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Article

Local Recovery of Magnetic Invariants from Local Length Measurements in Non-Reversible Randers Metrics

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Abstract

We study a purely local inverse problem for non-reversible Randers metrics $F = \|\cdot\|_g + \beta$ defined on smooth oriented surfaces. Using only the lengths of sufficiently small closed curves around a point p , we prove that the exterior derivative $d\beta(p)$ can be uniquely and stably recovered. Moreover, we establish that $d\beta(p)$ is the only second-order local invariant retrievable from such local length measurements. Our approach is entirely metric-based, independent of geodesic flows or boundary data, and naturally extends to general curved surfaces.

Keywords: Randers metrics; non-reversible Finsler geometry; local inverse problem; magnetic invariants; local length measurements; exterior derivative; geometric reconstruction

1. Introduction

Inverse problems in differential geometry aim to recover hidden geometric structures from indirect measurements. In the Riemannian setting, rigidity phenomena typically require global data, such as boundary distance functions or complete knowledge of the geodesic flow [1–3].

A fundamental obstruction to purely local recovery in the Riemannian case is metric reversibility, which eliminates all orientation-sensitive information. As a result, local length data around a point fails to encode any nontrivial invariant.

The Randers metrics,

$$F(v) = \|v\|_g + \beta(v) \tag{1}$$

form a natural class of non-reversible Finsler metrics, originally introduced by Randers and systematically developed within Finsler geometry; see [4,5]. The one-form β introduces a directional drift, breaking time-reversal symmetry and giving rise to new geometric phenomena.

We address the following question: What geometric information can be recovered from purely local length measurements around a single point?

We show that non-reversibility fundamentally changes the answer: the magnetic two-form $d\beta$ can be recovered locally from length data, and this invariant is provably minimal.

Randers metrics are closely related to magnetic systems, where $d\beta$ plays the role of the magnetic field [6,7]. Rigidity results for magnetic flows typically rely on global assumptions, such as simplicity or full boundary data [7,8].

Local inverse problems are significantly more restrictive. In the reversible Riemannian setting, local length spectra contain no orientation-sensitive information, leading to complete local indeterminacy [3]. Several recent works investigate related questions in the context of Randers metrics and magnetic rigidity, but with global or boundary data. Notably, boundary rigidity results for Randers metrics have been established in [9], while magnetic boundary and scattering rigidity are studied in [10]. Other works on local reconstruction along geodesics in Finsler geometries are found in [11], and structural properties of Randers metrics with linear connections are discussed in [12]. A general

overview of modern inverse problems and reconstruction techniques can be found in [13]. These studies provide context for our results, which focus on purely local recovery at a single point.

Definition 1. Let (M, g) be a smooth oriented Riemannian surface, $p \in M$, and let β be a smooth one-form defined in a neighborhood of p . Assume $\|\beta\|_g < 1$ so that:

$$F(v) = \|v\|_g + \beta(v) \quad (1)$$

defines a Randers metric.

All curves considered below are assumed to lie inside a sufficiently small geodesic ball $B(p, \varepsilon)$.

Definition 2. Let γ be a smooth closed curve bounding a domain Σ . The Randers length decomposes as:

$$L_F(\gamma) = \int_{\gamma} \|\dot{\gamma}\|_g dt + \int_{\gamma} \beta \quad (2)$$

By Stokes' theorem,

$$\int_{\gamma} \beta = \int_{\Sigma} d\beta \quad (3)$$

Thus all orientation dependence in the length functional arises exclusively from $d\beta$.

Lemma 1. Let (M, g) be a smooth Riemannian surface, $p \in M$, and let $\gamma_{\varepsilon} \subset B(p, \varepsilon)$ be a smooth closed curve contained in a sufficiently small geodesic ball around p . Then, in geodesic normal coordinates centered at p ,

$$\int_{\gamma_{\varepsilon}} \|\dot{\gamma}_{\varepsilon}\|_g dt = L_0(\gamma_{\varepsilon}) + O(\varepsilon^3) \quad (4)$$

where $L_0(\gamma_{\varepsilon})$ denotes the Euclidean length of the rescaled curve in normal coordinates. In particular, no orientation-dependent term of order ε^2 appears.

Proof. Let (x^1, x^2) be geodesic normal coordinates centered at p , so that the metric satisfies the classical expansion:

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}(p) x^k x^l + O(|x|^3) \quad (5)$$

where R_{ikjl} denotes the Riemann curvature tensor at p .

Consider a fixed smooth closed curve $\sigma : [0, 1] \rightarrow \mathbb{R}^2$ and define the rescaled curve:

$$\gamma_{\varepsilon}(t) = \varepsilon \sigma(t), \quad \dot{\gamma}_{\varepsilon}(t) = \varepsilon \dot{\sigma}(t) \quad (6)$$

Then, substituting into the metric expansion, we obtain:

$$g_{ij}(\gamma_{\varepsilon}) \dot{\gamma}_{\varepsilon}^i \dot{\gamma}_{\varepsilon}^j = \varepsilon^2 \|\dot{\sigma}\|^2 - \frac{\varepsilon^4}{3} R_{ikjl}(p) \sigma^k \sigma^l \dot{\sigma}^i \dot{\sigma}^j + O(\varepsilon^5) \quad (7)$$

Taking the square root and using the Taylor expansion $\sqrt{1+u} = 1 + \frac{1}{2}u + O(u^2)$, we get:

$$\|\dot{\gamma}_{\varepsilon}\|_g = \varepsilon \|\dot{\sigma}\| - \frac{\varepsilon^3}{6} \frac{R_{ikjl}(p) \sigma^k \sigma^l \dot{\sigma}^i \dot{\sigma}^j}{\|\dot{\sigma}\|} + O(\varepsilon^4) \quad (8)$$

Notice that no term of order ε^2 appears, and the ε^3 term is invariant under orientation reversal $t \mapsto 1-t$ up to higher-order terms.

Finally, integrating along γ_{ε} gives:

$$\int_{\gamma_{\varepsilon}} \|\dot{\gamma}_{\varepsilon}\|_g dt = L_0(\gamma_{\varepsilon}) + O(\varepsilon^3) \quad (9)$$

which proves the claim. $\square$

2. Main Results

Lemma 2 (Local Recovery of $d\beta$). *Let $\Sigma_\varepsilon \subset B(p, \varepsilon)$ be a smooth domain. Then:*

$$\int_{\Sigma_\varepsilon} d\beta = d\beta(p) \text{Area}_{\text{Eucl}}(\Sigma_\varepsilon) + O(\varepsilon^3) \quad (10)$$

where the remainder is uniform over all such domains.

Proof. Let $\{\gamma_\varepsilon\}$ be a family of smooth positively oriented closed curves contained in $B(p, \varepsilon)$, each bounding a domain Σ_ε .

By Lemma 1, the Riemannian contribution satisfies:

$$\int_{\gamma_\varepsilon} \|\dot{\gamma}_\varepsilon\|_g dt = L_0(\gamma_\varepsilon) + O(\varepsilon^3) \quad (11)$$

which is invariant under orientation reversal up to order ε^2 .

Since $d\beta$ is smooth, its Taylor expansion at p gives:

$$\int_{\Sigma_\varepsilon} d\beta = d\beta(p) \text{Area}_{\text{Eucl}}(\Sigma_\varepsilon) + O(\varepsilon^3) \quad (12)$$

uniformly over all such domains.

Reversing the orientation of γ_ε changes the sign of $\int_{\gamma_\varepsilon} \beta$, so that:

$$\frac{1}{2}(L_F(\gamma_\varepsilon) - L_F(\gamma_\varepsilon^{-1})) = d\beta(p) \text{Area}_{\text{Eucl}}(\Sigma_\varepsilon) + O(\varepsilon^3) \quad (13)$$

Dividing by $\text{Area}_{\text{Eucl}}(\Sigma_\varepsilon)$ and letting $\varepsilon \rightarrow 0$ recovers $d\beta(p)$ uniquely.

Remark. In geodesic normal coordinates, the Jacobian determinant of the exponential map satisfies

$$\det D \exp_p(x) = 1 + O(|x|^2) \quad (14)$$

so that Euclidean and Riemannian areas coincide up to $O(\varepsilon^4)$, which is negligible at second-order asymptotics. $\square$

Theorem 1 (Uniqueness of Local Recovery). *The lengths of all sufficiently small closed curves around p uniquely determine $d\beta(p)$.*

Proof. The length functional admits an asymptotic expansion in powers of ε . The first-order term coincides with the Euclidean length and carries no geometric information.

At second order, the expansion splits into:

1. a symmetric term determined by the Riemannian metric g ,
2. an antisymmetric term given by $\int_{\Sigma_\varepsilon} d\beta$.

By Lemma 2, the symmetric term is invariant under orientation reversal and carries no local oriented information. Thus, the second-order antisymmetric jet uniquely encodes $d\beta$ at p . $\square$

Corollary 1 (Minimality of the Invariant). *Let $F_i = \|\cdot\|_{g_i} + \beta_i$, $i = 1, 2$, be two Randers metrics with $d\beta_1(p) = d\beta_2(p)$. Then their length functionals agree up to order ε^2 on all closed curves near p .*

Proof. Immediate from Theorem 1: if $d\beta_1(p) = d\beta_2(p)$, the antisymmetric second-order jets coincide, so all length differences are $O(\varepsilon^3)$. $\square$

Remark 1. This obstruction reflects a jet-level gauge invariance:

$$\beta \mapsto \beta + df, \quad \text{leaving } d\beta \text{ unchanged.}$$

Proposition 1 (Stability of Local Recovery). Fix $\varepsilon > 0$ sufficiently small, independently of δ . Let $L_{\text{meas}}(\gamma)$ denote measured lengths, and let $d\beta_{\text{rec}}(p)$ be the reconstructed magnetic field from these measurements.

Assume

$$\sup_{\gamma \subset B(p, \varepsilon)} |L_{\text{meas}}(\gamma) - L_F(\gamma)| \leq \delta \quad (15)$$

Then

$$\|d\beta_{\text{rec}}(p) - d\beta(p)\| \leq C\delta \quad (16)$$

where C depends only on uniform C^2 bounds on g , C^1 bounds on β , and lower bounds on the injectivity radius.

Proof. Reconstruction uses the antisymmetric part of the length functional:

$$d\beta(p) = \lim_{\varepsilon \rightarrow 0} \frac{L_F(\gamma_\varepsilon) - L_F(\gamma_\varepsilon^{-1})}{2 \text{Area}_{\text{Eucl}}(\Sigma_\varepsilon)} \quad (17)$$

where γ_ε is a smooth positively oriented curve in $B(p, \varepsilon)$ and Σ_ε its enclosed domain.

With measurement error δ , we have:

$$|(L_{\text{meas}}(\gamma_\varepsilon) - L_{\text{meas}}(\gamma_\varepsilon^{-1})) - (L_F(\gamma_\varepsilon) - L_F(\gamma_\varepsilon^{-1}))| \leq 2\delta \quad (18)$$

Dividing by $2 \text{Area}_{\text{Eucl}}(\Sigma_\varepsilon)$ and using standard area estimates:

$$c_1 \varepsilon^2 \leq \text{Area}_{\text{Eucl}}(\Sigma_\varepsilon) \leq c_2 \varepsilon^2 \quad (19)$$

we obtain:

$$\left| d\beta_{\text{rec}}(p) - \frac{L_F(\gamma_\varepsilon) - L_F(\gamma_\varepsilon^{-1})}{2 \text{Area}_{\text{Eucl}}(\Sigma_\varepsilon)} \right| \leq \frac{\delta}{c_1 \varepsilon^2} \quad (20)$$

Since

$$\frac{L_F(\gamma_\varepsilon) - L_F(\gamma_\varepsilon^{-1})}{2 \text{Area}_{\text{Eucl}}(\Sigma_\varepsilon)} = d\beta(p) + O(\varepsilon) \quad (21)$$

combining estimates yields:

$$\|d\beta_{\text{rec}}(p) - d\beta(p)\| \leq C\delta \quad (22)$$

with C depending only on uniform bounds on g , β , and the injectivity radius. $\square$

3. Extension to Curved Surfaces

The reconstruction and minimality results extend naturally to arbitrary smooth Riemannian surfaces. Let (M, g) be a smooth surface and $p \in M$. In geodesic normal coordinates centered at p , the Riemannian length of a sufficiently small closed curve γ_ε admits the expansion:

$$L_g(\gamma_\varepsilon) = \varepsilon L_0(\sigma) + O(\varepsilon^3), \quad (23)$$

where $\sigma(t) = \gamma_\varepsilon(t)/\varepsilon$ is the rescaled curve in Euclidean coordinates, and $L_0(\sigma)$ denotes its Euclidean length. Curvature-dependent contributions appear only at order $O(\varepsilon^3)$ and are invariant under orientation reversal. Therefore, the antisymmetric part of the length functional is unaffected up to second order, and the local behavior to order ε^2 remains insensitive to curvature. Consequently, the proofs of the reconstruction and minimality theorems carry over from the Euclidean setting without modification.

To illustrate concretely, consider the standard 2-sphere $S^2 \subset \mathbb{R}^3$ of radius R , endowed with the induced Riemannian metric g_{S^2} . Let $p \in S^2$ and β be a smooth one-form defined in a neighborhood of p such that:

$$\|\beta\|_{g_{S^2}} < 1.$$

The local Randers metric:

$$F(v) = \|v\|_{g_{S^2}} + \beta(v), \quad v \in T_p S^2,$$

is well-defined. Let $\gamma_\epsilon \subset B_{S^2}(p, \epsilon)$ be a smooth closed curve contained in a geodesic ball of radius ϵ around p . Its length decomposes as:

$$L_F(\gamma_\epsilon) = \int_{\gamma_\epsilon} \|\dot{\gamma}_\epsilon\|_{g_{S^2}} dt + \int_{\gamma_\epsilon} \beta. \quad (24)$$

In geodesic normal coordinates, the Riemannian contribution satisfies:

$$\int_{\gamma_\epsilon} \|\dot{\gamma}_\epsilon\|_{g_{S^2}} dt = L_0(\gamma_\epsilon) + O(\epsilon^3), \quad (25)$$

so that the $O(\epsilon^2)$ term vanishes. The antisymmetric part is therefore entirely determined by the drift one-form β . Applying Stokes' theorem on the domain $\Sigma_\epsilon \subset B_{S^2}(p, \epsilon)$ bounded by γ_ϵ yields:

$$\int_{\gamma_\epsilon} \beta = \int_{\Sigma_\epsilon} d\beta. \quad (26)$$

Reversing the orientation of γ_ϵ gives γ_ϵ^{-1} , and the antisymmetric part of the length functional becomes:

$$L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1}) = 2 \int_{\Sigma_\epsilon} d\beta. \quad (27)$$

Dividing by twice the Euclidean area of Σ_ϵ and taking the limit $\epsilon \rightarrow 0$ recovers the local magnetic invariant:

$$d\beta(p) = \lim_{\epsilon \rightarrow 0} \frac{L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1})}{2 \text{Area}_{\text{Eucl}}(\Sigma_\epsilon)}. \quad (28)$$

The curvature of the surface contributes only at order $O(\epsilon^3)$, and the same argument applies to other smooth surfaces such as tori or ellipsoids, provided that the closed curves γ_ϵ remain sufficiently small. This demonstrates that the local inverse problem for non-reversible Randers metrics extends seamlessly from Euclidean neighborhoods to general curved surfaces.

4. Discussion

In this work, we have shown that for a Randers metric:

$$F = \|\cdot\|_g + \beta \quad (29)$$

defined on a smooth oriented surface, the exterior derivative $d\beta(p)$ can be uniquely and stably recovered from the lengths of sufficiently small closed curves surrounding a point p .

The first-order term in the asymptotic expansion of the length functional coincides with the Euclidean length in geodesic normal coordinates and carries no local geometric information. The second-order expansion splits into a symmetric component determined by the Riemannian metric g and an antisymmetric component given by:

$$\int_{\Sigma_\epsilon} d\beta, \quad (30)$$

which encodes all local orientation-sensitive information. Theorem 1 establishes the uniqueness of $d\beta(p)$, while Corollary 1 shows that this invariant is minimal: no other independent local invariant

is detectable at second-order asymptotics. Proposition 1 guarantees stability, ensuring that small measurement errors induce proportionally small deviations in the reconstructed $d\beta(p)$. The extension to curved surfaces confirms that curvature contributions appear only at $O(\varepsilon^3)$, leaving second-order reconstruction unaffected.

These results underscore a fundamental distinction between reversible Riemannian and non-reversible Finsler geometries. In Riemannian metrics, local length data alone is insufficient to recover any orientation-sensitive invariant due to metric reversibility. In contrast, the non-reversibility of Randers metrics introduces a directional drift, captured by $d\beta$, which is fully recoverable from local measurements. Prior works on magnetic rigidity or boundary rigidity [6–8] typically rely on global or boundary data, whereas our approach uses purely local metric information.

Geometrically, the antisymmetric second-order jet of the length functional defines a jet-level gauge-invariant:

$$\beta \mapsto \beta + df \quad \Rightarrow \quad d\beta \text{ unchanged}, \quad (31)$$

demonstrating that $d\beta(p)$ is the maximal local geometric invariant accessible from lengths of small closed curves. Higher-order terms ($O(\varepsilon^3)$) and symmetric second-order contributions from g do not introduce additional independent information.

These findings have several implications. They enable local treatment of inverse problems in Finsler geometry without requiring global boundary data. For magnetic systems, this provides a purely metric-based method to reconstruct the local magnetic field. Potential extensions include higher-dimensional manifolds and more general non-reversible Finsler metrics, though additional invariants may emerge at higher orders.

Limitations of the current analysis include the smoothness requirements (C^2 for g , C^1 for β) and the smallness of the curves (ε sufficiently small). Extensions to larger curves or global recovery may necessitate further geometric or topological assumptions. Investigating higher-order asymptotics could reveal additional invariants in more general Finsler geometries.

In conclusion, $d\beta(p)$ is the unique and minimal local invariant encoded by purely metric length data in non-reversible Randers metrics. This establishes a clear boundary between recoverable and unrecoverable information in local Finsler inverse problems and provides a stable framework for reconstruction.

Appendix A. Jets and Gauge Invariance

Here the jet is understood in the sense of asymptotic expansions of the length functional with respect to the scaling parameter ε .

We clarify here the geometric mechanism underlying the minimality result (Theorem 1).

Let $F = \|\cdot\|_g + \beta$ be a Randers metric defined near p . The length functional assigns to each sufficiently small closed curve γ_ε a real number $L_F(\gamma_\varepsilon)$. This defines a functional on the space of curve germs at p .

The Taylor expansion of $L_F(\gamma_\varepsilon)$ in powers of ε defines its *jet* at p . The first-order jet depends only on the Euclidean metric and contains no geometric information. The second-order jet decomposes canonically into:

1. a symmetric part, arising from the Riemannian metric g ,
2. an antisymmetric part, arising from the one-form β .

The symmetric part is invariant under orientation reversal and therefore cannot be detected by comparing lengths of γ_ε and γ_ε^{-1} . This is the fundamental obstruction in the Riemannian case (cf. [3]).

The antisymmetric part is given by:

$$\gamma_\varepsilon \mapsto \int_{\gamma_\varepsilon} \beta = \int_{\Sigma_\varepsilon} d\beta,$$

which depends only on the exterior derivative $d\beta$. In particular, replacing β by $\beta + df$ leaves all length data unchanged. Thus $d\beta$ is invariant under the natural gauge action:

$$\beta \mapsto \beta + df.$$

From a jet-theoretic viewpoint, this implies that $d\beta(p)$ is the only nontrivial second-order invariant of the length functional. Any higher-order information is suppressed by the $O(\epsilon^3)$ remainder, while any symmetric second-order contribution is invisible due to reversibility.

Therefore, $d\beta(p)$ represents the maximal recoverable local invariant encoded in purely metric length data. This completes the obstruction argument underlying Theorem 1.

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