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Article

On the Method for Proving RH Using the Alcantara-Bode Equivalence (II)

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Abstract

This study was inspired by Alcantara-Bode's equivalent to the Riemann Hypothesis published in 1993, the equivalent formulation consisting in the injectivity of an integral operator connected to Riemann Zeta function. Surprisingly, the research on this line has not continued, an explanation would be the lack of criteria for the injectivity of integral operators. This paper aims to fill this gap by proposing a functional-numerical analysis solution exploiting the operator positivity properties on dense sets. The main theorem says that a linear, bounded operator strict positive definite on a dense set of a separable Hilbert space, has its null space containing only the null element, equivalently, it is injective. Having in mind to obtain a generic and useful criterion, we gradually changed the hypothesis of the strict positivity of the operator on a dense set to the involvement at the end, of the associated Hermitian operator that is semi positive on the whole space requesting additional properties related to the positivity of operator approximations on finite dimension subspaces. Then, in order to apply the criterion for Hermitian Hilbert-Schmidt operators, we choose an adequate dense set allowing to obtain operator sparse matrix representations. The criterion applied to the associated Hermitian of the Alcantara-Bode integral operator, showed that the equivalent holds, so the Riemann Hypothesis is true.

Keywords: integral operators; Hilbert spaces; approximation methods; multi-level discretizations; Riemann Hypothesis

MSC: 31A10, 45P05, 47G10, 65R99, 11M26

1. Introduction

The method is built on the result obtained with Theorem 1 consisting in: if a linear bounded operator is strict positive on a dense set, then it is injective.

Weakening to positive only the hypothesis (Theorem 2) and later easing again to requests by considering operators semi positive definite on a dense set, the method could be formulated as follows (Lemma 1): a linear bounded operator semi positive on the dense set whose positivity parameters of the operator approximations are bounded, is strict positive on the dense set. Corollary of Lemma 1 considering the associated Hermitian operator instead, that is semi positive on the whole space and has the same null space with the original one, allows us to bypass the check of positivity.

2. On Linear Operators Injectivity

Let H be a separable Hilbert space and denote with $\mathcal{L}(H)$ the class of the linear bounded operators on H . If $T \in \mathcal{L}(H)$ has no zeros on a dense set $S \subset H$, then its 'eligible' zeros are all in the difference set $E := H \setminus S$, i.e. $N_T \subset E$. In our analysis we will take in consideration only the collection of eligible zeros that are on the unit sphere, without restricting the generality once for an element that is not null $w \in H$, both w and $w/\|w\|$ are or are not together in N_T .

Theorem 1. If $T \in \mathcal{L}(H)$ is strict positive on a dense set of a separable Hilbert space then $N_T = \{0\}$, equivalently T is injective.

Proof of Theorem 1.

The set $S \subset H$ is dense if its closure coincides with H . Then, if $w \in E := H \setminus S$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that $\|w - u_{\varepsilon,w}\| < \varepsilon$. Now, (1) results as follows. If $\|w\| \geq \|u_{\varepsilon,w}\|$:

$$0 \leq \|w\| - \|u_{\varepsilon,w}\| = \|w - u_{\varepsilon,w} + u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| \leq \|w - u_{\varepsilon,w}\| + \|u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| < \varepsilon.$$

If $\|u_{\varepsilon,w}\| \geq \|w\|$ instead, then:

$$0 \leq \|u_{\varepsilon,w}\| - \|w\| = \|u_{\varepsilon,w} - w + w\| - \|w\| \leq \|w - u_{\varepsilon,w}\| < \varepsilon.$$

Therefore, given $w \in E$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that

$$|\|w\| - \|u_{\varepsilon,w}\|| < \varepsilon \quad (1)$$

Let w be an eligible element from the unit sphere, $\|w\| = 1$ and take $\varepsilon_n = 1/n$. Then there exists at least one element $u_{\varepsilon_n,w} \in S$ such that $\|u_{\varepsilon_n,w} - w\| < \varepsilon_n$ holds. From (1), $|1 - \|u_{\varepsilon_n,w}\|| < 1/n$ showing that, for any choices of a sequence approximating w , $u_{\varepsilon_n,w} \in S, n \geq 1$, it verifies $\|u_{\varepsilon_n,w}\| \rightarrow 1$.

If $T \in \mathcal{L}(H)$ is strict positive on S , then there exists $\alpha > 0$ such that $\forall u \in S, \langle Tu, u \rangle \geq \alpha \|u\|^2$. Suppose that there exists $w \in E \cap N_T, \|w\| = 1$ and consider a sequence of approximations of w , $u_{\varepsilon_n,w} \in S, n \geq 1$ that, as we showed, has its normed sequence converging to 1. From the positivity of T on dense set S , follows:

$$\alpha \|u_{\varepsilon_n,w}\|^2 \leq \langle Tu_{\varepsilon_n,w}, u_{\varepsilon_n,w} \rangle = \langle T(u_{\varepsilon_n,w} - w), u_{\varepsilon_n,w} \rangle < \varepsilon_n \|T\| \|u_{\varepsilon_n,w}\| \quad (2)$$

With $c = \|T\|/\alpha$, we obtain $\|u_{\varepsilon_n,w}\| \leq c/n$. Then, $\|u_{\varepsilon_n,w}\| \rightarrow 0$ with $n \rightarrow \infty$, contradicting its convergence $\|u_{\varepsilon_n,w}\| \rightarrow 1$ with $n \rightarrow \infty$.

This occurs for any choice of the sequence of approximations of w , verifying

$$\|w - u_{\varepsilon_n,w}\| < \varepsilon_n, n \geq 1, \text{ when } Tw = 0.$$

Thus $w \notin N_T$, valid for any $w \in E, \|w\| = 1$, proving the theorem because no zeros of T there are in S either. $\square$

In order to obtain a criterion useful as a numerical analysis method, we moved the investigation of the strict positivity on a dense set to positivity of operator approximations on a family of finite dimension subspaces whose union is dense.

Suppose that the dense set S is the result of an union of finite dimension subspaces of a family F : $S = \cup_{n \geq 1} S_n, \bar{S} = H$. It is not mandatory but will ease our proofs considering that the subspaces are including: $S_n \subset S_{n+1}, n \geq 1$.

Given $T \in \mathcal{L}(H)$, let $\{T_n; n \geq 1\}$ be a sequence of its operator approximations on the family F having the following properties:

- i) $\epsilon_n := \|T - T_n\| \rightarrow 0$ with $n \rightarrow \infty$; (convergence of operator approximations)
- ii) $\langle T_n v, v \rangle \geq \alpha_n \|v\|^2, \forall v \in S_n, S_n \in F$ (operator approximations strict positivity)

In the hypotheses i)-ii), gradually easing the strict positivity property of the operator on the dense set, we obtained the following results:

• **Theorem 2.** If T is positive on S , $\langle Tu, u \rangle > 0$ for any $u \in S$ and the set of positivity parameters is bounded, i.e. there exists $\alpha > 0$ such that

$$\text{iii) } \alpha_n \geq \alpha > 0, n \geq 1, \text{ then } N_T = \{0\}.$$

• **Lemma 1.** If T is semi positive on S , $\langle Tu, u \rangle \geq 0$ for any $u \in S$ and the set of positivity parameters is bounded, i.e. there exists $\alpha > 0$ such that

$$\text{iii) } \alpha_n \geq \alpha > 0, n \geq 1, \text{ then } T \text{ is strict positive on the dense set.}$$

• **Corollary.** If the sequence of finite rank approximations on a dense set S of a Hermitian Hilbert-Schmidt operator $Q \in \mathcal{L}(H)$ verifies:

$\langle Q_n v, v \rangle \geq \alpha_n \|v\|^2 \geq \alpha \|v\|^2, \forall v \in S_n$, on any $S_n \in F$, where $\alpha > 0$ is a constant, then Q is strict positive on S .

Note. For $T \in \mathcal{L}(H)$ non Hermitian, $Q := T^*T$ is the associated Hermitian operator, so it is semi positive on any dense set ($\langle Qu, u \rangle = \|Tu\|^2 \geq 0, \forall u \in H$) allowing to use through Corollary, Lemma 1 without checking the semi positivity. Because $N_T = N_Q$, the result obtained for associated Hermitian will be transferred to original operator T . Then: a criterion for the injectivity of $T \in \mathcal{L}(H)$ Hilbert-Schmidt operator, consists in the existence of a bound of the positivity parameters of finite rank operator approximations of its associated Hermitian, verifying i)-ii) on a family of finite dimension subspaces whose union is dense.

Proofs

Observation 1. Let $\beta_n(u) := \|u - u_n\|$ be the normed residuum of element $u \in E$ after its orthogonal projection onto S_n . Then, $\beta_n(u) \rightarrow 0$ with $n \rightarrow \infty$.

Proof.

Given $\epsilon > 0$, from the density of the set S in H there exists $u_\epsilon \in S$ verifying $\|u - u_\epsilon\| < \epsilon$, as per the observations made in the proof of the Theorem 1. Let S_{n_ϵ} be the coarsest subspace, i.e. with the smallest dimension, from the family of subspaces containing u_ϵ . Because the best approximation of u in S_{n_ϵ} is its orthogonal projection, we obtain

$$\beta_{n_\epsilon}(u) := \|u - P_{n_\epsilon} u\| \leq \|u - u_\epsilon\| < \epsilon,$$

valid for every $\epsilon > 0$, proving our assertion. Rewriting,

$\beta_n(u) := \|u - P_n u\| = \|(I - P_n)u\| \leq \|I - P_n\| \|u\| \rightarrow 0$ for $n \rightarrow \infty$ for any $u \in H$ with P_n the orthogonal projection onto S_n . $\square$

Proof of Theorem 2.

Being positive on S , the operator does not have zeros in the dense set. Otherwise, if there exists $w \in S$ such that $Tw = 0$ then $\langle Tw, w \rangle = 0$ contradicting its positivity.

For $u \in E := H \setminus S$, denoting the not null orthogonal projection over coarser subspace S_n by $u_n := P_n u, n \geq n_0 := n_0(u)$, then from (ii),

$$0 < \alpha_n \|u_n\|^2 \leq \langle T_n u_n, u_n \rangle \leq \|T_n u_n\| \|u_n\|. \quad (3)$$

If for a such $u \in E, \|u\| = 1$ we have $u \in N_T$, with $\beta_n := \beta_n(u)$ its (normed) residuum, the estimation of $\|T_n u_n\|$ becomes:

$$\begin{aligned} \|T_n u_n\| &= \|(T_n u_n - T u_n) + (T u_n - T u)\| \leq (\|T - T_n\| \|u_n\| + \|T\| \|u - u_n\|) \\ &= (\epsilon_n \|u_n\| + \|T\| \beta_n) \end{aligned}$$

Because $\|u_n\| \rightarrow 1, \epsilon_n \rightarrow 0$ (from i)) and $\beta_n \rightarrow 0$ (from Observation 1), then $\|T_n u_n\| \rightarrow 0$. Now, from iii) and (3),

$$\alpha \|u_n\|^2 \leq (\epsilon_n + \|T\| \beta_n / \|u_n\|) \|u_n\|^2.$$

From $1 = \|u\|^2 = \|u_n\|^2 + \beta_n^2(u)$, results $\beta_n / \|u_n\| = \beta_n / \sqrt{1 - \beta_n^2} \rightarrow 0$. So,

$$0 < \alpha \leq (\epsilon_n + \|T\| \beta_n / \|u_n\|) \rightarrow 0.$$

The inequality is violated from a range $n_1 \geq n_0$, involving $u \notin N_T$. This conclusion is valid for any supposed zero of T in E . Since an element $u \notin S$ then $u \in E$, the eligible set containing also linear combinations between elements from dense and eligible sets. Follows: T has no zeros in the eligible set and, it has no zeros in the dense set, obtaining $N_T = \{0\}$. $\square$

Proof of Lemma 1.

Being Hilbert-Schmidt, the operator admits a sequence of approximations on a dense set. From the convergence to zero of the sequence $\epsilon_n, n \geq 1$ there exists a parameter ϵ_0 such that $\epsilon_0 := \max_n \{\epsilon_n; \epsilon_n < \alpha\}$, corresponding to a subspace $S_{n_0}, n_0 < \infty$. This parameter is independent of any $v \in S$ and, because of the inclusion property, for any $n < n_0$ we have $S_n \subset S_{n_0}$. We could

consider S_{n_0} to be S_1 discarding a finite number of subspaces or, we could consider v to be inside of S_{n_0} . Then:

$$\alpha_n \geq \alpha > \epsilon_0 \geq \epsilon_n \text{ for } n \geq 1, \text{ resulting } (\alpha_n - \epsilon_n) > (\alpha - \epsilon_0) > 0 \forall n \geq 1.$$

For an arbitrary $v \in S$ there exists a coarser subspace (i.e. with a smaller dimension) $S_n, n \geq n_1 := n_1(v)$, for which $v \in S_n$. For it, from the semi positivity property:

$$0 \leq \langle Tv, v \rangle = \langle T_n v, v \rangle + \langle (T - T_n)v, v \rangle.$$

Since T_n is positive on S_n and T is semi positive, $T_n - T$ or $T - T_n$ is positive for that v and, the second inner product in the right side of the equality is real valued.

If $\langle (T_n - T)v, v \rangle \geq 0$, then $0 \leq \langle (T_n - T)v, v \rangle \leq \epsilon_n \|v\|^2$ and,

$$\begin{aligned} \langle T_n v, v \rangle - \langle (T_n - T)v, v \rangle &\geq \langle T_n v, v \rangle - \epsilon_n \|v\|^2 \geq \alpha_n \|v\|^2 - \epsilon_n \|v\|^2 \\ &= (\alpha_n - \epsilon_n) \|v\|^2 > (\alpha - \epsilon_0) \|v\|^2. \text{ Follows} \\ \langle Tv, v \rangle &> (\alpha - \epsilon_0) \|v\|^2. \end{aligned}$$

If instead $\langle (T_n - T)v, v \rangle < 0$, then $\langle Tv, v \rangle$ is a sum of two positive quantities so, greater than any of them:

$$\langle Tv, v \rangle \geq \alpha_n \|v\|^2 \geq \alpha \|v\|^2 > (\alpha - \epsilon_0) \|v\|^2.$$

Thus, taking $\alpha(T) = (\alpha - \epsilon_0) > 0$, $\langle Tv, v \rangle > \alpha(T) \|v\|^2$ holds $\forall v \in S$.

Follows: T_ρ is strict positive on the dense set S and from Theorem 1, $N_T = \{0\}$. $\square$

Proof of Corollary.

Being Hermitian, the operator verifies $\langle Qv, v \rangle \geq 0$, for every $v \in H$. Being Hilbert-Schmidt it is compact and so, it could be approximated on a dense family of finite dimension subspaces. In the hypotheses i)-ii),

$\langle Qv, v \rangle = \langle Q_n v, v \rangle - \langle (Q_n - Q)v, v \rangle \geq 0$ for any $v \in S$. Following the steps from the proof of Lemma 1 we obtain that:

$$\langle Qv, v \rangle \geq (\alpha - \epsilon_0) \|v\|^2 \text{ i.e., } Q \text{ is strict positive on the dense set. } \square$$

3. Operator Approximations on L^2

Let $H := L^2(0, 1)$. The semi-open intervals of equal lengths $h = 2^{-m}, m \in N, nh = 1, \Delta_{h,k} = ((k-1)/2^m, k/2^m], k = 1, n-1$ together with the open $\Delta_{h,n}$ are defining for $m \geq 1$ a partition of $(0, 1)$, $k=1, n, nh = 2^m, nh = 1$. Consider the interval indicator functions having the supports these intervals ($k=1, n, nh=1$):

$$I_{h,k}(t) = 1 \text{ for } t \in \Delta_{h,k} \text{ and } 0 \text{ otherwise} \quad (4)$$

The family F of finite dimensional subspaces S_h that are the linear spans of interval indicator functions of the h -partitions defined by (4) with disjoint supports, $S_h = \text{span}\{I_{h,k}; k = 1, n, nh = 1\}$, built on a multi-level structure, are including $S_h \subset S_{h/2}$ by halving the mesh h . In fact, the property is obtained from (4) observing that any $I_{h,i} \in S_h, i = 1, n, nh = 1$ could be embedded into $S_{h/2}$ by rewriting it as

$$S_h \ni I_{h,i} = I_{h/2, 2i-1} + I_{h/2, 2i} \in S_{h/2}.$$

With the observation that the set $S := \cup_{n \geq 2} S_h, nh = 1$ is dense in $H := L^2(0, 1)$, well known in literature, the prerequisites needed for investigation of injectivity of operators from $\mathcal{L}(H)$ are met. Choosing open subintervals or closed subintervals for partitioning the domain, the union of the subspaces generated in both cases are also dense, as is observed in [9]: if one from three sets is dense, then other two sets are dense. The option for semi-open subintervals partitions ensures the subspaces including property. On another side, for a such partitioning, every pair of indicator subinterval functions has disjoint supports obtaining as result sparse diagonal matrix representations of the operator on finite dimension subspaces spanned by corresponding indicator subinterval functions.

Citing [5], (pg 986), the integral operator $P_h^r, n \geq 1$ having the kernel function:

$$r_h(y, x) = h^{-1} \sum_{k=1, n} I_{h,k}(y) I_{h,k}(x) \quad (5)$$

is a finite rank integral operator orthogonal projection having the spectrum $\{0, 1\}$ with the eigenvalue 1 of the multiplicity n ($nh=1$) corresponding to the orthogonal eigenfunctions $I_{h,k}, k = 1, n$. Then, $\forall u \in H, P_h^r u \in S_h$ and, $(P_h^r)^2 = P_h^r$ for $n \geq 2, nh = 1$. Indeed, for any $u \in H$

$P_h^r u = \int_0^1 r_h u(x) dx = h^{-1} \sum_{k=1,n} c_k I_{h,k} \in S_h$, the constants $c_k, k=1,n$ being $c_k = \int_0^1 u(x) I_{h,k}(x) dx := \langle u, I_{h,k} \rangle$, i.e. P_h^r is an orthogonal projection onto S_h .

Let $T_\rho \in \mathcal{L}(H)$ be a Hilbert-Schmidt integral operator. Its integral operator approximation on S_h denoted by T_{ρ_h} is a finite rank operator approximation, with the kernel function ([5]):

$$\rho_h(y, x) = h^{-1} \sum_{k=1,n} I_{h,k}(y) \rho(y, x) I_{h,k}(x) := h^{-1} \sum_{k=1,n} \rho_h^k(y, x) \quad (6)$$

where the pieces $\rho_h^k, k = 1, n$ of the kernel function ρ_h in the sum have disjoint supports in $L^2(0, 1)^2$, namely $\Delta_{h,k} \times \Delta_{h,k}, k = 1, n, nh = 1$.

The operator approximations of T_ρ are obtained as follows:

$$\begin{aligned} P_h^r(\rho u) &= \int_0^1 r(y, x) \rho(y, x) u(x) dx = h^{-1} \int_0^1 \sum_{k=1,n} I_{h,k}(y) I_{h,k}(x) \rho(y, x) u(x) dx \\ &= \int_0^1 [h^{-1} \sum_{k=1,n} I_{h,k}(y) \rho(y, x) I_{h,k}(x)] u(x) dx \\ &= \int_0^1 \rho_h(y, x) u(x) dx. \end{aligned}$$

$T_{\rho_h} u = P_h^r(\rho u)$. The strong convergence of the sequence of the operator approximations $\{T_{\rho_h}; n \geq 2, nh = 1\}$ follows from:

$$\begin{aligned} T_\rho u - T_{\rho_h} u &= \int_0^1 (\rho u - \rho_h u) dx = \int_0^1 (I - P_h^r)(\rho u) dx. \text{ So} \\ \|T_\rho u - T_{\rho_h} u\| &\leq \|I - P_h^r\| \|\rho\| \|u\| \end{aligned}$$

Since $\{P_h^r; n \geq 2, nh = 1\}$ is a sequence of orthogonal projections on a family of including subspaces whose union is dense, $\|I - P_h^r\| \rightarrow 0$ with $n \rightarrow \infty, nh = 1$, then we have $\epsilon_n := \|T_\rho - T_{\rho_h}\| \rightarrow 0$ for $n \rightarrow \infty, nh=1$, that is the strong convergence of the finite rank approximations, i.e. property i) in Theorem 2/Lemma 1/Corollary.

Note. In both papers [5] and [6] the authors used finite rank approximations like in (6) were dealing with better estimations of the rate of convergence to 0 of the operator's eigenvalues of the integral operators with kernel like Mercer type ([8]).

Remark 1. The matrix representation $M_h^r(T_\rho)$ of T_ρ is a 1-diagonal matrix.

Evaluating $T_{\rho_h} v$ for $v = I_{h,i}$, we obtain

$$\begin{aligned} (T_{\rho_h} I_{h,i})(y) &= h^{-1} \sum_{k=1,n} \int_{\Delta_{h,k}} [I_{h,k}(y) \rho(y, x) I_{h,k}(x)] I_{h,i}(x) dx \\ &= h^{-1} \left[\int_{\Delta_{h,i}} \rho(y, x) I_{h,i}(x) dx \right] I_{h,i}(y) := h^{-1} w(y) I_{h,i}(y) \end{aligned}$$

where $w \notin S_h$ and the support of the function $w I_{h,i}$ is $\Delta_{h,i}$.

In fact, for a function $w \notin S_h, (w I_{h,i}) \notin S_h$ but, once we observe that $I_{h,j} = \overline{I_{h,j}} = I_{h,j}^2$, the following relations are trivial for any element $I_{h,j}, j \neq i$ from orthogonal basis of S_h :

$\langle w I_{h,i}, I_{h,j} \rangle = \langle w I_{h,i}, I_{h,j}^2 \rangle = \langle I_{h,j} w I_{h,i}, I_{h,j} \rangle = 0$ because from $\Delta_{h,i} \cap \Delta_{h,j} = \emptyset$ for $i \neq j$ evaluating the quantity $(I_{h,j} w I_{h,i})(y), i \neq j$ it is zero for any $y \in (0, 1)$. Then with $w = \left[\int_{\Delta_{h,i}} \rho(y, x) I_{h,i}(x) dx \right]$,

$d_{ij}^h := \langle T_{\rho_h} I_{h,i}, I_{h,j} \rangle = \langle w(y) I_{h,i}(y), I_{h,j}(y) \rangle = 0$ and, the matrix representation of the finite rank operator T_{ρ_h} in the basis $\{I_{h,k}, k = 1, n\}$ of S_h , is:

$M_h^r(T_\rho) = h^{-1} \text{diag}[d_{kk}^h]_{k=1,n}$, a sparse 1-diagonal matrix with the diagonal entries given by:

$$d_{kk}^h = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} I_{h,k}(y) \rho(y, x) I_{h,k}(x) dx dy := \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho(y, x) dx dy \quad (7)$$

Remark 2. If the diagonal entries of the matrix representations are strict positive valued, $d_{kk}^h > 0, \forall k = 1, n, nh = 1$, then T_{ρ_h} is strict positive on S_h .

Proof.

Given $v_h = \sum_{k=1,n} c_k I_{h,k} \in S_h$, from $\|v_h\|^2 = h \sum_{k=1,n} c_k \overline{c_k}$, we obtain:

$\langle T_{\rho_h} v_h, v_h \rangle = \sum_{k=1,n} c_k \overline{c_k} d_{kk}^h \geq \alpha_h(T_{\rho_h}) \|v_h\|^2$ where $\alpha_h(T_{\rho_h})$ is the positivity parameter of the finite rank operator approximation T_{ρ_h} ,

$$\alpha_h(T_{\rho_h}) = h^{-2} \min_{(k=1,n)} d_{kk}^h \quad (8)$$

Follows: if there exists a constant $\alpha > 0$ such that $\alpha_h(T_{\rho_h}) \geq \alpha, n \geq 2, nh = 1$ and T_ρ is at least semi positive on S or is Hermitian, then T_ρ is strict positive on the dense set.

4. Injectivity of Alcantara-Bode's Operator

The Hilbert-Schmidt integral operator T_ρ , having the kernel function $\rho(y, x) = \{y/x\}$ the fractional part of the quantity between brackets,

$$(T_\rho u)(y) = \int_0^1 \rho(y, x) u(x) dx, \quad u \in L^2(0, 1)$$

has been used by Alcantara-Bode [2] in his paper on the equivalent formulation of the Riemann Hypothesis. Riemann conjectured ([11]) that the Riemann Zeta function defined by the infinite sum:

$$\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \dots$$

has its non trivial zeros $s = \sigma + it$ on the vertical line $\sigma = 1/2$. A connection between the Riemann Zeta function and the integral operator T_ρ could be found in [4] after reformulating as: $(T_\rho x^{s-1})(\theta) := \int_0^1 \rho(\theta/x) x^{s-1} dx$, the left term in the equality below

$$\int_0^1 \rho(\theta/x) x^{s-1} dx = \theta/(s-1) - \theta^s \zeta(s)/s, \quad \sigma > 0, s = \sigma + it.$$

The kernel function $\rho \in L^2(0, 1)^2$ is continue almost everywhere, the discontinuities in $(0, 1)^2$ consisting in a set of numerable one dimensional lines of the form $y = kx, k \in \mathbb{N}$, being of Lebesgue measure zero and so, Riemann integrable.

We will prove that $N_{T_{\hat{\rho}}} = \{0\}$ knowing that $N_{T_\rho} = N_{T_{\hat{\rho}}}$, where $T_{\hat{\rho}} = (T_\rho^* T_\rho)$ is the associated Hermitian integral operator having the kernel function

$$\hat{\rho}(y, x) := \int_0^1 \rho(y, t) \overline{\rho(t, x)} dt = \int_0^1 \rho(y, t) \rho(t, x) dt = \int_0^1 \varphi(y, x, t) dt$$

We are in the Corollary scenario, the associated Hermitian being a Hilbert-Schmidt operator semi positive on the dense set and its finite rank approximations are satisfying the strong convergence condition i) from Theorem 2/Lemma 1, positioning ourselves in the environment of the previous section. For finding the positivity parameters sequence of finite rank approximations, we will proceed with the computation of the diagonal entries (7) in the matrix representations.

Then, for $y, x \in \Delta_{h,k}, t \in \Delta_{h,j}$ the function φ to be integrated in variable t has the form

$$\begin{aligned} \varphi(y, x, t) &= (y/t - \lfloor y/t \rfloor)(t/x - \lfloor t/x \rfloor) \\ &= y/x - (t/x) \lfloor y/t \rfloor = (1/x)(y - t \lfloor y/t \rfloor) \quad \text{for } j < k, (t < kh): \\ &= y/x - (y/t) \lfloor t/x \rfloor = y(1/x - 1/t \lfloor t/x \rfloor) \quad \text{for } j > k, (t > kh): \end{aligned}$$

where $\lfloor \cdot \rfloor$ is the floor function.

The strategy consists in approximating for every $\Delta_{h,k}^2, k = 1, n$ the integral in variable t then to find the minimum value that multiplied n times will be an inferior bound for the sum of the components of the k -entry in the 1-diagonal matrix representation. So, the components to be evaluated are the integrals in variable t on $\Delta_{h,j}$, for $j = 1, n$, as follows: $d_{kk}^{h,j=k}, d_{kk}^{h,j < k}$ and $d_{kk}^{h,j > k}$. Then,

$$\hat{d}_{kk}^h = \sum_{j=1,n} d_{kk}^{h,j} \geq n \cdot \min_{j=1,n} d_{kk}^{h,j}.$$

The the computed components of a generic k entry in the matrix representation are evaluated in the paragraph below showing that the minimum value is obtained for $d_{kk}^{h,j=1}$ bounded inferior by $\frac{ln2}{16} h^3$ (see (9) below) for which the entry in the matrix is bounded inferior by ((11)),

$$\hat{d}_{kk}^h \geq n \cdot d_{kk}^{h,j=1} \geq \frac{ln2}{16} h^2, nh = 1, \forall k = 1, n. \text{ Hence, for any } n \geq 2, nh = 1, \text{ the positivity parameter}$$

on S_h is given by

$$\alpha_h(T_{\hat{\rho}_h}) = h^{-2} \min_{k=1,n} \hat{d}_{kk}^h \geq \frac{ln2}{16}.$$

Follows: the Corollary holds for the Hermitian operator of T_ρ taking $\alpha = \frac{ln2}{16}$.

Theorem 3. Alcantara-Bode integral operator is injective, equivalently $N_{T_\rho} = \{0\}$. Thus, Riemann Hypothesis holds.

Proof.

The associated Hermitian operator of the Hilbert-Schmidt integral operator T_ρ is semi positive definite on the dense set S , it allowing a sequence of strict positive finite rank approximations whose positivity parameters are inferior bounded by a constant. This constant is valued from (11) by applying (8) to:

$\alpha_h(T_{\hat{\rho}_h}) \geq \frac{\ln 2}{16} > 0$. Then, for every $v \in S$ there exists a subspace S_h containing it, for which $\langle T_{\hat{\rho}_h} v, v \rangle \geq \frac{\ln 2}{16} \|v\|^2$ and so, by Lemma 1 the integral operator is strict positive on the dense set. Then, its null space is $N_{T_{\hat{\rho}}} = \{0\}$. From $N_{T_\rho} = N_{T_{\hat{\rho}}}$, $N_{T_\rho} = \{0\}$ equivalently, the Alcantara-Bode integral operator is injective and RH holds. $\square$

5. Computation of Positivity Parameters

A.) *The case of associated Hermitian.* The following are the estimates of the components of the diagonal entries.

1) For $y \in ((k-1)h, kh]$ and $t \in ((j-1)h, jh], j < k$. With $z, t \in \Delta_{k-j}, y \in \Delta_k, y = z + (k-j-1)h$,

$\int_{\Delta_k} (y - t \lfloor y/t \rfloor) dy = \int_{\Delta_k} y dy - \int_{\Delta_k} t \lfloor (z + (k-j-1)h)/t \rfloor dz$. Then, integrating in variable t followed by integration in x , we obtain:

$$\begin{aligned} d_{kk}^{h,j < k} &:= \int_{\Delta_k} \int_{\Delta_k} \int_{\Delta_{k-j}} 1/x(y - t \lfloor y/t \rfloor) dx dy dt \\ &= h^3 \ln(k/(k-1))^{k-1} [1/2 + 1/(k-1/2) + 4/3((k-1)) + j/(k-1)] \end{aligned}$$

obtaining the monotony of the sequence $d_{kk}^{h,j < k} \uparrow$ when $j \uparrow$ the smaller in the sequence being obtained for $j=2$:

$$d_{kk}^{h,j=2} = h^3 \ln(k/(k-1))^{k-1} [1/2 + j/(k-1) + 1/(k-1/2) + 4/3((k-1))]$$

where $\ln(k/(k-1))^{k-1} \uparrow 1$ smallest value being $\ln 2$ obtained for $k=2$ and the sequence $d_{kk}^{h,2 \leq j \leq (k-1)} \uparrow$ with $j \uparrow$.

For obtaining an estimate of $d_{kk}^{h,j=1}$ we will compare it with the estimate on the the upper grid $h/2$: $d_{kk}^{h,j=1} > d_{kk}^{h/2,j=1} + d_{kk}^{h/2,j=2} > d_{kk}^{h/2,j=2} \geq d_{kk}^{h,j=2}/8$

Then, $\min_{1 \leq j < k} \{d_{kk}^{h,j < k}\} = d_{kk}^{h,j=1}$, and,

$$d_{kk}^{h,j=1} \geq 1/8 d_{kk}^{h,j=2} \geq (\ln 2/8) h^3 (1/2 + 1/(k-1/2) + 10/(3(k-1))). \text{ Further,}$$

$$d_{kk}^{h,j=1} \geq h^3 \ln 2 / 16 \quad (9)$$

2) For $x \in ((k-1)h, kh], t \in \Delta_{h,j}$ for $j \geq k$.

Then $j/(k-1) \geq \lfloor t/x \rfloor \geq (j-1)/k$ and,

$$\int_{\Delta_{h,j > k}} t \lfloor t/x \rfloor dt \geq \int_{\Delta_{h,j > k}} (1/t)(j-1)/k = 1/k \ln(j/(j-1))^{j-1} \uparrow \text{ with } j \uparrow n \text{ proving the increasing}$$

monotony of $d_{kk}^{h,j > k}$ with $j \uparrow n$

For $j = k+1$, $\lfloor t/x \rfloor = 1$ and, the minimum value is given by:

$$\begin{aligned} d_{kk}^{h,j=k+1} &= \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \int_{\Delta_{h,k+1}} (y(1/x - 1/t \lfloor t/x \rfloor)) dx dy dt \\ &= h^3 (2k-1) \ln(k+1)/(k-1). \text{ Follows,} \end{aligned}$$

$$\min_{(j;j > k)} d_{kk}^{h,j > k} = d_{kk}^{h,j=k+1} \approx h^3 3.3 \quad (10)$$

$$\begin{aligned} 3) d_{kk}^{h,j=k} &= \int_{\Delta_k} \int_{\Delta_k} \int_{\Delta_k} I_{h,k}(y) \varphi(y, x) I_{h,k}(x) dx dy dt \\ &= \int_{\Delta_k} (h^2 (k-1/2) \cdot 1/t - kh + t) (t \cdot \ln(k/(k-1)) - t + (k-1)h) dt \end{aligned}$$

Ordering the product and integrating in t , for $k \geq 2$ results:

$$d_{kk}^{h,j=k} \geq h^3 [\ln 2 (2k/3 - 1/3) + 4/3] \geq h^3 (2 \ln 2 / 3 + 4/3). \text{ Then}$$

$$\hat{d}_{kk}^h \geq n \cdot d_{kk}^{h,j=1} \geq h^2 \ln 2 / 16 \quad (11)$$

and the positivity constant is $\alpha = \ln 2/16 > 0$ independent from the mesh h .

B.) On T_ρ positivity.

The entries in the diagonal matrix representation $M_h^r(T_\rho)$ of the finite rank integral operator T_{ρ_h} are given by: $d_{kk}^h = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \{y/x\} dx dy$, valued in [1]:

$d_{11}^h = h^2(3 - 2\gamma)/4$; $d_{kk}^h = \frac{h^2}{2}(-1 + \frac{2k-1}{k-1} \ln(\frac{k}{k-1})^{k-1})$, for $k \geq 2$ where γ is the Euler-Mascheroni constant, and:

$$\alpha_h(T_{\rho_h}) = h^{-2}d_{11}^h = (3 - 2\gamma)/4 > 0, \text{ for any } h, nh = 1.$$

To use Theorem 2 or Lemma 1 we should prove that T_ρ is at least semi positive on S , process that we bypassed using Corollary for its associated Hermitian operator.

Note. In [1] the author used in the Injectivity Criteria the finite rank operator approximations involving the adjoint operator restrictions. At that time, the theory presented in [9] were not finalised. We consider this paper as a continuation of [9], pointing out a robust criterion for the injectivity of liner bounded operators.

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