ otherwise, $c(a) = P(b)$ and $p(a) = c(b)\}$.

Where $E_{cu}(HFQ_n)$ is called cube edge, which in a cluster of HFQ_n and $E_{cr}(HFQ_n)$ is called cross edge, which connecting two vertices in two distinct clusters of HFQ_n.

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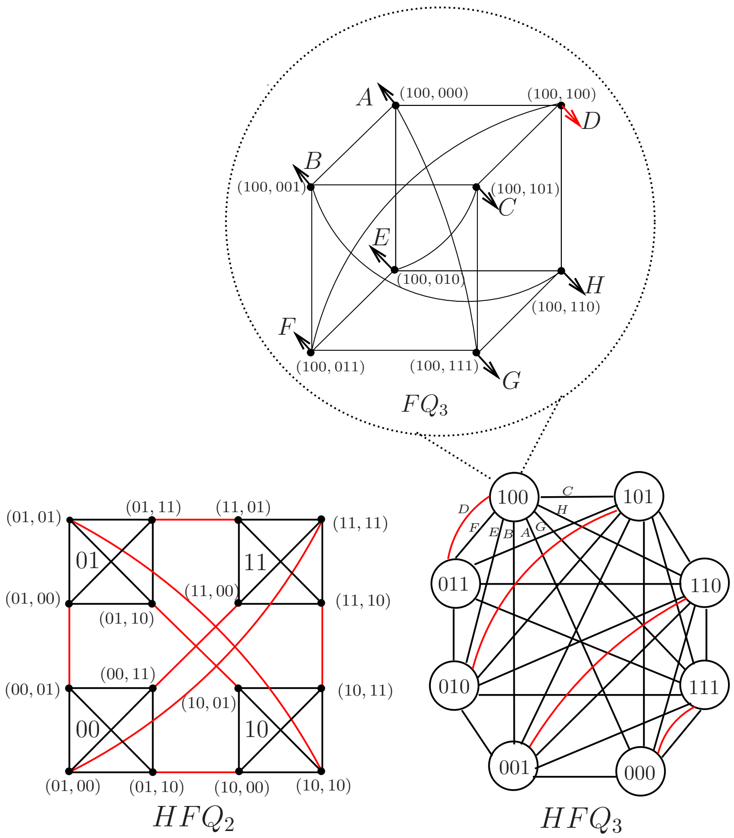

Figure 1. The 2, 3-dimension hierarchical folded hypercube.

Lemma 2.1. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HFQ_n for $n \geq 7$, we have*

(1) *For $i \in [2^n]$, let $b \in V(C_i)$ with $c(b) = p(b)$. The external neighbors of different vertices in $V(C_i) \backslash \{b\}$ distributes in distinct clusters of HFQ_n. Besides, if $a \in V(C_i) \backslash \{b\}$ with $p(a) = \overline{p(b)}$, the external neighbor of a distributes in the same cluster with b.*

(2) *For $a \in V(C_i)$ and $b \in V(C_j)$, there exist two cross edges between C_i and C_j for $i \neq j$ and $i, j \in [2^n]$ if and only if $c(a) = \overline{c(b)}$; otherwise there is just one cross edge.*

(3) *For any two vertices $a, b \in V(C_i)$ in HFQ_n for $i \in [2^n]$, the external neighbor of a and b can not be the same vertex.*

Proof. (1) Let $b_1, b_2 \in V(C_i) \backslash \{b\}$ and $b_1 \neq b_2$. By the definition of HFQ_n, $c(b_1) \neq p(b_1)$, $c(b_2) \neq p(b_2)$ and $p(b_1) \neq p(b_2)$. Therefore, the external neighbors of b_1 and b_2 are $(p(b_1), c(b_1))$ and $(p(b_2), c(b_2))$, respectively. Since $p(b_1) \neq p(b_2)$, the external neighbor of b_1 and b_2 distribute in different clusters of HFQ_n. In view of $b \in V(C_i)$ and $c(b) = p(b)$, the external neighbor of b is $\overline{c(b)} = \overline{p(b)}$. Let $a \in V(C_i) \backslash \{b\}$, then $c(a) \neq p(a)$ and the external neighbor of a is $(p(a), c(a))$. When $(p(a) = \overline{p(b)})$, the external neighbors of a and b belong to the same cluster of HFQ_n.

(2) By (1), it is clear that the result holds.

(3) Let $a, b \in V(C_i)$ for $i \in [2^n]$ and suppose that a and b have a same external neighbor, denote by x, then a and b are the two external neighbors of x, a contradiction.

□

Lemma 2.2. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HFQ_n for $n \geq 7$. Then, arbitrary vertex $b = (c(b), p(b)) \in V(C_i)$ for $i \in [2^n]$, $|N_{C_i}(b)| = n + 1$, we have*

(1) *The external neighbors of the distinct vertices in $N_{C_i}(b)$ distribute in different clusters of HFQ_n.*

(2) *When $c(b) = p(b)(c(b) = \overline{p(b)})$, there is one and only one vertex b_i in $N_{C_i}(b)$ with the external neighbor belongs to the same cluster with the vertex v and $b_i = (c(b), \overline{p(b)})(c(b),$*

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$p(b)$). Otherwise, all the vertices in $N_{C_i}(b)$ with the external neighbors distribute in different clusters with the vertex b.

Proof. (1) w.l.o.g., let $b \in V(C_1)$. Recall that $C_1 \cong \overline{FQ_n}$, which is $(n+1)$-regular, therefore $|N_{C_1}(v)| = n+1$. As $n \geq 7$, for any $b_1, b_2 \in N_{C_1}(b)$, $p(b_1) \neq \overline{p(b_2)}$. By (1) of Lemma 1, the external neighbors of vertices in $N_{C_1}(b)$ distribute in different clusters of HFQ_n.

(2) By the definition of HFQ_n, the result can be obtained directly.
□

Lemma 2.3. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HFQ_n and $N = HFQ_n[\bigcup_{j=1}^{k} V(C_{i_j})]$ for $i_j \in V(2^n)$, where $k \geq 1$ and $n \geq 7$, then N is connected.*

Proof. w.l.o.g., let $N = HFQ_n[\bigcup_{j=1}^{k} V(C_j)]$. By (2) of Lemma 1, there exists at least one cross edge between arbitrary two different clusters of HFQ_n. Therefore, N is connected.
□

Lemma 2.4. *[20] The folded hypercube FQ_n is $(n+1)$-regular and $\kappa(FQ_n) = n+1$ for $n \geq 2$.*

Lemma 2.5. *[21] The hierarchical folded hypercube HFQ_n is $(n+2)$-regular and $\kappa(HFQ_n) = n+2$ for $n \geq 2$.*

3. The 4-set tree connectivity of hierarchical folded hypercube

In this part, we determine the k-set tree connectivity of folded hypercubes HFQ_n, when $k = 4$. That is, $\kappa_4(HFQ_n) = n+1$ where $n \geq 7$.

The following lemma provides a sharp upper bound for our main result.

Lemma 3.1. *[5] Let G be a connected graph of order $|V(G)|$ with minimum degree $\delta(G)$. Then $\kappa_k(G) \leq \delta(G)$ for $3 \leq k \leq |V(G)|$. In addition, if there exist two adjacent vertices of degree $\delta(G)$, then $\kappa_k(G) \leq \delta(G) - 1$ for $3 \leq k \leq |V(G)|$.*

The following three Lemmas describe several clear and crucial properties on k-connected graphs, which provide great convenience when we construct internally disjoint paths.

Lemma 3.2. *[1] Let G be a k-connected graph and u and v be arbitrary two distinct vertices of G. Then there are k internally disjoint paths joining x and y in G.*

Lemma 3.3. *[1] Let $G = (V, E)$ be a k-connected graph, arbitrary vertex $u \in V$, let $Y \subseteq V \backslash \{u\}$ be a vertex subset of V and $|Y| \geq k$. Then, there exists a k-fan in G from u to Y.*

Lemma 3.4. *[1] Let $G = (V, E)$ be a k-connected graph and $U, W \subseteq V$ be two disjoint vertex subsets with $|U| \geq k$ and $|W| \geq k$. Then, there exist k disjoint (U, W)-paths in G.*

The following two Theorems on the folded hypercube FQ_n are very beneficial to our proof of the main result.

Theorem 3.1. *[25] $\kappa_3(FQ_n) = n$, where $n \geq 2$.*

Theorem 3.2. *[26] $\kappa_4(FQ_n) = n$, where $n \geq 7$.*

Lemma 3.5. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HFQ_n for $n \geq 7$ and $S = \{x, y, z, w\} \subseteq V(HFQ_n)$. Let $|S \cap V(C_i)| = 3$ and $|S \cap V(C_j)| = 1$ for different $i, j \in [2^n]$, then there exist $n+1$ internally disjoint S-trees in HFQ_n.*

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Proof. w.l.o.g., suppose that $|S \cap V(C_1)| = 3$ and $|S \cap V(C_2)| = 1$. Let $\{x, y, z\} \subseteq V(C_1)$ and $w \in V(C_2)$. See Figure 2. By Theorem 3.1, there are n internally disjoint trees $\widehat{T}_1, \widehat{T}_2, \ldots, \widehat{T}_n$ connecting x, y, z. Recall that $C_i \cong FQ_n$, for each $i \in [2^n]$. By Lemma 2.4, $\kappa(FQ_n) = n+1$, there are $n+1$ neighbors $x_1, x_2, \ldots, x_{n+1}$ of x in C_1. Let $x_i \in V(\widehat{T}_i)$ for $i \in [n]$ and $X = \{x_1, x_2, \ldots, x_{n+1}, x\}$.

Note that $X = \{x_1, x_2, \ldots, x_{n+1}, x\}$, it is potential that $y \in X$ or $z \in X$. To avoid repetition, we only consider the case of $y \notin X$ and $z \notin X$, as the discussion about $y \in X$ or $z \in X$ is similar. By (2) of Lemma 2.2, there are at most two vertices of X with the external neighbors belonging to C_2, and the two vertices must be x and x_i, for some $i \in [n+1]$. To obtain the main result, we need two consider following three cases.

Case 1. There are two vertices in X with the external neighbors belonging to C_2.

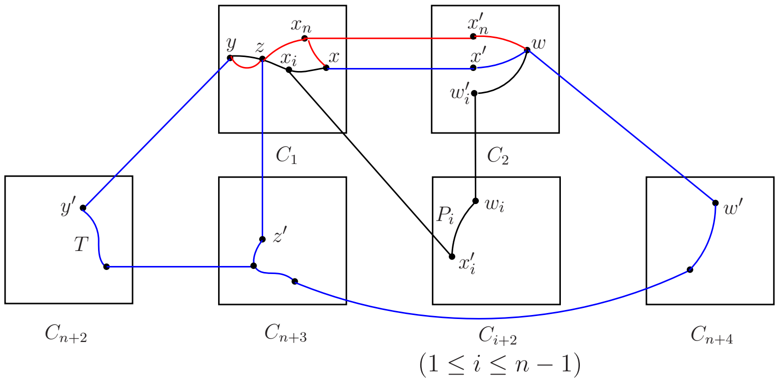

Figure 2. The explanation of Case 1.

w.l.o.g., assume that the two vertices are x and x_n, that is $x', x'_n \in V(C_2)$. Let $x_i \in V(C_{i+2})$ and there exists a vertex $w_i \in V(C_{i+2})$ and such that $w'_i \in V(C_2)$ for $i \in [n-1]$. Since C_{i+2} is connected, there exists a path P_i joining x'_i and w_i in C_{i+2}, for $i \in [n-1]$. Let $W = \{w'_1, \ldots, w'_{n-1}, x'_n, x'\}$. By Lemma 3.3, there are $n+1$ disjoint paths $P'_1, P'_2, \ldots, P'_{n+1}$ from w to W, and such that $x'_i \in V(P'_i)$, $x'_n \in V(P'_n)$, $x' \in V(P'_{n+1})$ for $i \in [n-1]$. w.l.o.g., suppose that $y' \in V(C_{n+2})$, $z' \in V(C_{n+3})$, $w' \in V(C_{n+4})$. By Lemma 2.3, $HFQ_n[\cup_{i=n+2}^{n+4} V(C_i)]$ is connected, then there is a tree T connecting y', z' and w'. Let $T_i = \widehat{T}_i \cup x_i x'_i \cup P_i \cup w_i w'_i \cup P'_i$ for $i \in [n-1]$, $T_n = \widehat{T}_n \cup x_n x'_n \cup P'_n$ and $T_{n+1} = xx' \cup P'_{n+1} \cup yy' \cup zz' \cup ww' \cup T$, then $T_1, T_2, \ldots, T_{n+1}$ are $n+1$ internally disjoint S-trees.

Case 2. There is only one vertex in X with the external neighbor belonging to C_2.

Case 2.1. $x' \in V(C_2)$.

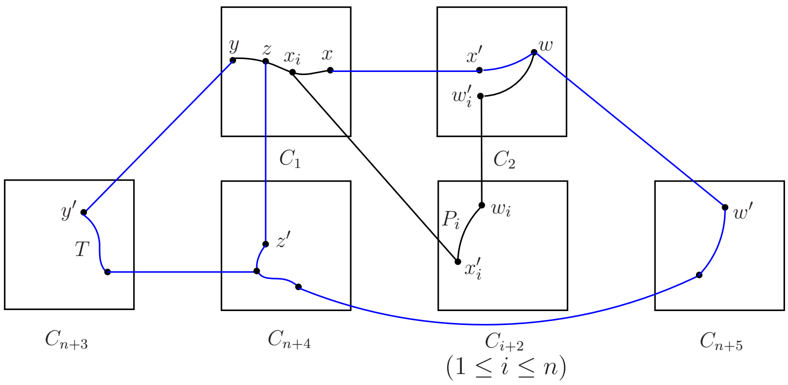

Figure 3. The explanation of Case 2.1.

Recall that, we consider that $y, z \notin X$, by (2) of Lemma 2.2, $y', z' \notin V(C_2)$. Let $x'_i \in V(C_{i+2})$ for $i \in [n]$. Since C_{i+2} is connected, there is a path P_i joining x'_i and w_i in C_{i+2}, and such that $w'_i \in V(C_2)$ for $i \in [n]$. Let $W = \{w'_1, \ldots, w'_n, x'\}$. By Lemma 3.3, there are $n+1$ disjoint paths $P'_1, P'_2, \ldots, P'_{n+1}$ between w and W, and subject to $x'_i \in V(P'_i)$, $x' \in V(P'_{n+1})$ for $i \in [n]$. w.l.o.g., assume

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that $y' \in V(C_{n+3})$, $z' \in V(C_{n+4})$, $w' \in V(C_{n+5})$. By Lemma 2.3, $HFQ_n[\cup_{i=n+3}^{n+5} V(C_i)]$ is connected, then there is a tree T connecting y', z' and w'. Let $T_i = \widehat{T}_i \cup x_i x_i' \cup P_i \cup w_i w_i' \cup P_i'$ for $i \in [n]$ and $T_{n+1} = xx' \cup P'_{n+1} \cup yy' \cup zz' \cup ww' \cup T$, then $T_1, T_2, \ldots, T_{n+1}$ are $n+1$ internally disjoint S-trees.

Case 2.2. $x_i' \in V(C_2)$ for certain $i \in [n+1]$.

w.l.o.g., suppose that $x_n' \in V(C_2)$. By Lemma 2.1, at most one vertex of y' and z' belongs to C_2.

Case 2.2.1. $y', z' \notin V(C_2)$.

Let $x_i' \in V(C_{i+2})$ for $i \in [n-1]$. Since C_{i+2} is connected, there exists a path P_i joining x_i' and w_i in C_{i+2}, and such that $w_i' \in V(C_2)$ for $i \in [n-1]$. w.l.o.g., assume that $x' \in V(C_{n+2})$. Note that, y' or z' may belongs to the same cluster C_{i+2} with some x_i' for some $i \in [n-1]$. We need to discuss the following two situations.

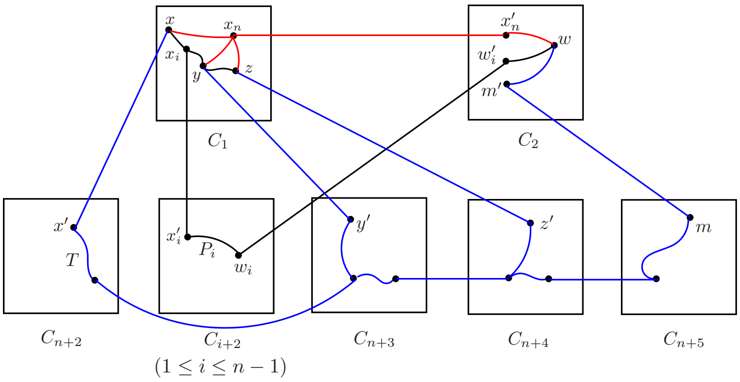

Figure 4. The explanation about $y', z' \notin V(C_{i+2})$ for $i \in [n-1]$ of Case 2.2.1.

When $y', z' \notin V(C_{i+2})$ for $i \in [n-1]$. w.l.o.g., suppose that $y' \in V(C_{n+3})$, $z' \in V(C_{n+3})$, $m \in V(C_{n+5})$ and such that $m' \in V(C_2)$. Let $W = \{w_1', w_2', \ldots, w_n', m'\}$. By Lemma 3.3, there exist $n+1$ disjoint paths $P_1', P_2', \ldots, P_{n+1}'$ from w to W, and such that P'_{n+1} be the path joining w and m'. By Lemma 2.3, $HFQ_n[\cup_{i=n+2}^{n+5} V(C_i)]$ is connected, then there exists a tree T connecting x', y', z' and m. Let $T_i = \widehat{T}_i \cup x_i x_i' \cup P_i \cup w_i w_i' \cup P_i'$ for $i \in [n-1]$, $T_n = \widehat{T}_n \cup x_n x_n' \cup P_n'$ and $T_{n+1} = xx' \cup yy' \cup zz' \cup P'_{n+1} \cup mm' \cup T$, then $T_1, T_2, \ldots, T_{n+1}$ are $n+1$ internally disjoint S-trees.

When $y' \in V(C_{i+2})$ or $z' \in V(C_{i+2})$ for certain $i \in [n-1]$.

w.l.o.g., suppose that $y' \in V(C_{n+1})$ and $z' \in V(C_{n+3})$. That is, $x'_{n-1}, y' \in V(C_{n+1})$. Let $w_{n-1}, g \in V(C_{n+1})$ and $w'_{n-1} \in V(C_2)$, $g' \in V(C_{n+3})$. Let $A = \{x'_{n-1}, y'\}$, $B = \{w_{n-1}, g\}$ and $A, B \subseteq V(C_{n+1})$. By Lemma 3.4, there exist two internally disjoint (A, B)-paths, saying P_{n-1} and Q, subject to P_{n-1} be the path from x'_{n-1} to w_{n-1} and P be the path from y' to g. Similar to the case of $y', z' \notin V(C_{i+2})$ for $i \in [n-1]$, we get n internally disjoint S-trees $T_1, T_2, \ldots, T_n$. Let $m \in V(C_{n+4})$ and $m' \in V(C_2)$. By Lemma 2.3, $HFQ_n[\cup_{i=n+2}^{n+5} V(C_i)]$ is connected, then there exists a tree T connecting y', z', g' and m. Let $T_{n+1} = xx' \cup yy' \cup zz' \cup Q \cup gg' \cup P'_{n+1} \cup mm' \cup T$.

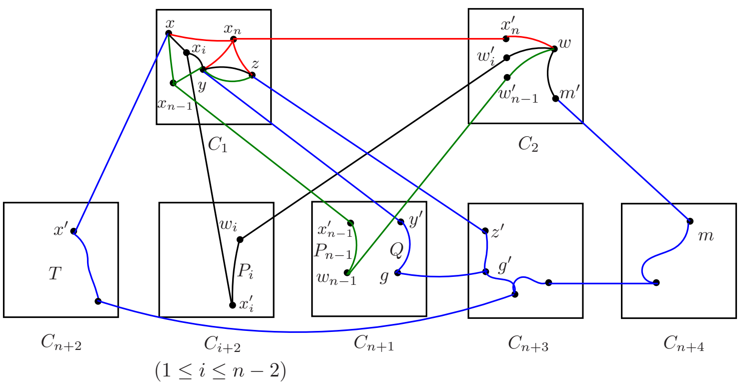

Figure 5. The explanation about $y' \in V(C_{i+2})$ for some $i \in [n-1]$ of Case 2.2.1.

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Case 2.2.2. $y' \in V(C_2)$ or $z' \in V(C_2)$.

w.l.o.g., assume that $y' \in V(C_2)$, by (2) of Lemma 2.2, y must be the neighbor of x_n in C_1. Then The proof is similar to that of Case 1.

Case 3. None of the vertex in X with their external neighbor belongs to C_2.

By (2) of Lemma 2.1, y' and z' may belong to C_2. We consider the following three subcases to obtain the result.

Case 3.1. $y', z' \in V(C_2)$.

By (2) of Lemma 2.2, y is adjacent to z in C_1 and y' is adjacent to z' in C_2. Let $x_i' \in V(C_{i+2})$ for $i \in [n-1]$. Since C_{i+2} is connected, there is a path P_i joining x_i' and w_i in C_{i+2}, and subject to $w_i' \in V(C_2)$ for $i \in [n]$. w.l.o.g., assume that $x' \in V(C_{n+3})$, $w' \in V(C_{n+4})$. By Lemma 2.3, $HFQ_n[\cup_{i=n+3}^{n+4} V(C_i)]$ is connected, then there exists a tree T connecting x' and w'. Let $W = \{w_1', w_2', \ldots, w_n', y'\}$. By Lemma 3.3, there exist $n+1$ disjoint paths $P_1', P_2', \ldots, P_{n+1}'$ from w to W, and such that P_{n+1}' is the path joining y' and w. Let $T_i = \widehat{T_i} \cup x_i x_i' \cup P_i \cup w_i w_i' \cup P_i'$ for $i \in [n]$.

When $z' \in V(P_i')$, for certain $i \in [n+1]$. w.l.o.g., suppose that $z' \in V(P_{n+1}')$. Let $T_{n+1} = xx' \cup yy' \cup zz' \cup ww' \cup P_{n+1}' \cup T$.

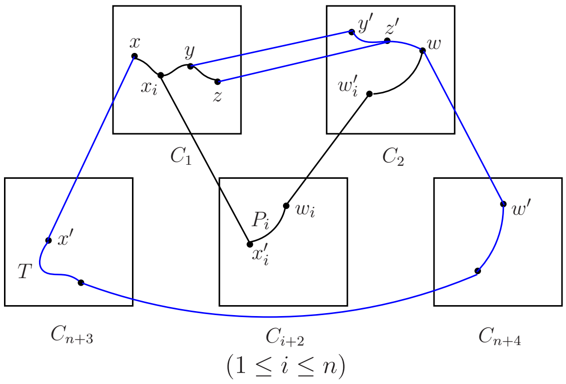

Figure 6. The explanation about $z' \in V(p_{n+1}')$ of Case 3.1.

When $z' \notin V(P_i')$, for $i \in [n+1]$. Since y' is adjacent to z', let $R = P_{n+1}' \cup y'z'$. Obviously, $P_1', P_2', \ldots, P_n', R$ are disjoint paths. Let $T_{n+1} = T \cup xx' \cup yy' \cup zz' \cup ww' \cup R$.

Case 3.2. $y' \in V(C_2)$ or $z' \in V(C_2)$.

Is clear that, similar to that of Case 2.1, we can get $n+1$ internally disjoint S-trees.

Case 3.3. $y' \notin V(C_2)$ and $z' \notin V(C_2)$.

Note that, y' or z' may belongs to the same cluster C_{i+2} with x_i' for some $i \in [n]$.

When $y', z' \notin V(C_{i+2})$ for $i \in [n]$. w.l.o.g., assume that $y' \in V(C_{n+4})$ and $z' \in V(C_{n+5})$. Similar to that of Case 3.1, we get n internally disjoint S-trees. By Lemma 2.3, $HFQ_n[\cup_{i=n+3}^{n+5} V(C_i)]$ is connected, then there exists a tree T connecting x', y', z' and m. Let $T_{n+1} = xx' \cup yy' \cup zz' \cup mm' \cup P_{n+1}' \cup T$.

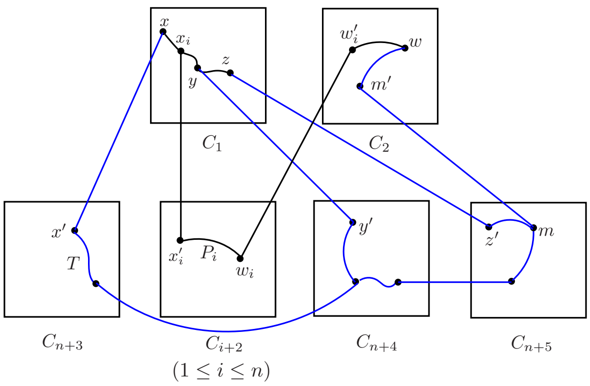

Figure 7. The explanation of Case 3.3.

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When $y' \in V(C_{i+2})$ or $z' \in V(C_{i+2})$ for certain $i \in [n]$. w.l.o.g., suppose that $y' \in V(C_{n+2})$ and $z' \in V(C_{n+4})$. That is, $x'_n, y' \in V(C_{n+2})$. Let $w_n, f \in V(C_{n+2})$ and $w'_n \in V(C_2)$, $f' \in V(C_{n+4})$. Let $C = \{x'_n, y'\}$, $D = \{w_n, f\}$ and $C, D \subseteq V(C_n)$. By Lemma 3.4, there exist two internally disjoint (C, D)-paths, saying P_n and R, subject to P_n be the path joining x'_n and w_n, subject to R be the path joining y' and f. Similar to the subcase of $y', z' \notin V(C_{i+2})$ for $i \in [n]$, we get $T_1, T_2, \ldots, T_n$ n internally disjoint S-trees. Let $m, z' \in V(C_{n+4})$ and $m' \in V(C_2)$. By Lemma 2.3, $HFQ_n[\cup_{i=n+3}^{n+5} V(C_i)]$ is connected, then there exists a tree T connecting x', z', f' and m. Let $T_{n+1} = xx' \cup yy' \cup zz' \cup R \cup ff' \cup P'_{n+1} \cup mm' \cup T$.
□

Lemma 3.6. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HFQ_n for $n \geq 7$ and $S = \{x, y, z, w\} \subseteq V(HFQ_n)$. Let $|S \cap V(C_i)| = 2$ and $|S \cap V(C_j)| = 2$ for different $i, j \in [2^n]$, then there exist $n+1$ internally disjoint S-trees in HFQ_n.*

Proof. w.l.o.g., suppose that $|S \cap V(C_1)| = 2$, $|S \cap V(C_2)| = 2$. Let $x, y \in V(C_1)$ and $z, w \in V(C_2)$. By Lemma 2.5, $\kappa(C_1) = \kappa(C_2) = n+1$, then there exist $n+1$ internally disjoint paths $P_1, P_2, \ldots, P_{n+1}$ joining x and y in C_1 and $n+1$ internally disjoint paths $P'_1, P'_2, \ldots, P'_{n+1}$ joining z and w in C_2. Let $x_i \in V(P_i) \cap N(x)$ and $z_i \in V(P'_i) \cap N(z)$ for $i \in [n+1]$. Let $\widehat{X} = \{x, x_1, x_2, \ldots, x_{n+1}\}$ and $\widehat{Z} = \{z, z_1, z_2, \ldots, z_{n+1}\}$. Select n vertices from $\widehat{X}$, namely X, subject to the external neighbor of arbitrary vertex in X doesn't distribute in C_2. As the same way, select n vertices from $\widehat{Z}$, namely Z, subject to the external neighbor of arbitrary vertex in Z doesn't distribute in C_1. By Lemma 2.2, it can be done. w.l.o.g., suppose that $X = \{x_1, x_2, \ldots, x_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$. By (1) of Lemma 2.2, any vertex in $X(resp.Z)$ with its external neighbor belongs to different clusters. Let vertex subset $X' = \{x'_1, x'_2, \ldots, x'_{n+1}\}$ and vertex subset $Z' = \{z'_1, z'_2, \ldots, z'_{n+1}\}$.

Case 1. There are two vertices in $\widehat{X}$ with the external neighbors distribute in C_2.

By (2) of Lemma 2.2, two vertices $x', x'_{n+1} \in V(C_2)$. By (1) of Lemma 2.2, any vertex in $X(resp.Z)$ with its external neighbor belongs to different clusters. To avoid repetition, we just consider the vertices in $X' \cup Z'$ distribute in distinct clusters, as the discussion of some vertex x'_i belongs to the same cluster with z'_i is similar. Let $x'_i \in V(C_{i+2})$ and $z'_i \in V(C_{n+i+2})$ for $i \in [n]$. By Lemma 2.3, $HFQ_n[\cup_{j=i+2}^{n+i+2} V(C_j)]$ is connected, then there exists a tree $\widehat{T}_i$ connecting x'_i and z'_i. Let $T_i = P_i \cup x_i x'_i \cup \widehat{T}_i \cup P'_i \cup z_i z'_i$ for $i \in [n]$.

When x' or x'_{n+1} distributes in $V(P'_i)$ for some $i \in [n+1]$. w.l.o.g., assume that $x'_{n+1} \in V(P'_{n+1})$. Let $T_{n+1} = P_{n+1} \cup x_{n+1} x'_{n+1} \cup P'_{n+1}$.

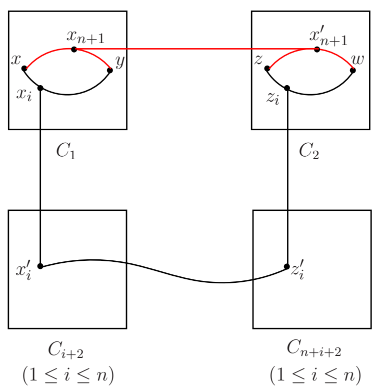

Figure 8. The explanation about x' or x'_{n+1} distributes in $V(P'_i)$ of Case 1.

When x' and x'_{n+1} don't belong to $V(P'_i)$ for $i \in [n+1]$. Since C_2 is connected, there is a path R joining x' and z, that is $R \cap N_{C_2}(z) \neq \oslash$. w.l.o.g., let $R \cap N_{C_2}(z) = \{z_{n+1}\}$. Let $T_{n+1} = P_{n+1} \cup xx' \cup R \cup P'_{n+1}$.

Case 2. There is only one vertex in $\widehat{X}$ with the external neighbor distributing in C_2.

The proof is similar to that of Case 1.

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Case 3. None of the vertex in $\widehat{X}$ with their external neighbor distributes in C_2.

To avoid repetition, we just consider the vertices in $X' \cup Z'$ distribute in distinct clusters, as the discussion of some vertex x_i' belongs to the same cluster with z_i' is similar. This can be done for $n \geq 7$. Let $x_i' \in V(C_{i+2})$ and $z_i' \in V(C_{n+i+3})$ for $i \in [n+1]$. By 2 of Lemma 2.1, C_{i+2} is connected to C_{n+i+3}, then there is a tree $\widehat{T}_i$ connecting x_i' and z_i'. Let $T_i = P_i \cup x_i x_i' \cup \widehat{T}_i \cup z_i z_i'$ for $i \in [n+1]$.

□

Lemma 3.7. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HFQ_n for $n \geq 7$ and $S = \{x, y, z, w\} \subseteq V(HFQ_n)$. Let $|S \cap V(C_i)| = 2$, $|S \cap V(C_j)| = 1$ and $|S \cap V(C_k)| = 1$ for different $i, j, k \in [2^n]$, then there exist $n+1$ internally disjoint S-trees in HFQ_n.*

Proof. w.l.o.g., suppose that $|S \cap V(C_1)| = 2$, $|S \cap V(C_2)| = 1$ and $|S \cap V(C_3)| = 1$. Let $\{x, y\} \subseteq V(C_1)$, $z \in V(C_2)$ and $w \in V(C_3)$. By Lemma 2.5, $\kappa(C_1) = n+1$, then there are $n+1$ internally disjoint paths $P_1, P_2, \ldots, P_{n+1}$ joining x and y in C_1. Let $x_i \in V(P_i) \cap N(x)$ for $i \in [n+1]$ and $X = \{x_1, x_2, \ldots, x_{n+1}, x\}$. Note that, it is possible that $y \in X$. In fact, to avoid repetition, we only consider the case of $y \notin X$.

Case 1. There exist three cross edges between X and $V(C_2 \cup C_3)$.

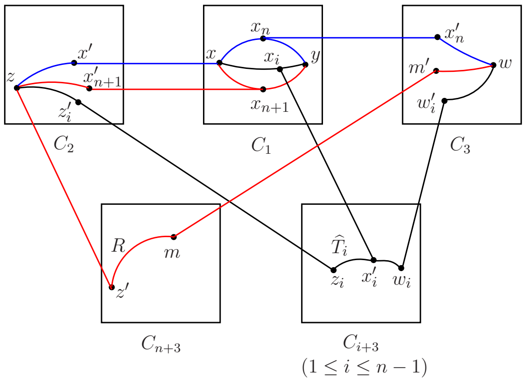

Figure 9. The explanation of Case 1.

w.l.o.g., assume that there are two cross edges between X and C_2 and one cross edge between X and C_3. By Lemma 2.2, suppose that $x', x'_{n+1} \in V(C_2)$ and $x'_n \in V(C_3)$. Let $x_i' \in V(C_{i+3})$, where $i \in [n-1]$. Since C_{i+3} is connected, there is a tree $\widehat{T}_i$ connecting z_i, w_i and x_i' in C_{i+3} and such that $z_i' \in V(C_2)$, $w_i' \in V(C_3)$. There are two cross edges between C_1 and C_2, then $z' \notin HFQ_n[V(C_1 \cup C_2 \cup C_{i+3})]$ for $i \in [n-1]$. w.l.o.g., let $z' \in V(C_{n+3})$. Since C_{n+3} is connected, there is a path R joining z' and m in C_{n+3} and such that $m' \in V(C_3)$. Let $Z = \{z_1', z_2', \ldots, z'_{n-1}, x', x'_{n+1}\}$ and $W = \{w_1', w_2', \ldots, w'_{n-1}, x'_n, m'\}$. By Lemma 3.3, there exist $n+1$ internally disjoint paths $P_1', P_2', \ldots, P'_{n+1}$ between z and Z, subject to $z_i' \in V(P_i')$, $x' \in V(P_n')$ and $x'_{n+1} \in V((P'_{n+1})$ for $i \in [n-1]$. Similarly, there are $n+1$ internally disjoint paths $\widehat{P}_1, \widehat{P}_2, \ldots, \widehat{P}_{n+1}$ from w to W, such that $w_i' \in V(\widehat{P}_i)$, $x'_n \in V(\widehat{P}_n)$ and $m' \in V(\widehat{P}_{n+1})$ for $i \in [n-1]$. Let $T_i = P_i \cup x_i x_i' \cup \widehat{T}_i \cup z_i z_i' \cup P_i' \cup w_i w_i' \cup \widehat{P}_i$ for $i \in [n-1]$, $T_n = P_n \cup x x' \cup P_n' \cup x_n x_n' \cup \widehat{P}_n$ and $T_{n+1} = P_{n+1} \cup x_{n+1} x'_{n+1} \cup P'_{n+1} \cup z z' \cup R \cup m m' \cup \widehat{P}_{n+1}$.

Case 2. There exist two cross edges between X and $V(C_2 \cup C_3)$.

Case 2.1. There are two cross edges between X and C_2.

w.l.o.g., suppose that $x', x'_{n+1} \in V(C_2)$. Similar to Case 1, we by replacing subset $Z = \{z_1', z_2', \ldots, z'_{n-1}, x', x'_{n+1}\}$ to subset $Z = \{z_1', z_2', \ldots, z_n', x'_{n+1}\}$ and replacing subset $W = \{w_1', w_2', \ldots, w'_{n-1}, x'_n, m'\}$ to subset $W = \{w_1', w_2', \ldots, w_n', m'\}$, we get $T_1, T_2, T_{n-1}, T_{n+1}$ n internally disjoint S-trees. As for the n-th S-tree, we changed $T_n = P_n \cup x x' \cup P_n' \cup x_n x_n' \cup \widehat{P}_n$ into $T_n = P_i \cup x_n x_n' \cup \widehat{T}_n \cup z_n z_n' \cup P_n' \cup w_n w_n' \cup \widehat{P}_n$.

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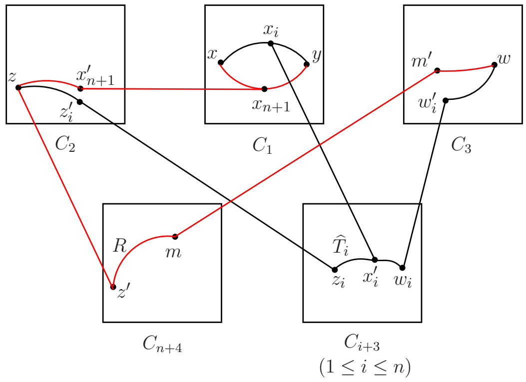

Figure 10. The explanation of Case 2.1.

Case 2.2. There is one cross edge between X and C_2, the other between X and C_3.

Case 2.2.1. $x', x'_i \in V(C_2 \cup C_3)$ for certain $i \in [n+1]$.

w.l.o.g., assume that $x' \in V(C_2)$, $x'_{n+1} \in V(C_3)$. By Lemma 2.2, the vertices in $\{x'_1, x'_2, \ldots, x'_n\}$ distribute in different clusters. Let $x'_i \in V(C_{i+3})$ for $i \in [n]$. Since C_{i+3} is connected, there exists a tree $\widehat{T}_i$ connecting x'_i, z_i, w_i and such that $z'_i \in V(C_2)$, $w'_i \in V(C_3)$. Let $Z = \{z'_1, z'_2, \ldots, z'_n, x'\}$ and $W = \{w'_1, w'_2, \ldots, w'_n, x'_{n+1}\}$. By Lemma 3.3, there exist $n+1$ internally disjoint paths $P'_1, P'_2, \ldots, P'_{n+1}$ from z and Z, $n+1$ internally disjoint paths $\widehat{P}_1, \widehat{P}_2, \ldots, \widehat{P}_{n+1}$ from w to W and subject to $z'_i \in V(P'_i)$, $x' \in V(P'_{n+1})$, $w'_i \in V(\widehat{P}_i)$, $x'_{n+1} \in V(\widehat{P}_{n+1})$, where $i \in [n]$. Let $T_i = P_i \cup x_i x'_i \cup \widehat{T}_i \cup z_i z'_i \cup P'_i \cup w_i w'_i \cup \widehat{P}_i$, where $i \in [n]$. Let $T_{n+1} = P_{n+1} \cup xx' \cup P'_{n+1} \cup x_{n+1}x'_{n+1} \cup \widehat{P}_{n+1}$.

Case 2.2.2. $x'_i, x'_j \in V(C_2 \cup C_3)$, for $i \neq j$ and $i, j \in [n+1]$.

w.l.o.g., suppose that $x'_n \in V(C_2)$, $x'_{n+1} \in V(C_3)$ and $x'_i \in V(C_{i+3})$, where $i \in [n-1]$.

When $y' \in V(C_2)$, by 2 of Lemma 2.1, $x' \notin V(C_1 \cup C_2 \cup \ldots \cup C_{n+2})$. w.l.o.g., assume that $x' \in V(C_{n+3})$. As C_{n+3} is connected, there is a path Q joining x' and m, subject to $m' \in V(C_2)$. Let $Z = \{z'_1, z'_2, \ldots, z'_{n-1}, x'_n, m'\}$. Note that, it is possible that $z \in Z$. To avoid repetition, we just consider the case of $z \notin Z$, as the discussion of $z \in Z$ is similar. By (2) of Lemma 2.1, $z' \notin V(C_1 \cup C_2 \cup \ldots \cup C_{n+3})$. w.l.o.g., let $z' \in V(C_{n+4})$. As C_{n+4} is connected, there is a path R joining z' and g and subject to $g' \in V(C_3)$. Let $W = \{w'_1, w'_2, \ldots, w'_{n-1}, x'_{n+1}, g'\}$. By the Lemma 2.4, $\kappa(C_2) = \kappa(C_3) = n+1$. By the Lemma 3.3, there are $P'_1, P'_2, \ldots, P'_{n+1}$ $n+1$ internally disjoint paths between z and Z and $\widehat{P}_1, \widehat{P}_2, \ldots, \widehat{P}_{n+1}$ $n+1$ internally disjoint paths between w and W such that $z'_i \in V(P'_i)$, $x'_n \in V(P'_n)$, $m' \in V(P'_{n+1})$ and $w'_i \in V(\widehat{P}_i)$, $g' \in V(\widehat{P}_n)$, $x'_{n+1} \in V(\widehat{P}_{n+1})$ for $i \in [n-1]$. Let $T_i = P_i \cup x_i x'_i \cup \widehat{T}_i \cup z_i z'_i \cup P'_i \cup w_i w'_i \cup \widehat{P}_i$, where $i \in [n-1]$, $T_n = P_n \cup x_n x'_n \cup P'_n \cup zz' \cup R \cup gg' \cup \widehat{P}_n$ and $T_{n+1} = P_{n+1} \cup x_{n+1}x'_{n+1} \cup \widehat{P}_{n+1} \cup xx' \cup Q \cup mm' \cup P'_{n+1}$.

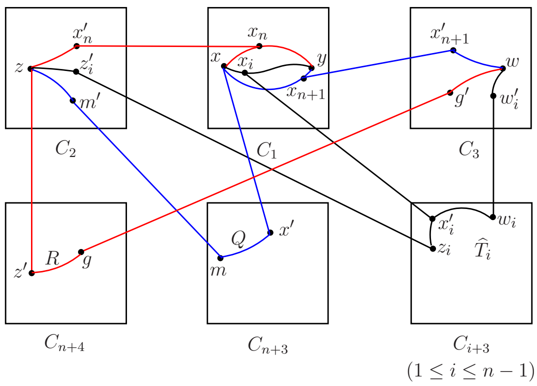

Figure 11. The explanation of $y' \in V(C_2)$ of Case 2.2.2.

When $y' \in V(C_3)$, similar to the situation of $y' \in V(C_2)$, we obtain $n+1$ internally disjoint S-trees.

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When $y' \in V(C_{i+3})$, for certain $i \in [n-1]$. Similar to the situation of $y' \in V(C_2)$, we obtain $n+1$ internally disjoint S-trees.

When $y' \notin V(HFQ_n) \setminus \cup_{i=1}^{n+2} V(C_i)$. w.l.o.g., assume that $y' \in V(C_{n+3})$. Since C_{n+3} is connected, there is a path M joining y' and m, subject to $m' \in V(C_2)$. Let $Z = \{z'_1, z'_2, \ldots, z'_{n-1}, x'_n, m'\}$ and $W = \{w'_1, w'_2, \ldots, w'_{n-1}, g, x'_{n+1}\}$ and subject to $g' \in V(C_2)$. By the Lemma 2.4, $\kappa(C_2) = \kappa(C_3) = n+1$. By the Lemma 3.3, there are $P'_1, P'_2, \ldots, P'_{n+1}$ $n+1$ internally disjoint paths between z and Z, $\widehat{P}_1, \widehat{P}_2, \ldots, \widehat{P}_{n+1}$ $n+1$ internally disjoint paths from w to W such that $z'_i \in V(P'_i)$, $x'_n \in V(P'_n)$, $m' \in V(P'_{n+1})$ and $w'_i \in V(\widehat{P}_i)$, $g' \in V(\widehat{P}_n)$, $x'_{n+1} \in V(\widehat{P}_{n+1})$, where $i \in [n-1]$. We obtain $T_1, T_2, \ldots, T_{n-1}$ $n-1$ internally disjoint S-trees, similar to the situation of $y' \in V(C_2)$. Let $T_{n+1} = P_{n+1} \cup x_{n+1}x'_{n+1} \cup \widehat{P}_{n+1} \cup yy' \cup M \cup mm' \cup P'_{n+1}$. For the n-th tree, we consider the following two situations. First, assume that $m' \in V(P'_i)$ for certain $i \in [n+1]$. w.l.o.g., suppose that $g' \in V(P'_n)$. Let $T_n = P_n \cup x_n x'_n \cup P'_n \cup gg' \cup \widehat{P}_n$. Next, suppose that $g' \notin V(P'_i)$ for $i \in [n+1]$. As C_2 is connected, there is a path Q joining z and g'. That is, $R \cap N(z) \neq \oslash$. w.l.o.g., let $R \cap N(z) = \{z_n\}$, that is $R \cap V(P'_n) = \{z_n\}$. Let $T_n = P_n \cup x_n x'_n \cup P'_n \cup R \cup gg' \cup \widehat{P}_n$.

Case 3. There exist one cross edge between X and $V(C_2 \cup C_3)$.

Suppose that the cross edge is between X and $V(C_2)$.

Case 3.1. $x' \in V(C_2)$.

By Lemma 2.2, the vertices in $\{x'_1, x'_2, \ldots, x'_{n+1}\}$ distribute in distinct clusters of HFQ_n. Let $x'_i \in V(C_{i+3})$, where $i \in [n+1]$. Since C_{i+3} is connected, there is a tree $\widehat{T}_i$ connecting x'_i, z_i, w_i and subject to $z'_i \in V(C_2)$, $w'_i \in V(C_3)$, where $i \in [n+1]$. Let subset $Z = \{z'_1, z'_2, \ldots, z'_{n+1}\}$ and subset $W = \{w'_1, w'_2, \ldots, w'_{n+1}\}$. By Lemma 2.4, $\kappa(C_2) = \kappa(C_3) = n+1$. By Lemma 3.3, there are $P'_1, P'_2, \ldots, P'_{n+1}$ $n+1$ internally disjoint paths between z and Z, $\widehat{P}_1, \widehat{P}_2, \ldots, \widehat{P}_{n+1}$ $n+1$ internally disjoint paths between w and W. Let $T_i = P_i \cup x_i x'_i \cup \widehat{T}_i \cup z_i z'_i \cup P'_i \cup w_i w'_i \cup \widehat{P}_i$ for $i \in [n+1]$.

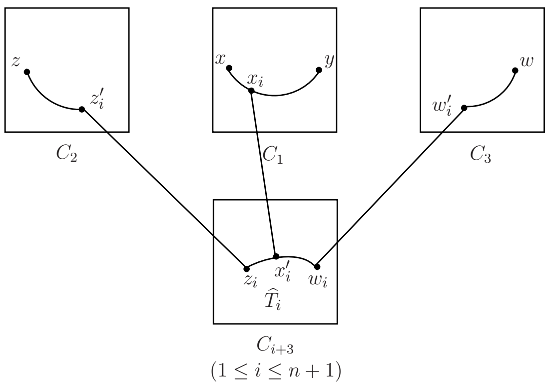

Figure 12. The explanation of Case 3.1.

Case 3.2. $x'_i \in V(C_2)$ for some $i \in [n+1]$.

w.l.o.g., suppose that $x'_{n+1} \in V(C_2)$ and $x'_i \in V(C_{i+3})$ for $i \in [n]$.

When $x' \in V(C_{n+4})$. By Lemma 2.2, the vertices in $\{x'_1, x'_2, \ldots, x'_n\}$ distribute in different clusters of HFQ_n. Let $x'_i \in V(C_{i+3})$ for $i \in [n]$. Since C_{i+3} is connected, then there is a tree $\widehat{T}_i$ connecting x'_i, z_i, w_i and subject to $z'_i \in V(C_2)$, $w'_i \in V(C_3)$, where $i \in [n]$. Let $Z = \{z'_1, z'_2, \ldots, z'_n, x'_{n+1}\}$, $W = \{w'_1, w'_2, \ldots, w'_n, m\}$ and subject to $m' \in V(C_{n+4})$. By Lemma 2.4, $\kappa(C_2) = \kappa(C_3) = n+1$. By Lemma 3.3, there are $P'_1, P'_2, \ldots, P'_{n+1}$ $n+1$ internally disjoint paths between z and Z subject to $z'_i \in V(P'_i)$, $x'_{n+1} \in V(P'_{n+1})$, $n+1$ internally disjoint paths $\widehat{P}_1, \widehat{P}_2, \ldots, \widehat{P}_{n+1}$ from w to W subject to $w'_i \in V(\widehat{P}_i)$, $m' \in V(\widehat{P}_{n+1})$ for $i \in [n]$. Since C_{n+4} is connected, then there exist a path R joining x' and m'. Let $T_i = P_i \cup x_i x'_i \cup \widehat{T}_i \cup z_i z'_i \cup P'_i \cup w_i w'_i \cup \widehat{P}_i$, where $i \in [n]$. Let $T_{n+1} = P_{n+1} \cup x_{n+1}x'_{n+1} \cup P'_{n+1} \cup xx' \cup R \cup mm' \cup \widehat{P}_{n+1}$.

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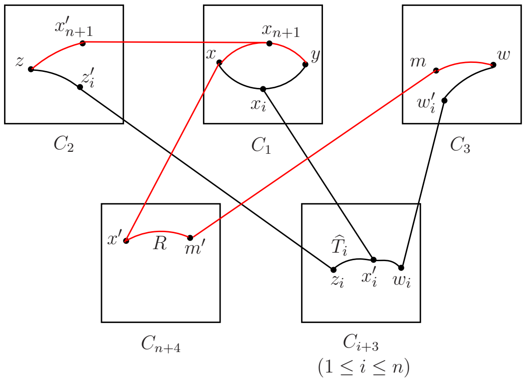

Figure 13. The explanation of $x' \in V(C_{n+4})$ of Case 3.2.

When $x' \in V(C_{i+3})$, by Lemma 2.2, $y' \in V(C_3)$ or $y' \in HFQ_n[\cup_{i=n+4}^{2^n} V(C_i)]$, similar to the case of $x' \in V(C_{n+4})$, we get $n+1$ internally disjoint S-trees.

Case 4. There is no cross edge between X and $V(C_2 \cup C_3)$.

Similar to that of Case 3.1, we get $n+1$ internally disjoint S-trees.

□

Lemma 3.8. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HFQ_n for $n \geq 7$ and $S = \{x, y, z, w\} \subseteq V(HFQ_n)$. Let $|S \cap V(C_i)| = 1$, $|S \cap V(C_j)| = 1$, $|S \cap V(C_k)| = 1$ and $|S \cap V(C_l)| = 1$ for different $i, j, k, l \in [2^n]$, then there exist $n+1$ internally disjoint S-trees in HFQ_n.*

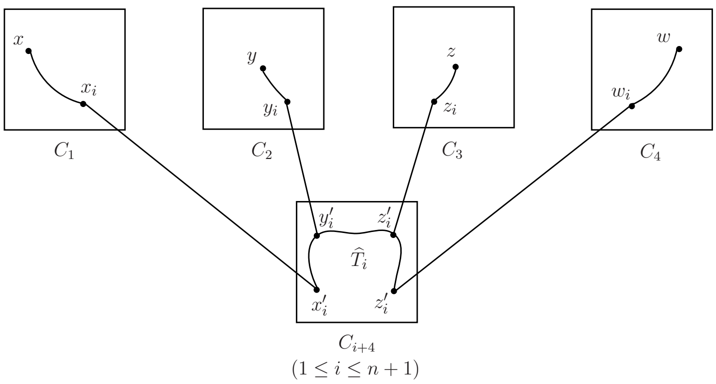

Figure 14. The explanation of Lemma 3.8.

Proof. w.l.o.g., suppose that $|S \cap V(C_1)| = 1$, $|S \cap V(C_2)| = 1$, $|S \cap V(C_3)| = 1$ and $|S \cap V(C_4)| = 1$. Let $x \in V(C_1)$, $y \in V(C_2)$, $z \in V(C_3)$ and $w \in V(C_4)$. Let $X = \{x_1, x_2, \ldots, x_{n+1}\} \subseteq V(C_1)$, $Y = \{y_1, y_2, \ldots, y_{n+1}\} \subseteq V(C_2)$, $Z = \{z_1, z_2, \ldots, z_{n+1}\} \subseteq V(C_3)$, $W = \{w_1, w_2, \ldots, w_{n+1}\} \subseteq V(C_4)$. By Lemma 2.5, $\kappa(C_1) = \kappa(C_2) = \kappa(C_3) = \kappa(C_4) = n+1$, then there are $n+1$ internally disjoint paths $P_1, P_2, \ldots, P_{n+1}$ from x and X in C_1, $n+1$ internally disjoint paths $P'_1, P'_2, \ldots, P'_{n+1}$ from y and Y in C_2, $n+1$ internally disjoint paths $\widehat{P}_1, \widehat{P}_2, \ldots, \widehat{P}_{n+1}$ from z and Z in C_3, $n+1$ internally disjoint paths $\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_{n+1}$ from z and Z in C_4, such that $x_i \in V(P_i)$, $y_i \in V(P'_i)$, $z_i \in V(\widehat{P}_i)$, $z_i \in V(\widetilde{P}_i)$ and such that $\{x'_i, y'_i, z'_i, w'_i\} \subseteq V(C_{i+4})$ for $i \in [n+1]$. Since C_{i+4} is connected, there exists a tree $\widehat{T}_i$ connecting x'_i, y'_i, z'_i, w'_i in C_{i+4} for $i \in [n+1]$. Let $T_i = \widehat{T}_i \cup x_i x'_i \cup y_i y'_i \cup z_i z'_i \cup w_i w'_i \cup P_i \cup P'_i \cup \widehat{P}_i \cup \widetilde{P}_i$ for $i \in [n+1]$.

□

Theorem 3.3. $\kappa_4(HFQ_n) = n+1$, *for* $n \geq 7$.

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Proof. By Lemma 2.5, $\kappa(HFQ_n) = n+2$. By Lemma 3.1, $\kappa_4(HFQ_n) \leq \delta(HFQ_n) - 1 = n+1$. By Lemma 3.5, Lemma 3.6, Lemma 3.7, Lemma 3.8, we obtain that $\kappa_4(HFQ_n) \geq n+1$. Thus, the result holds.

□

4. Concluding remarks

The hierarchical folded hypercube HFQ_n is a more popular topology for designing the internetworks due to its regularity, symmetry and many other unique properties. As one of the important variants of the hypercube, hierarchical folded hypercube has received more attention and many research results on hierarchical folded hypercube have emerged. We mainly study the 4-set tree connectivity of hierarchical folded hypercube and obtain that $\kappa_4(HFQ_n) = n+1$ for $n \geq 7$. Currently, there are few results about k-set tree connectivity of networks when $k = 4$ and most of the results are about $k = 3$. Our future work would like to consider the k-set tree connectivity of some network for $k \geq 5$ and it will be a great challenge.

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