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Article

Pipe Flow of Viscoplastic Fluids and Analytical Predictions of Concrete Pumping Based on the Shear-stress-dependent Parabolic Model

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Abstract: This study investigates the Hagen–Poiseuille pipe flow of viscoplastic fluids, focusing on analytical predictions of concrete pumping using the shear-stress-dependent parabolic model. Research highlights encompass solving the steady laminar pipe flow for viscoplastic fluids described by the parabolic model, extending analytical studies to a nonlinear rheological model with easily accessible experimental parameters, presenting detailed results for the two-fluid parabolic model, and introducing computational apps implementing theoretical findings. The parabolic model outperforms linear models, such as the Bingham model, in accuracy by accounting for the nonlinearity in the shear stress and shear rate relations. The influence of rheological parameters on these relations is analyzed, and their versatility is demonstrated by a Wolfram Mathematica-based application program. This approach is adaptable for other models with shear stress as the independent variable, offering valuable insights into viscoplastic fluid flows.

Keywords: Pipe flow; Viscoplastic fluid; Concrete pumping; Lubrication Layer; Rheology; Yield stress; shear-thickening

1. Introduction

Viscoplastic fluids are a type of non-Newtonian fluids that exhibit yield stress. For these fluids to flow, applied stress must exceed a critical value known as “yield stress” [1–6]. The yield stress is therefore characterized as the material’s resistance to the initiation of flow. According to historical records [1], viscoplasticity was first discovered in the 1890s by Schwedoff when he used a Couette instrument to do experiments on colloidal gelatin solutions. His findings revealed a nonlinear relationship between the torque and angular velocity in this instrument, making his experiments to be the first set of measurements of non-Newtonian behavior. To describe his results, he had to include a yield stress value. Following this, the work of Bingham and Green in the 1920s led to broad acknowledgment that some fluids display yield stress behavior [1,7–9]. Many industrially important materials, such as concentrated suspensions [10,11], red mud residues [12], pastes [13], food products [14], emulsions [15], foams [16], waxy crude oils [17], fiber-reinforced plastics [18], and other composites, are viscoplastic. Concrete is another example of viscoplastic fluid [19–22], which is the most widely used man-made material in the world. It is indispensable for numerous major infrastructure developments, from buildings, roads, bridges, and high-speed rail facilities to renewable energy applications.

Concrete pumping is a common placement technique that makes it possible to deliver concrete considerably more quickly, which speeds up construction and thus reduces costs [23–25]. At the same time, the effective application of pumping technology can offer excellent reliability and productivity during the construction process while reducing the danger of failure involving human error. Therefore, there is a significant economic benefit to understanding and precisely defining the pumping behavior of concrete [25]. However,

pumping concrete is not a simple process, and many accidents occur every year as a result of pump or pipeline blockages, blowouts, or breaks [23,24]. Consequently, a prior prediction of concrete flow rates, which influence construction process duration, is required to apply concrete pumping to large-scale construction projects [26].

Using known rheological models, several attempts have been made to estimate concrete flow in pumping. The majority of approaches to concrete pumping have considered the yield stress and the presence of the unsheared zone by presuming that concrete behaves as a Bingham fluid or Herschel–Bulkley fluid. But without taking into account the fact that there is a lubricating layer at the interface between concrete and the pipe, these approaches nearly always failed to estimate correctly pumping flow rate on a wide range of concrete fluidity [26]. The possible mechanisms that contribute to the formation of the lubrication layer (LL) include the wall effect phenomenon caused by particle–wall interactions and shear-induced particle migration (SIPM) [27–32].

Kaplan *et al.* [33,34] conducted a pioneering study on predicting pumping performance, which proved that the lubricating layer is important in facilitating concrete pumping because it has much lower viscosity and yield stress than concrete [30,31]. In 2013, Kwon *et al.* [31,35], followed by Khatib and Khayat [36], developed analytical predictions on concrete pumping that includes the rheological properties of the lubrication layer fluid. Prior theoretical work of Kaplan [33], Kwon *et al.* [35], as well as Khatib and Khayat [36] models used the classical Bingham model to characterize the rheological behavior of bulk concrete and the lubrication layer fluid, which sets a linear relationship between shear stress (τ) and shear rate ($\dot{\gamma}$) as given by

$$\begin{cases} \tau = \tau_0 + \mu_p \dot{\gamma} & \text{for } \tau > \tau_0 \\ \dot{\gamma} = 0 & \text{for } \tau \leq \tau_0 \end{cases} \quad (1)$$

where $\tau_0 \geq 0$ and $\mu_p > 0$ are the two parameters of the Bingham model: yield stress and plastic viscosity, respectively.

Although the Bingham model is frequently used for cement-based materials, it has been extensively reported that the relationship between shear stress and shear rate for fresh concrete, mortar, and cement paste is not exactly linear [20,37–40]. Since the Bingham model does not account for this non-linearity, two commonly used rheological models to describe the shear-thickening behavior of fresh self-compacting concrete are the Herschel–Bulkley and the modified Bingham models [37,38,40]. Recently, Zhaidarbek *et al.* [41] developed a flow rate–pressure drop relation for both dual-fluid Herschel–Bulkley and the dual-fluid modified Bingham models, where the two fluids are bulk concrete and the lubrication layer fluid. The constitutive equation for the Herschel–Bulkley model is given by

$$\begin{cases} \tau = \tau_0 + K \dot{\gamma}^n & \text{for } \tau > \tau_0 \\ \dot{\gamma} = 0 & \text{for } \tau \leq \tau_0 \end{cases} \quad (2)$$

Herschel–Bulkley model has three parameters: yield stress $\tau_0 \geq 0$, flow consistency index $K > 0$, and flow behavior index (dimensionless) $n > 0$. For $n < 1$, the fluid is shear-thinning, and for $n > 1$, the fluid is shear-thickening. If $n = 1$, the Herschel–Bulkley fluid model reduces to the Bingham model with plastic viscosity $\mu_p = K$.

The constitutive equation of the modified Bingham model is

$$\begin{cases} \tau = \tau_0 + \mu_p \dot{\gamma} + A_2 \dot{\gamma}^2 & \text{for } \tau > \tau_0 \\ \dot{\gamma} = 0 & \text{for } \tau \leq \tau_0 \end{cases} \quad (3)$$

which also has three parameters: yield stress $\tau_0 \geq 0$, plastic viscosity $\mu_p > 0$, and a second-order coefficient A_2 that describes the deviations from the linear relation between shear stress and shear rate as described by Bingham model in Eq. (1). For $A_2 > 0$, the modified Bingham model shows a shear-thickening behavior, and for $A_2 < 0$, the model displays a

shear-thinning behavior. Similarly, for $A_2 = 0$, the modified Bingham model reduces to the Bingham model in Eq. (1).

Although there seems to be a sufficient study on the pumping behavior of concrete and lubrication layer properties based on nonlinear rheological models, the proposed models can be difficult to employ in rheological experiments. As shown by Li *et al.* [42] by solving the Couette inverse program, the resulting relationships between torque and rotational speed in a coaxial cylinder rheometer for the Herschel–Bulkley model and the modified Bingham model are rather bulky and are difficult to apply in regression analysis of experimental data. In comparison, the shear-stress-dependent Parabolic model, which contains a nonlinear term between shear stress and shear rate, has a resulting torque–angular frequency relation described by a simple quadratic function with easy-to-access parameters. Therefore, according to Li *et al.* [42], among all the commonly used three-parameter rheological models for yield-stress fluids, the parabolic model is the most suitable for the analysis of the torque–angular frequency data from a rotational rheometer.

The parabolic model was first introduced in 1985 by Atzeni *et al.* [43] as a parabolic-type empirical law that relates the shear rate to the stress as

$$\dot{\gamma} = a + b\tau + c\tau^2 \tag{4}$$

It was noted that the parabolic model offered a good fit with the experimental data along with Herschel–Bulkley and Eyring’s models and important rheological parameters like yield stress (τ_0) and viscosity (μ_p) can easily be derived with this model. Li *et al.* [42] emphasized that the parabolic model’s introduction as an inverse function makes it easier to solve the Couette inverse problem based on this model, which deals with the relationship between rotational speed and torque. In a more recent study [44], the modified Bingham model and the parabolic model were compared, and the results showed that the parabolic model provided a more precise characterization of the paste’s flowing performance. The rheological parameters for pastes obtained based on the modified Bingham model lacked credibility in comparison to the parabolic model.

Traditionally, most of the constitutive models use shear rate as the independent variable and are expressed in the form of $\tau = f(\dot{\gamma})$. However, such constitutive models may also be expressed in the form of $\dot{\gamma} = \dot{\gamma}(\tau) = f^{-1}(\tau)$ by finding their inverse functions. There are models, such as the Ellis model [45], the Meter model [46], and the Parabolic model [43] that are commonly expressed in the form of $\dot{\gamma} = \dot{\gamma}(\tau)$ using the shear stress as the independent variable. Among them, the parabolic model has yield stress and can describe viscoplastic fluids such as cement pastes [42–44,47].

Despite the listed advantages of the parabolic model over other commonly used three-parameter rheological models, studies on the shear-stress-dependent parabolic model [42–44,47] are scarce. In particular, there is a lack of detailed analysis of the parabolic model, and the pipe flow problem of the parabolic fluid has not been solved so far both for single- and dual-fluid cases. The main research objectives of this paper are to: (1) Construct a theoretical framework suitable to solve the Hagen–Poiseuille pipe flow problem for viscoplastic fluids with shear-stress-dependent constitutive models; (2) Advance the state-of-the-art in rheology-based analytical models for predicting flow rate vs. pressure drop relationships in concrete pumping; (3) Using the parabolic model as a representative case, solve the associated Hagen–Poiseuille pipe flow problem and develop analytical models for concrete pumping predictions.

The structure of this paper is as follows: Section 2 presents the detailed analysis of the parabolic model including the derivation of the Fanning friction factor for the model, a ready-to-use method for obtaining the flow rate–pressure drop relation for the steady Hagen–Poiseuille coaxial flow of two immiscible, incompressible fluids under no-slip boundary conditions for all constitutive models that use shear stress as the independent variables, and key analytical results for the parabolic model. Section 3 presents selected graphical representations of analytical results for the parabolic model along with some

discussion. Section 4 includes the limitations of the present work and future directions. Finally, in section 5, we summarize the work.

2. Theoretical Framework and Derivations

2.1. Analysis of the parabolic model

Generally speaking, constitutive models for incompressible viscoplastic fluids can be classified as those using the shear rate as the independent variable, i.e., $\tau = \tau(\dot{\gamma})$, and those using the shear stress as the independent variable, i.e., $\dot{\gamma} = \dot{\gamma}(\tau)$. While it is a common practice to introduce the viscosity function (SI unit: $\text{Pa} \cdot \text{s}$) as

$$\eta(\dot{\gamma}) = \tau(\dot{\gamma}) / \dot{\gamma} \quad (5)$$

for the former class of constitutive models, it is more convenient to introduce the fluidity function (SI unit: $\text{Pa}^{-1} \cdot \text{s}^{-1}$) [1,48] as

$$\zeta(\tau) = \dot{\gamma}(\tau) / \tau \quad (6)$$

for the latter class of constitutive models.

The term in Eq. (4) with coefficient c brings in nonlinear dependence of the shear rate on the shear stress when $c \neq 0$. A careful analysis of the model shows that the fluid is Bingham plastic if $c = 0$, yield-pseudoplastic (i.e., shear thinning) if $c > 0$, and is yield-dilatant (also referred to as shear-thickening) if $c < 0$ as shown in Figure 1.

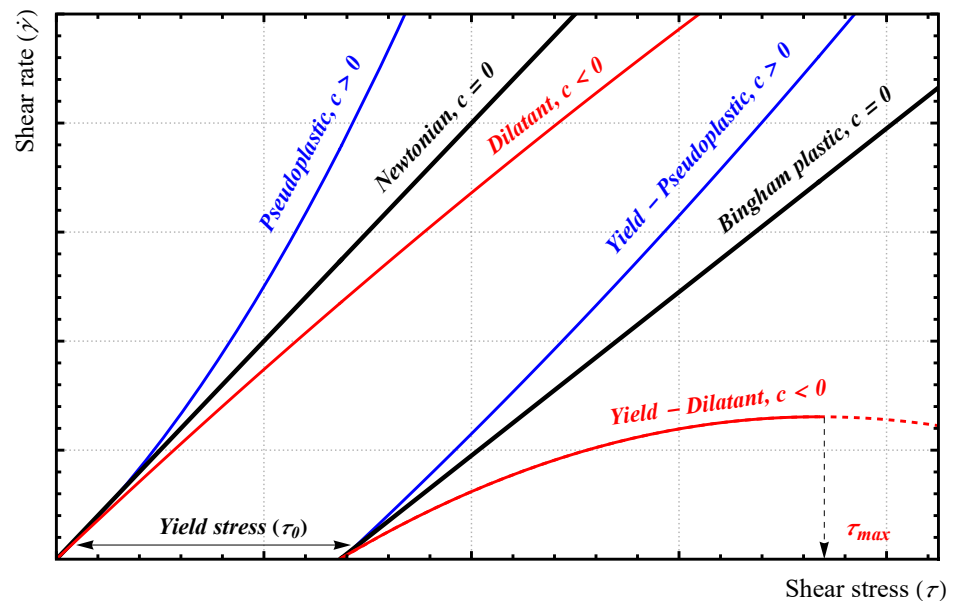


Figure 1. Graphical representation of the parabolic model

From Figure 1, the correct form of the parabolic model should be expressed as

$$\dot{\gamma} = \begin{cases} 0 & \text{for } \tau \leq \tau_0 \\ a + b\tau + c\tau^2 & \text{for } \tau_0 < \tau < \tau_{\max} \end{cases} \quad (7)$$

where τ_0 is the yield stress and is obtained by requiring the shear rate given by Eq. (7) to be a continuous function. We have

$$\tau_0 = \begin{cases} (-b + \sqrt{b^2 - 4ac}) / (2c) & \text{for } c \neq 0 \\ -a/b & \text{for } c = 0 \end{cases} \quad (8)$$

The above expression requires $b^2 - 4ac \geq 0$. The term $\tau_{\max}$ in Eq. (7) corresponds to an upper limit in the shear stress required by the model for the case of $c < 0$, and

$$\tau_{\max} = \begin{cases} +\infty & \text{for } c \geq 0 \\ -b/(2c) & \text{for } c < 0 \end{cases} \quad (9)$$

Note that $\tau_{\max} = +\infty$ is equivalent to saying that there is no restriction on the upper limit of shear stress in the model. Moreover, note that at $a = 0$, we have $\tau_0 = 0$, i.e., the fluid does not have a yield stress.

The parabolic model, given by in Eq. (7), belongs to the class of constitutive models that use the shear stress as the independent variable, and its fluidity function is given by

$$\zeta(\tau) = \begin{cases} 0 & \text{for } \tau \leq \tau_0 \\ a\tau^{-1} + b + c\tau & \text{for } \tau_0 < \tau < \tau_{\max} \end{cases} \quad (10)$$

In the case of $c = 0$, the model reduces to the classical Bingham model with yield stress τ_0^B and plastic viscosity μ_p^B , where

$$\tau_0^B = -a/b, \quad \mu_p^B = 1/b \quad (11)$$

Thus, it is required that $b > 0$ and $a \leq 0$ for the model to be of practical use.

Present here the ratio between the true yield stress of the parabolic model, τ_0 , and τ_0^B :

$$\frac{\tau_0}{\tau_0^B} = \frac{-1 + \sqrt{1 + 4\alpha}}{2\alpha} \quad (12)$$

where α is a dimensionless number defined as

$$\alpha = (-a)c/b^2 \quad (13)$$

In the case of $c \rightarrow 0$, we have $\alpha \rightarrow 0$ and $\tau_0/\tau_0^B \rightarrow 1$. The relationship between τ_0/τ_0^B vs. α for $\alpha \in [-1/4, 2]$ is shown in Figure 2.

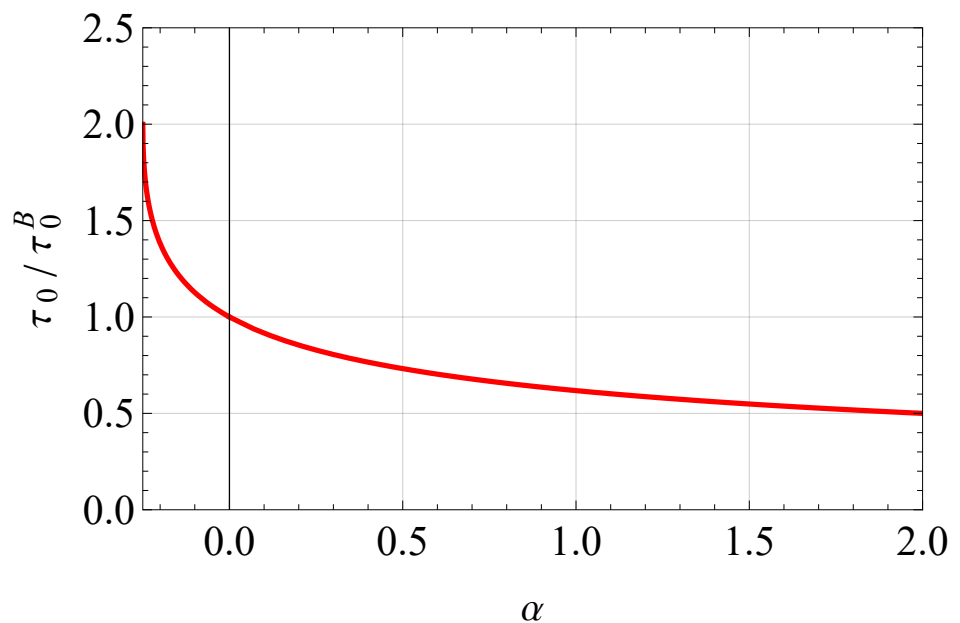


Figure 2. Relationship between τ_0/τ_0^B and α

Table 1. The SI units and dimensions in terms of the MLT system associated with the variables and parameters of the parabolic model.

Physical quality	Symbol	SI unit	Dimension
Shear rate	$\dot{\gamma}$	s^{-1}	T^{-1}
Shear stress	τ	Pa	$\text{M} \cdot \text{L}^{-1} \cdot \text{T}^{-2}$
Yield stress	τ_0	Pa	$\text{M} \cdot \text{L}^{-1} \cdot \text{T}^{-2}$
a -parameter	a	s^{-1}	T^{-1}
b -parameter	b	$\text{Pa}^{-1} \cdot \text{s}^{-1}$	$\text{M}^{-1} \cdot \text{L} \cdot \text{T}$
c -parameter	c	$\text{Pa}^{-2} \cdot \text{s}^{-1}$	$\text{M}^{-2} \cdot \text{L}^2 \cdot \text{T}^3$

2.2. Hagen–Poiseuille equation for non-Newtonian pipe flow: Single fluid

Hagen–Poiseuille equation is a physical law that gives the pressure drop in an incompressible Newtonian fluid in laminar flow flowing through a long cylindrical pipe of a constant cross-section [49]. This section will introduce the general method for obtaining the Hagen–Poiseuille equation for constitutive models that use shear stress as the independent variable.

Let G be the pressure drop per unit length of the pipe, i.e.,

$$G = (\mathcal{P}_0 - \mathcal{P}_L)/L \tag{14}$$

where $\mathcal{P}_0 - \mathcal{P}_L > 0$ is the pressure drop; $\mathcal{P}_0$ and $\mathcal{P}_L$ are, respectively, the “modified” pressures [50] at $z = 0$ and $z = L$. The shear stress (also known as momentum flux) distribution is obtained as

$$\tau_{rz}(r) = \frac{1}{2} \frac{\Delta \mathcal{P}}{L} r = \frac{1}{2} Gr \tag{15}$$

Here, $G \equiv \Delta \mathcal{P}/L$ is the pressure drop per unit length of the pipe given by Eq. (14). From Eq. (15) the shear stress at the pipe wall (where $r = R$) is obtained as

$$\tau_{\text{wall}} \equiv \tau_{rz}(r = R) = \frac{1}{2} GR \tag{16}$$

For the parabolic model, the wall shear rate is given as

$$\dot{\gamma}_{\text{wall}} = a + b\tau_{\text{wall}} + c\tau_{\text{wall}}^2 \tag{17}$$

The yield radius, also called “plug” radius, is found by solving for r from the equation of $\tau_{rz}(r) = \tau_0$. The result is

$$r_{\text{plug}} = \frac{2\tau_0}{G} = \frac{\tau_0}{\tau_{\text{wall}}} R \geq 0 \tag{18}$$

In the region $r < r_{\text{plug}}$ the velocity of the fluid is constant across the cross-section perpendicular to the axis of the pipe (i.e., plug flow).

According to its definition, the volumetric flow rate can be obtained from the velocity distribution u_z as

$$Q = \int_0^R 2\pi r u_z(r) dr \tag{19}$$

Carrying out the integration by parts yields

$$Q/\pi = \left(r^2 u_z\right)\Big|_{r=0}^{r=R} + \int_0^R r^2 \left(-\frac{du_z}{dr}\right) dr \tag{20}$$

Assuming the no-slip velocity at the wall of the tube, i.e., $u_z(r = R) = 0$, the first term on the right-hand side of Eq. (20) is zero at both limits of the integration. Thus, we have

$$Q/\pi = \int_0^R r^2 \left(-\frac{du_z}{dr} \right) dr \quad (21)$$

In the plug-flow region, $r \in [0, r_{\text{plug}}]$, we have $-du_z/dr = 0$. In the shear-flow region, $r \in [r_{\text{plug}}, R]$, we have $-du_z/dr \equiv \dot{\gamma}$, which, for the parabolic model, can be expressed as a function of the shear stress, τ_{rz} . For the Hagen–Poiseuille flow in a pipe, we have a linear shear stress distribution of the form:

$$\tau_{rz}/\tau_{\text{wall}} = r/R \quad (22)$$

Thus, we have $r = \tau_{rz}R\tau_{\text{wall}}^{-1}$ and $dr = R\tau_{\text{wall}}^{-1}d\tau_{rz}$. Substituting the expression for r and dr into Eq. (21), the volumetric flow rate is now given by

$$Q/\pi = \int_{\tau_0}^{\tau_{\text{wall}}} \left(\tau_{rz}R\tau_{\text{wall}}^{-1} \right)^2 \dot{\gamma} \left(R\tau_{\text{wall}}^{-1}d\tau_{rz} \right) \quad (23)$$

After simplification, we have

$$Q = \frac{\pi R^3}{\tau_{\text{wall}}^3} \int_{\tau_0}^{\tau_{\text{wall}}} \tau_{rz}^2 \dot{\gamma} d\tau_{rz} = \frac{\pi R^3}{\tau_{\text{wall}}^3} \int_{\tau_0}^{\tau_{\text{wall}}} \tau_{rz}^3 \zeta(\tau_{rz}) d\tau_{rz} \quad (24)$$

where the variable of integration is the shear stress, τ_{rz} , and the shear rate, expressed as a function of the shear stress, depends on the rheological properties of the fluid; $\zeta(\tau_{rz})$ is the fluidity function as introduced in Eq. (6).

Introducing the dimensionless variables given in Table 2 to simplify the expressions, the shear rate and integral in Eq. (24) can be rewritten as

$$s = \frac{\dot{\gamma}}{\dot{\gamma}_{\text{wall}}} = \tilde{a} + \tilde{b}\tilde{\tau} + \tilde{c}\tilde{\tau}^2 \quad (25)$$

$$\tilde{u}_{\text{avg}} = \int_{\tilde{r}_{\text{plug}}}^1 \tilde{\tau}^2 s(\tilde{\tau}) d\tilde{\tau} \quad (26)$$

where dimensionless plug radius is found using the definitions of yield and wall shear stresses

$$\tilde{r}_{\text{plug}} = \frac{\tau_0}{\tau_{\text{wall}}} = \frac{GR/2}{Gr_{\text{plug}}/2} = \frac{r_{\text{plug}}}{R} \quad (27)$$

The Hagen–Poiseuille equation for the single fluid case using the parabolic model in terms of dimensionless variables summarized in Table 2, is thus calculated from

$$Q_{\text{HP}} = \pi R^3 \dot{\gamma}_{\text{wall}} \tilde{u}_{\text{avg}} \quad (28)$$

where $\dot{\gamma}_{\text{wall}}$ is given by Eq. (17) and $\tilde{u}_{\text{avg}}$ is given by Eq. (26).

From Eq. (26), the dimensionless average velocity for the parabolic model is found to be

$$\tilde{u}_{\text{avg}} = \frac{\tilde{a}}{3} \left(1 - \tilde{r}_{\text{plug}}^3 \right) + \frac{\tilde{b}}{4} \left(1 - \tilde{r}_{\text{plug}}^4 \right) + \frac{\tilde{c}}{5} \left(1 - \tilde{r}_{\text{plug}}^5 \right) \quad (29)$$

By substituting a, b parameters from Table 2 and setting $c = 0$, the Eq. (28) reduces to the Hagen–Poiseuille equation for the Bingham model as

$$Q_{\text{HP}}^{\text{B}} = \frac{\pi R^4 G}{8/b} \left[1 + \frac{4}{3} \frac{a/b}{\tau_{\text{wall}}} + \frac{1}{3} \left(\frac{a/b}{\tau_{\text{wall}}} \right)^4 \right] \quad (30)$$

Table 2. Nondimensionalization used in this work, following the same approach as that used in Ref. [51].

Quality	Symbol	Unit	Dimensionless variable
Radial distance	r	R	$\tilde{r} = r/R \in [0, 1]$
Shear stress	τ_{rz}	τ_{wall}	$\tilde{\tau} = \tau_{rz}(r)/\tau_{\text{wall}} = \tilde{r}$
Shear rate	$\dot{\gamma}$	$\dot{\gamma}_{\text{wall}}$	$s = \dot{\gamma}/\dot{\gamma}_{\text{wall}} \in [0, 1]$
a -parameter	a	$\dot{\gamma}_{\text{wall}}$	$\tilde{a} = a/\dot{\gamma}_{\text{wall}}$
b -parameter	b	$\tau_{\text{wall}}^{-1}\dot{\gamma}_{\text{wall}}$	$\tilde{b} = b/(\tau_{\text{wall}}^{-1}\dot{\gamma}_{\text{wall}})$
c -parameter	c	$\tau_{\text{wall}}^{-2}\dot{\gamma}_{\text{wall}}$	$\tilde{c} = c/(\tau_{\text{wall}}^{-2}\dot{\gamma}_{\text{wall}})$
Plug radius	r_{plug}	R	$\tilde{r}_{\text{plug}} = r_{\text{plug}}/R$
Velocity	u	$R\dot{\gamma}_{\text{wall}}$	$\tilde{u} = u/(R\dot{\gamma}_{\text{wall}})$

By substituting τ_0^B and μ_p^B Bingham model parameters given in Eq. (11), Eq. (30) can be rewritten as

$$Q_{\text{HP}}^B = \frac{\pi R^4 G}{8\mu_p^B} \left[1 - \frac{4}{3} \frac{\tau_0^B}{\tau_{\text{wall}}} + \frac{1}{3} \left(\frac{\tau_0^B}{\tau_{\text{wall}}} \right)^4 \right] \quad (31)$$

where $\tau_{\text{wall}} = GR/2$ as given in Eq. (16). Eq. (31) is known as the Buckingham-Reiner Equation [50,52].

The same flow rate–pressure drop relation can be obtained using a more traditional approach, namely, by integrating the velocity distribution. For the parabolic model, the governing ordinary differential equations (ODEs) for bulk concrete in shearing flow region, where $r_{\text{plug}} \leq r \leq R$, and plug flow region, where $r \leq r_{\text{plug}}$, are given by

$$\dot{\gamma} = -\frac{du_z(r)}{dr} = \begin{cases} a + b\tau + c\tau^2 & \text{for } r_{\text{plug}} \leq r \leq R \\ 0 & \text{for } r \leq r_{\text{plug}} \end{cases} \quad (32)$$

Introducing $\tilde{r} = r/R$ and $\tilde{u} = u/(R\dot{\gamma}_{\text{wall}})$ (see Table 2), Eq. (32) may be rewritten as

$$-\frac{d\tilde{u}}{d\tilde{r}} = \begin{cases} \tilde{a} + \tilde{b}\tilde{r} + \tilde{c}\tilde{r}^2 & \text{for } \tilde{r}_{\text{plug}} \leq \tilde{r} \leq 1 \\ 0 & \text{for } \tilde{r} \leq \tilde{r}_{\text{plug}} \end{cases} \quad (33)$$

By solving Eq. (32) with the no-slip boundary condition at the wall, i.e., $\tilde{u}(\tilde{r} = 1) = 0$, and at plug flow/shearing flow interface, the velocity profile at shearing and plug flow regions is obtained as

$$\tilde{u}(\tilde{r}) = \begin{cases} \tilde{a}(1 - \tilde{r}) + \frac{\tilde{b}}{2}(1 - \tilde{r}^2) + \frac{\tilde{c}}{3}(1 - \tilde{r}^3) & \text{for } \tilde{r}_{\text{plug}} \leq \tilde{r} \leq 1 \\ \tilde{a}(1 - \tilde{r}_{\text{plug}}) + \frac{\tilde{b}}{2}(1 - \tilde{r}_{\text{plug}}^2) + \frac{\tilde{c}}{3}(1 - \tilde{r}_{\text{plug}}^3) & \text{for } \tilde{r} \leq \tilde{r}_{\text{plug}} \end{cases} \quad (34)$$

For laminar flow in a pipe, once the steady velocity profile is known, $u_z(r)$, the volumetric flow rate, Q , is obtained as

$$Q = \int_0^R u_z(r) 2\pi r dr = \pi R^3 \dot{\gamma}_{\text{wall}} \tilde{u}_{\text{avg}} \quad (35)$$

where the dimensionless average flow velocity is given by

$$\tilde{u}_{\text{avg}} = \int_0^1 \tilde{u}(\tilde{r}) 2\tilde{r} d\tilde{r} = \tilde{u}_{\text{plug}} \tilde{r}_{\text{plug}}^2 + \int_{\tilde{r}_{\text{plug}}}^1 \tilde{u}(\tilde{r}) 2\tilde{r} d\tilde{r} = \tilde{V}_{\text{plug}} + \tilde{V}_{\text{shear}} \quad (36)$$

Here, the contributions from the shearing flow region and the plug-flow region are given by

$$\begin{aligned}\tilde{V}_{\text{shear}} &= \int_{\tilde{r}_{\text{plug}}}^R \tilde{u}(\tilde{r}) 2\pi \tilde{r} d\tilde{r} \\ &= \frac{\tilde{a}}{3} + \frac{\tilde{b}}{4} + \frac{\tilde{c}}{5} - \left(\tilde{a} + \frac{\tilde{b}}{2} + \frac{\tilde{c}}{3}\right) \tilde{r}_{\text{plug}}^2 + \left(\frac{2\tilde{a}}{3} + \frac{\tilde{b}}{4} + \frac{2\tilde{c}}{15}\right) \tilde{r}_{\text{plug}}^3\end{aligned}\quad (37)$$

and

$$\begin{aligned}\tilde{V}_{\text{plug}} &= \tilde{u}_{\text{plug}} \tilde{r}_{\text{plug}}^2 \\ &= \tilde{r}_{\text{plug}}^2 \left[\tilde{a} \left(1 - \tilde{r}_{\text{plug}}\right) + \frac{\tilde{b}}{2} \left(1 - \tilde{r}_{\text{plug}}^2\right) + \frac{\tilde{c}}{3} \left(1 - \tilde{r}_{\text{plug}}^3\right) \right]\end{aligned}\quad (38)$$

Substituting the results in Eqs. (37) and (38) back to Eqs. (36) leads to the same expression as that given in Eq. (29), which also verifies the result.

The Fanning friction factor f_F for the pipe flow of a Bingham plastic is characterized by the Bingham Reynolds number (Re_B) and the Bingham number (Bi) [53], where

$$f_F = \frac{2\tau_{\text{wall}}}{\rho V^2}, \quad \text{Re}_B = \frac{\rho V D}{\mu_p^B}, \quad \text{Bi} = \frac{\tau_0^B}{\mu_p^B V/D}\quad (39)$$

and

$$f_F = \frac{16}{\text{Re}_B} \left[1 + \frac{\text{Bi}}{6} - \frac{1}{3} \frac{\text{Bi}^4}{(f \text{Re}_B)^3} \right]\quad (40)$$

In line with the above approach, it can be shown by dimensional analysis that the Fanning friction factor for the pipe flow of a parabolic model fluid is characterized by three dimensionless numbers, the Bingham Reynolds number (Re_B), the Bingham number (Bi), and an extra dimensionless number. We define

$$\text{Re}_B = \frac{\rho V D}{\mu_p^B} = b \rho V D\quad (41)$$

$$\text{Bi} = \frac{\tau_0^B}{\mu_p^B V/D} = (-a) D/V\quad (42)$$

$$\beta = \frac{c(\mu_p^B V/D)}{b} = \frac{c}{b^2} \frac{V}{D}\quad (43)$$

We see that the dimensionless number α introduced in Eq. (13) is the product of the Bingham number and the newly introduced dimensionless number β , i.e., $\alpha = \text{Bi}\beta$.

Can we derive an expression for $\Phi(f_F, \text{Re}_B, \text{Bi}, \beta) = 0$? Since the density of the fluid does not appear in the expression for the volume flow rate, we must have the product of f_F and Re_B in our final expression, i.e., What we are looking for is $\Phi(f_F \text{Re}_B, \text{Bi}, \beta) = 0$. The non-dimensional number β characterizes the nonlinearity effect in the parabolic model, and when $\beta = 0$, the expression for the Bingham model must be recovered. Recall that the product of the Fanning friction factor and the Reynolds number is also known as the Poiseuille number (Po) [54], i.e., $\text{Po} = f_F \text{Re}_B$. Defining

$$\chi = \chi(\text{Po}, \text{Bi}, \beta) = \frac{1}{\text{Po}\beta} \left(-1 + \sqrt{1 + 4\text{Bi}\beta} \right)\quad (44)$$

It can be shown that the friction factor that we are looking for is given by

$$\Phi(\text{Po}, \text{Bi}, \beta) = 1 + \frac{1}{6} \text{Bi} (1 - \chi^3) - \frac{\text{Po}}{16} (1 - \chi^4) - \frac{\text{Po}^2 \beta}{40} (1 - \chi^5) = 0\quad (45)$$

In the limit of $\beta \rightarrow 0$, it can be shown that Eq. (40) is recovered. In the limit of $\beta \rightarrow 0$ and $Bi \rightarrow 0$, we have $Po = 16$, the classical result for the pipe flow of a Newtonian fluid in the laminar flow regime.

To summarize the results on the friction factor, the Poiseuille number was plotted as a function of the Bingham number for different β values as shown in Figure 3. Since $Po = f_F Re_B$, at a constant Reynolds number, the plot of Po vs. Bi shows the influence of β on friction factor f_F . As is expected, the β parameter shows the deviations from the linear relation between Po and Bi as described by the Bingham model. For the shear-thickening case, $c < 0$, we have $\beta < 0$ and there is more friction. In the case of shear-thinning behavior, $c > 0$, we have $\beta > 0$, and as we increase β , the value of friction factor f_F decreases for the same Bingham number.

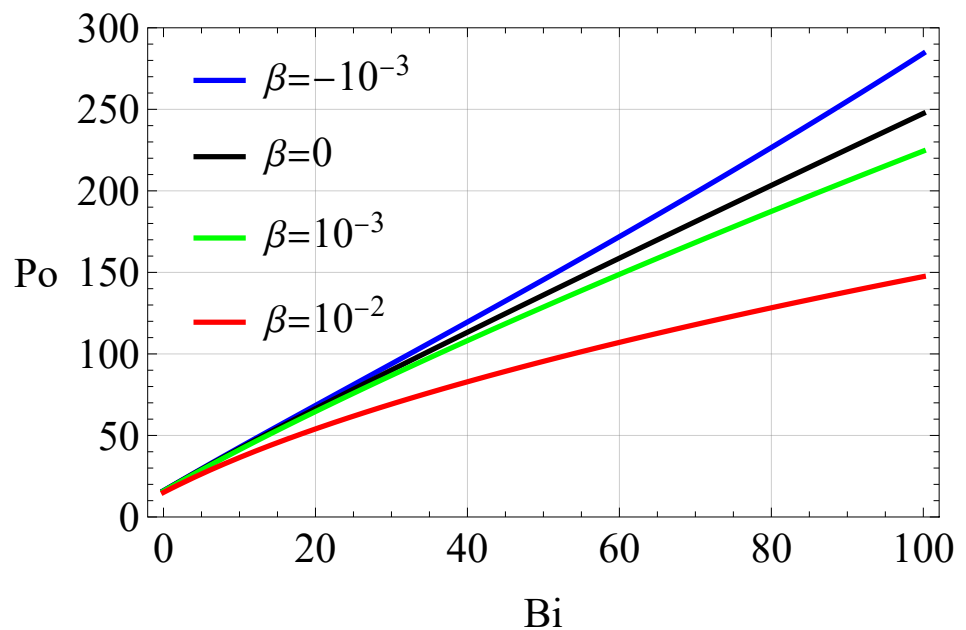


Figure 3. Plot of Po as a function of Bi for different β values

2.3. Hagen–Poiseuille equation for non-Newtonian pipe flow: Coaxial flow of two immiscible fluids

The existence of a lubrication layer resulted in a dual-fluid model in which bulk concrete and lubrication fluid with different rheological properties are in contact with each other and flow together, with a distinct interface separating them. This section will introduce the derivation of the flow rate expression for the co-axial pipe flow of dual fluids. The method is essentially the same as that detailed in our recent work [41].

Depending on the type of concrete that is pumped, we may have $r_{\text{plug}} > R - \ell$ or $r_{\text{plug}} < R - \ell$. In the case of $r_{\text{plug}} \geq R - \ell$ (which is common for the pumping of CVC [34])—hereinafter referred to as “Case I”, the bulk concrete experiences only plug flow. On the other hand, in the case of $r_{\text{plug}} < R - \ell$ —hereinafter referred to as “Case II”, the bulk concrete experiences two types of flows: plug flow in the region of $r \leq r_{\text{plug}}$ and shear flow in the region of $r_{\text{plug}} < r \leq R - \ell$, as is commonly the case for the pumping of SCC [34,36,55,56]. Here, the plug radius is given by

$$r_{\text{plug}} = 2\tau_{0,c}/G \quad (46)$$

where $\tau_{0,c}$ is the yield stress of bulk concrete.

From Eqs. (20) and (24), we see that if there is a slip velocity at the wall, let it be V_R , then the flow rate becomes

$$Q(\vec{\zeta}, R, G, V_R) = \pi R^2 V_R + \frac{\pi R^3}{\tau_{\text{wall}}^3} \int_{\tau_0}^{\tau_{\text{wall}}} \tau_{rz}^3 \zeta d\tau_{rz} \quad (47)$$

where $\tau_{\text{wall}} = GR/2$ depends on the G -parameter (pressure drop per unit pipe length) and the pipe radius R , and shear rate function depends on the rheological properties (denoted by $\vec{\zeta}$) of the fluid. Eq. (47) tells that the overall flow rate in the presence of a certain slip velocity (V_R) at the boundary ($r = R$) is the sum of the flow rate with zero velocity at the wall and an additional term of $\pi R^2 V_R$ that only depends on the cross-section area and the velocity at the boundary.

In the case of the steady co-axial pipe flow of dual fluids, the flow rate from the fluid in the lubrication layer is given by

$$Q_{\text{LL}} = Q_{\text{HP}}(\vec{\zeta}_{\text{LL}}, R, G, V_R = 0) - Q_{\text{HP}}(\vec{\zeta}_{\text{LL}}, R - \ell, G, V_R = V_{\text{Interface}}) \quad (48)$$

where $V_{\text{Interface}}$ is the flow velocity at $r = R - \ell$, which is the interface between the fluid in the lubrication layer and the bulk concrete. In deriving Eq. (48), we have applied the result of Eq. (47). If the bulk concrete experiences only plug flow (i.e., Case I), its contribution to the overall volume flow rate is given by

$$Q_{\text{C}} = \pi(R - \ell)^2 V_{\text{Interface}} \quad (49)$$

The overall flow rate–pressure drop relation is obtained as [41]

$$\begin{aligned} Q &= Q_{\text{LL}} + Q_{\text{C}} \\ &= Q_{\text{HP}}(\vec{\zeta}_{\text{LL}}, R, G) - Q_{\text{HP}}(\vec{\zeta}_{\text{LL}}, R - \ell, G) \end{aligned} \quad (50)$$

On the other hand, if there is also a sheared-concrete zone in the bulk concrete (i.e., Case II) in addition to the plug flow zone near the center of the pipe, the contribution from the bulk concrete to the overall volume flow rate can be expressed as

$$Q_{\text{C}} = Q_{\text{HP}}(\vec{\zeta}_{\text{C}}, R - \ell, G, V_R = V_{\text{Interface}}) \quad (51)$$

The overall flow rate–pressure drop relation is thus obtained as [41]

$$\begin{aligned} Q &= Q_{\text{LL}} + Q_{\text{C}} \\ &= Q_{\text{HP}}(\vec{\zeta}_{\text{LL}}, R, G) - Q_{\text{HP}}(\vec{\zeta}_{\text{LL}}, R - \ell, G) + Q_{\text{HP}}(\vec{\zeta}_{\text{C}}, R - \ell, G) \end{aligned} \quad (52)$$

The two cases can be unified into one by the use of the Heaviside step function. The result is

$$Q_{\text{tot}} = Q_{\text{HP}}(\vec{\zeta}_{\text{LL}}, R, G) - Q_{\text{HP}}(\vec{\zeta}_{\text{LL}}, R - \ell, G) + Q_{\text{HP}}(\vec{\zeta}_{\text{C}}, R - \ell, G)H(R - \ell - r_{\text{plug}}) \quad (53)$$

where $H(x)$ denotes the Heaviside function: $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x \leq 0$.

3. Results and Discussion

Following the analytical derivations from the previous section for both single and dual-fluid cases, it is possible to obtain the relations on the shear rate distribution, the velocity distribution, and the prediction of volume flow rate for the pressure loss per unit pipe length. For the single-fluid case, the input parameters that are necessary to make these predictions include $\{a, b, c, R, G\}$, where a , b , and c are the three parameters in the parabolic fluid model, R is the radius of the pipe, and G represents the pressure loss per unit pipe length. For the co-axial flow of dual fluids (referred to as the bulk concrete and the lubrication layer fluid in the context of concrete pumping), we shall need four more

parameters, the thickness of the lubrication layer, ℓ , and three parameters in the parabolic fluid model for the lubrication layer fluid. One can follow the approaches presented by Eq. (24) and Eq. (53) to derive the expressions to predict the volumetric flow rate for single and dual fluid cases respectively depending on the pressure drop per unit pipe length. In this section, we demonstrate the resulting distributions from the parabolic model and analyze the influence of the model's parameters. We consider some possible values for the a -parameter, b -parameter, and c -parameter with ranges derived by fitting the Parabolic model to the Herschel–Bulkley model explored in our recent work [41].

3.1. Parabolic model: Shear stress and shear rate distribution

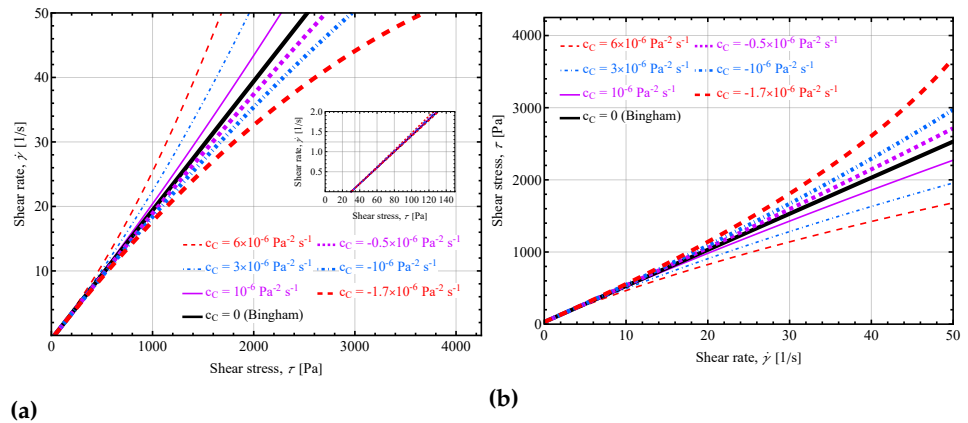


Figure 4. Influence of c_C -parameter of the bulk concrete: (a) on shear rate vs. shear stress curves predicted by the Parabolic model; (b) on shear stress vs. shear rate curves predicted by the Parabolic model. Unless otherwise specified in the figure, the following values were used: $a_C = -0.6 \text{ s}^{-1}$, $b_C = 0.02 \text{ Pa}^{-1} \text{ s}^{-1}$, $R = 62.5 \text{ mm}$, $\Delta P/L = 60 \text{ kPa/m}$.

Figure 4 shows the influence of the c_C -parameter of the bulk concrete for the curves showing the relationship between shear stress and shear stress using the Parabolic model. Figure 4a shows the shear rate vs. shear stress curve with the influence of this parameter, while Figure 4b presents the conventional approach of shear stress vs. shear rate curves with the shear rate being on the x-axis as the independent variable. From both figures the effect of varying the c_C -parameter can be clearly observed in how it influences the either shear thickening or shear thinning behavior of the fluid. With the increase of the analyzed parameter, the shear thinning behavior of the fluid is more prominent for both curves, where there is a more noticeable increase of shear stress with the increase in shear rate. The same is true for the analysis of shear thickening behavior, as with the decrease of the c_C -parameter, the shear stress grows slower with the increase in shear rate. Likewise, for the Figure 4b, as the value of the c_C -parameter is increased, the shear rate rises slower and as the parameter is decreased, the curves rise faster with the increasing shear stress corresponding to the shear thinning and shear thickening behavior respectively. It should be noted that Figure 4b demonstrates that the parabolic model can easily reproduce the shear stress vs. shear rate curve predicted by the Herschel–Bulkley model [41] and c -parameter of the parabolic model and n -parameter of the Herschel–Bulkley model are similar in nature.

In Figure 5, shear rate vs. shear stress curves are presented with the influence of other parameters from the Parabolic model: Figure 5a showing the effect of varying a_C -parameter on the curves, and Figure 5b including the effect of varying the b_C -parameter on the shear rate curves. For Figure 5a it can be observed that the most noticeable effect of varying the a_C -parameter is a change in yield stress values. With the decrease of this parameter, the curves' x-intercepts shift more to the right, indicating an increase in the values of yield stress. Such results are expected given the expressions for the yield stress derived from the Parabolic model as presented in Eq. (8). In the case of Figure 5a, b_C -parameter has a more significant influence on the shear rate vs. shear stress curve unlike a_C -parameter. This is

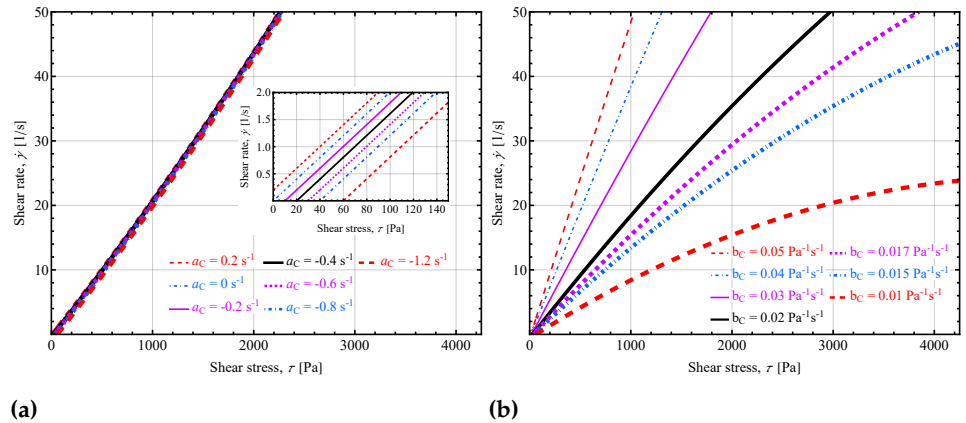


Figure 5. Shear rate vs. shear stress curves predicted by the Parabolic model: (a) influence of a_c -parameter of the bulk concrete; (b) influence of b_c -parameter of the bulk concrete. Unless otherwise specified in the figure, the following values were used: $a_c = -0.6 \text{ s}^{-1}$, $b_c = 0.02 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_c = -10^{-6} \text{ Pa}^{-2} \text{ s}^{-1}$, $R = 62.5 \text{ mm}$, $\Delta P / L = 60 \text{ kPa/m}$.

consistent with previously reported results [36,41,57] since b -parameter is inversely related to the plastic viscosity derived from the parabolic model when it reduces to the Bingham model as given by Eq. (11).

3.2. Single-fluid model

3.2.1. Shear rate and velocity distribution

Figure 6 shows the influence of the c_c -parameter of the bulk concrete on the velocity distribution and shear-rate distribution. Figures 6a and 6b show the shear flow and plug flow regions in the bulk concrete, but plug flow is not visible in the figure due to the small plug radius. This is common for the pumping of SCC [41]. In Figures 6a, as we increase the c -parameter, the velocity increases for the same radial distance and hence the volume flow rate. This is also true for the shear rate distribution shown in Figure 6b. The shear rate value increases with increasing c -parameter value. This is not surprising since increasing the c -parameter decreases the friction value. Thus as fluid is more shear-thinning, $c > 0$, the velocity and shear rate distributions are higher, and when fluid is more shear-thickening, $c < 0$, the corresponding velocity and shear rate values are lower.

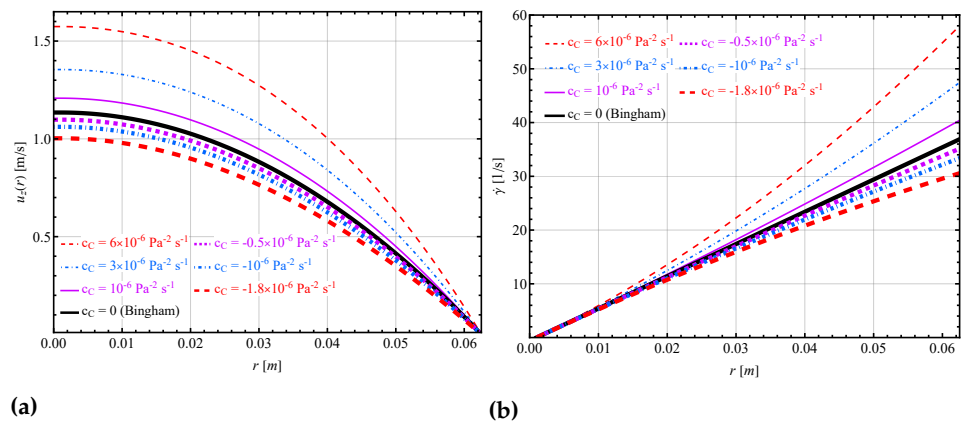


Figure 6. Influence of c_c -parameter of the bulk concrete: (a) on the velocity distribution; (b) on the shear rate distribution predicted by the Parabolic model. Unless otherwise specified in the figure, the following values were used: $a_c = -0.6 \text{ s}^{-1}$, $b_c = 0.02 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_c = -10^{-6} \text{ Pa}^{-2} \text{ s}^{-1}$, $R = 62.5 \text{ mm}$, $\Delta P / L = 60 \text{ kPa/m}$.

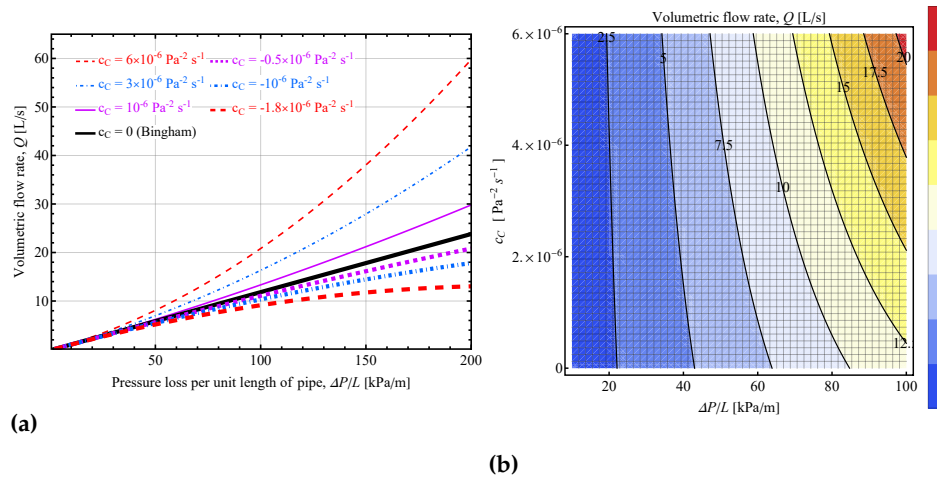


Figure 7. (a) Influence of c_c -parameter on volume flow rate vs pressure loss per unit pipe for the single fluid flow predicted by the Parabolic model; (b) contour plot of the volumetric flow rate (Q) as a function of the pressure loss per unit pipe length, $\Delta P/L$, as the x-axis and c_c -parameter on the y-axis. Unless otherwise specified in the figure, the following values were used: $a_c = -0.6 \text{ s}^{-1}$, $b_c = 0.02 \text{ Pa}^{-1} \text{ s}^{-1}$, $R = 62.5 \text{ mm}$.

3.2.2. Volumetric flow rate vs. pressure loss

In Figure 7 the volume flow rate relationship is shown: Figure 7a demonstrates how changing the values of the c_c -parameter affects the volume flow rate vs. pressure loss per unit pipe relation for the single fluid flow; Figure 7b presents the influence of the c_c -parameter on the volume flow rate while varying pressure loss per unit pipe length in contour form. Based on the Figure 7a, an increase in the value of c_c -parameter also results in a steeper increase in volume flow rate curves. Similarly to Figure 4b, Figure 7a shows the same influence of c -parameter of the parabolic model and n parameter of the Herschel-Bulkley model [41] on the flow rate–pressure drop relation. From Figure 7b, it is clearly seen that at low values of $\Delta P/L$, the influence of c -parameter is negligible, but as we increase $\Delta P/L$, its effect becomes predominant. As we increase the c -parameter from low to high, the volume flow rate increases for the same pressure drop. This is expected since as we increase the c -parameter, β value also increases, resulting in lower friction as given in Figure 3.

3.3. Dual fluid model

For the dual fluid case, the rheological properties of both bulk concrete and the lubrication layer of the fluid affect the volume flow rate predictions. Figure 8 shows the volume flow rate vs. pressure loss per unit pipe length for the dual fluid case predicted by the Parabolic model based on Eq. (53). Figure 8a presents the influence of c_c -parameter of the bulk concrete and Figure 8b presents the influence of c_c -parameter of the lubrication layer of the fluid. Both figures demonstrate the same trend as in Figure 7a, where for smaller values of pressure drop, the influence of c -parameter on the flow rate is negligible whereas for bigger values it becomes predominant. Comparing these two figures, it can be seen that both of these parameters have a similar effect on the volume flow rate curves, increasing the c -parameter increases the volume flow rate due to low friction.

Similar to Figure 8, Figure 9 presents the influence of a -parameter on the volume flow rate as a function of pressure loss per unit pipe length for bulk concrete and the lubrication layer of the fluid. Figure 9a shows the effect of a_c -parameter of the bulk concrete and Figure 9b shows the effect of a_{LL} -parameter of the lubrication layer of the fluid on the volume flow rate values. Both figures show the same trend that the volume flow rate is not affected by the variation in values of the a -parameter both for the bulk concrete and the

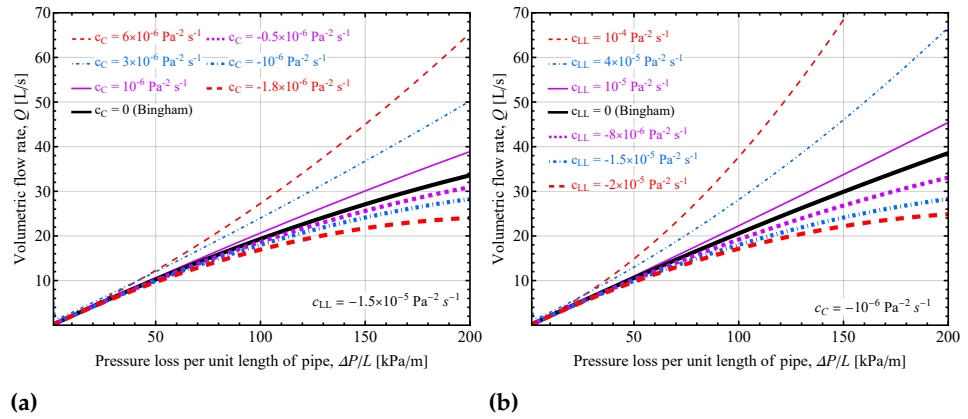


Figure 8. Volume flow rate vs. pressure loss per unit pipe length for the dual fluid flow predicted by the Parabolic model: (a) influence of c_C -parameter of the bulk concrete with $c_{LL} = -1.5 \cdot 10^{-5} \text{ Pa}^{-2} \text{ s}^{-1}$ for the fluid in the lubrication layer; (b) influence of c_{LL} -parameter of the fluid in the lubrication layer with $c_C = -10^{-6} \text{ Pa}^{-2} \text{ s}^{-1}$ for the bulk concrete. Unless otherwise specified in the figure, the following values were used: $a_C = -0.6 \text{ s}^{-1}$, $b_C = 0.02 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_C = -10^{-6} \text{ Pa}^{-2} \text{ s}^{-1}$, $a_{LL} = 3.5 \text{ s}^{-1}$, $b_{LL} = 0.2 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_{LL} = -1.5 \cdot 10^{-5} \text{ Pa}^{-2} \text{ s}^{-1}$, $R = 62.5 \text{ mm}$, $\ell = 1.5 \text{ mm}$.

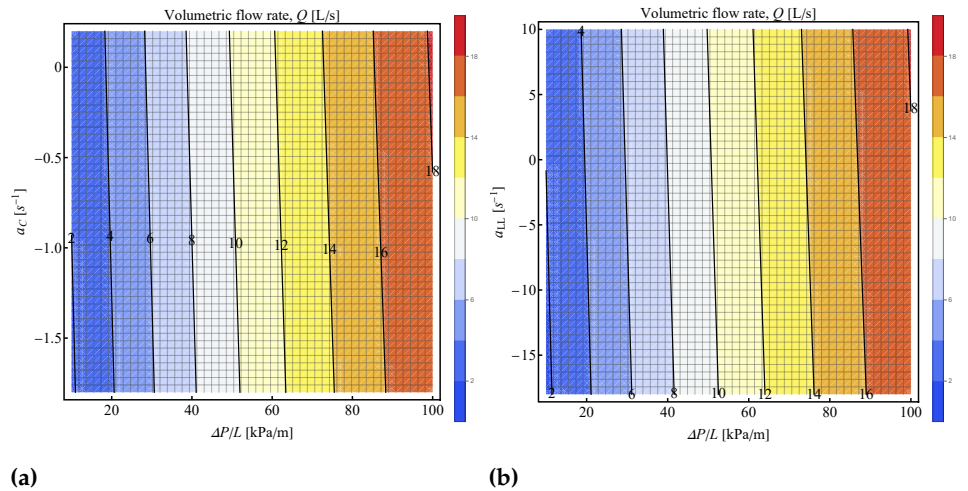


Figure 9. Contour plots of the volumetric flow rate (Q) as a function of the pressure loss per unit pipe length, $\Delta P/L$, as the x-axis and a -parameter on the y-axis: (a) a_C -parameter of the bulk concrete on the y-axis; (b) a_{LL} -parameter of the fluid in the lubrication layer on the y-axis. Unless otherwise specified in the figure, the following values were used: $a_C = -0.6 \text{ s}^{-1}$, $b_C = 0.02 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_C = -10^{-6} \text{ Pa}^{-2} \text{ s}^{-1}$, $a_{LL} = 3.5 \text{ s}^{-1}$, $b_{LL} = 0.2 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_{LL} = -1.5 \cdot 10^{-5} \text{ Pa}^{-2} \text{ s}^{-1}$, $R = 62.5 \text{ mm}$, $\ell = 1.5 \text{ mm}$.

lubrication layer of the fluid. This result is true for the whole range of pressure loss per unit pipe length used in the figures.

When considering Figure 10, the results are similar to that of Figure 7b. In Figure 10, contour plots for the volumetric flow rate as a function of pressure loss per unit length and b -parameter are presented: Figure 10a showing the influence of b_C -parameter of the bulk concrete and Figure 10b showing the influence of b_{LL} -parameter of the lubrication layer of the fluid. In the case of b_C -parameter of the bulk concrete in Figure 10a, it can be seen that while the effect of varying this parameter is not that prominent for the low values of $\Delta P/L$, it becomes more prominent towards the end of the used range of $\Delta P/L$ on the graph. The same is true for the b_{LL} -parameter of the lubrication layer of the fluid, that the influence of the parameter becomes more noticeable with the increase of values of $\Delta P/L$. It can be seen that at high values of $\Delta P/L$, larger the b_C and b_{LL} -parameters, the larger the volume flow are under a given pumping pressure. This is not surprising since the larger the b -parameter, the smaller the plastic viscosity of the fluid (Eq. (11)), and previously reported studies [41]

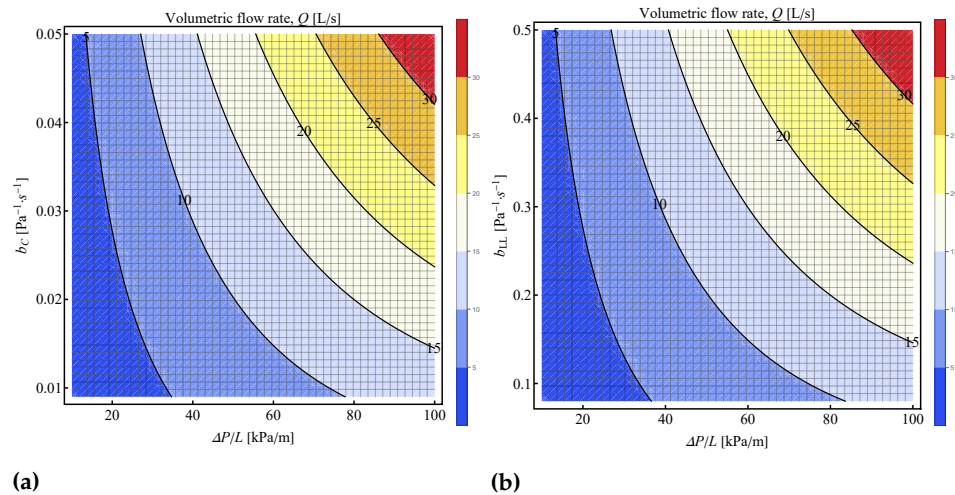


Figure 10. Contour plots of the volumetric flow rate (Q) as a function of the pressure loss per unit pipe length, $\Delta P/L$, as the x-axis and b -parameter on the y-axis: (a) b_C -parameter of the bulk concrete on the y-axis; (b) b_{LL} -parameter of the fluid in the lubrication layer on the y-axis. Unless otherwise specified in the figure, the following values were used: $a_C = -0.6 \text{ s}^{-1}$, $b_C = 0.02 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_C = -10^{-6} \text{ Pa}^{-2} \text{ s}^{-1}$, $a_{LL} = 3.5 \text{ s}^{-1}$, $b_{LL} = 0.2 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_{LL} = -1.5 \cdot 10^{-5} \text{ Pa}^{-2} \text{ s}^{-1}$, $R = 62.5 \text{ mm}$, $\ell = 1.5 \text{ mm}$.

have shown that for smaller values of $\mu_{p,C}$ and $\mu_{p,C}$, the volume flow rate is larger for the same pressure drop.

3.4. Computational Apps for the parabolic model

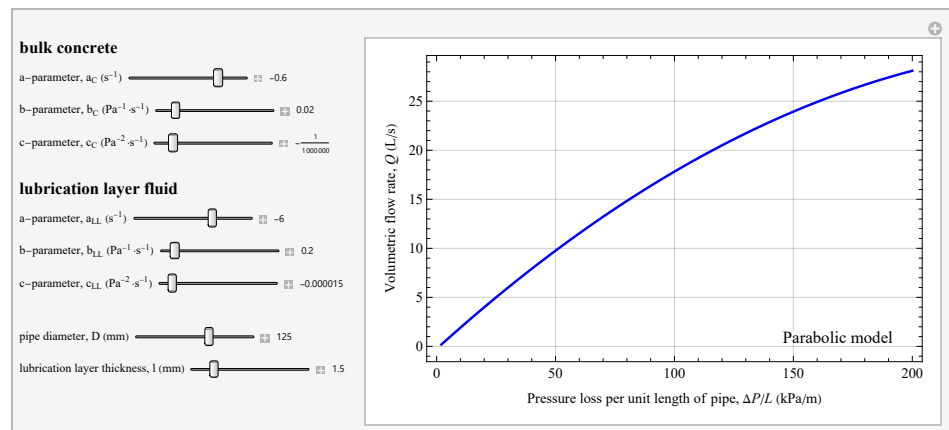


Figure 11. The graphical user interface of a Wolfram-based demonstration App that implements the dual-fluid Parabolic model developed in this work. The following values were used for the computational app in the figure: $a_C = -0.6 \text{ s}^{-1}$, $b_C = 0.02 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_C = -10^{-6} \text{ Pa}^{-2} \text{ s}^{-1}$, $a_{LL} = -6 \text{ s}^{-1}$, $b_{LL} = 0.2 \text{ Pa}^{-1} \text{ s}^{-1}$, $c_{LL} = -1.5 \cdot 10^{-5} \text{ Pa}^{-2} \text{ s}^{-1}$, $R = 62.5 \text{ mm}$, $\ell = 1.5 \text{ mm}$.

Using the analytical expressions derived for the single-fluid and dual-fluid cases and the necessary rheological parameters, one can predict the relationship between the volume flow rate and pressure loss per unit pipe of length. Furthermore, different software and programming languages can be used to easily obtain the necessary curves by inserting the parameters corresponding to the bulk concrete and lubrication layer.

To facilitate applications of our theoretical results in practice, a demonstration application using the Wolfram Mathematica software (version 13.2) has been developed with a sample graph shown in Figure 11. Users can manipulate the values of the rheological parameters for both bulk concrete and the lubrication layer of the fluid, the diameter of the pipe, and the lubrication layer thickness within the ranges of values written in the program.

The resulting graph shows the total volume flow rate vs. the pressure loss per unit pipe length. Furthermore, the application can be used for the analysis purposes of the relation on how the parameters of the bulk fluid, the lubrication layer, the pipe diameter, and the lubrication layer thickness influence the resulting curve.

4. Limitations of the present work

In this section, the main assumptions and limitations of the existing work are summarized along with the future study to be conducted. The main assumptions and limitations are similar to those reported by Kwon *et al.* [35], Khatib and Khayat [36], and Zhaidarbek *et al.* [41]. For the analytical derivations, the Hagen-Poiseuille equation is used as the governing law. Therefore, the derived equations are limited to the fully developed, steady, laminar, one-dimensional, and isothermal flow, with the fluids being incompressible, homogeneous, and experiencing no change in their properties during pumping. The rheological properties of the bulk concrete and lubrication layer of the fluid are considered separately and with time independence. The pipe diameter and thickness of the lubrication layer are assumed as constant, and the lubrication layer thickness is also independent of the fluid’s rheological properties. Finally, the difference in the densities of bulk concrete and lubrication layer fluid is considered as negligible.

Despite these limitations, the rheological characteristics of the fluid in the lubricating layer are challenging to estimate experimentally. For the parabolic model, there are four parameters for the lubrication layer fluid: a_{LL} -parameter, b_{LL} -parameter, c_{LL} -parameter, and lubrication layer thickness ℓ , that need to be determined in order to calculate the flow rate–pressure drop relation. One approach to solve this problem is to use a wall-slip model to approximate the dual-fluid flow rate–pressure drop relation. According to the wall-slip model, Eq. (53) can be simplified to

$$Q = Q_{HP}(\vec{\zeta}_C, R, G) + Q_s \tag{54}$$

where the first term is the single-fluid flow rate in Eq. (28) using the bulk concrete parameters and Q_s is the additional flow rate that occurs due to the slip effects on the surface. The latter is given by

$$Q_s = \pi R^2 u_s \tag{55}$$

where, u_s is the wall-slip velocity. Using the wall-slip model, the lubrication layer fluid properties can be characterized by just one parameter—the wall-slip velocity. The wall-slip velocity represents a macro-scale description of the wall’s boundary condition[58–62] and depends on the lubrication layer fluid at the wall. This velocity can be estimated in capillary tubes [16]. Consequently, a future goal is to validate the applicability of the wall-slip model in estimating the dual-fluid flow rate-pressure drop relationship in concrete pumping.

5. Conclusions

In this study, rheological models with shear stress as the independent variable are considered, contrasting the conventional approach of using shear rate as the independent variable. The parabolic model is analyzed by dimensional analysis and employed for analytical predictions of the flow rate vs. pressure drop relations in Hagen–Poiseuille pipe flow for a single fluid and in the co-axial flow of dual-fluids. A key advantage of the parabolic model is its ability to account for the nonlinearity of the shear stress and shear rate relations through the inclusion of the nonlinear c -parameter, leading to more accurate results compared to linear models such as the classical Bingham model.

Theoretical derivations have been conducted to derive analytical results for the shear rate distribution, velocity distribution, and volume flow rate relations for Hagen–Poiseuille flow in viscoplastic fluids. This method, which is applied to the parabolic model in this paper, can be generalized for other models with shear stress as the independent variable. This study presents not only the analytical expressions but also explores the effects of the a , b , and c parameters.

While studying the flow curves (i.e., shear rate vs. shear stress curves) for the parabolic model, it is demonstrated how the value of the c -parameter affects the curves: if the value is reduced to $c = 0$, the curve reduces to that of the Bingham model, while a positive c value corresponds to the shear thinning behavior, and a negative c value corresponds to the shear thickening behavior. The b parameter strongly affects the flow curves. In comparison, varying the a -parameter does not have a significant influence on the flow curves.

Subsequently, this study also analyzes the resulting relations based on the parabolic model and how they are influenced by the rheological parameters of the bulk concrete for the single fluid case and along with the lubrication layer for the dual fluid case. For the relation between the volume flow rate vs. pressure loss per unit length of pipe for the single-fluid case, the influence of the c -parameter becomes more prominent for the higher values of pressure drop while being negligible at its low values. For the volume flow rate – pressure drop relation in the dual fluid case, it has been found that varying the a -parameters for both the bulk concrete and lubrication layer fluid has little effect on the relation, while the b -parameters possess a strong effect on the relation with the increasing value of pressure loss per unit pipe length.

Finally, a demonstration App is developed, enabling users to obtain volume flow rate curves by inputting the rheological properties of the bulk concrete and lubrication layer. This application facilitates the study of the influence of rheological parameters on volume flow rate curves and allows for further data analysis.

Author Contributions: B. Zhaidarbek: Formal Analysis, Visualization, Software Development, Writing – Original Draft. K. Savitskaya: Software Development, Validation, Writing – Original Draft. Y. Wang: Conceptualization, Methodology, Supervision, Project Administration, Funding Acquisition, Writing – Review & Editing.

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Data Availability Statement: The data supporting the findings of this study are available within the article and its supplementary materials. The source codes and plotting scripts can be obtained from the corresponding author upon reasonable request.

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Conflicts of Interest: The authors confirm that there are no known competing financial interests or personal relationships which could be perceived as influencing the work presented in this paper.

Abbreviations

The following abbreviations are used in this manuscript:

CVC	Conventional Vibrated Concrete
LL	Lubrication Layer
ODE	Ordinary Differential Equation
SIPM	Shear-Induced Particle Migration
SCC	Self-Compacting Concrete, or Self-Consolidating Concrete

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