Technical Note

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Abstract

For the calibration of surface plate, the classical Moody method is still commonly used. In this method the straightness of a number of lines over a surface plate in a union-jack configuration are measured and combined to a flatness measurement. The measurement of two center lines is commonly omitted in the evaluation and only used to determine so-called closure errors. These two lines can be incorporated in the measurement evaluation in a least-squares sense, giving an 18% reduction of the uncertainty. A further reduction in the uncertainty is possible when using the gravity vector as a common reference, as can be done when using electronic levels. A least-squares evaluation of measurements taken in this way gives a further reduction of the uncertainty of 29% relative to the traditional evaluation according to the Moody method. This is illustrated with an actual measurement example and additional Monte-Carlo simulations.

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1. Introduction

The measurement of flatness, and in particular the flatness measurements of surface plates, is an established measurement problem in dimensional metrology. For flatness measurements of surface plates, traditional methods are based on either autocollimators, electronic levels, or laser interferometer systems operated in angular measurement mode, where the variation in the gradient between two sides of a base is used to calculate the surface flatness [1]. The measurement can take place on a rectangular grid, where it is recognized that diagonals must be measured if autocollimators or laser interferometer systems with angular optics (on a base) are used [2]. The configuration where measurements are taken for two diagonals and three lines in both the x- and y direction is called the Union Jack configuration because of its resemblance to the flag of the United Kingdom. The measuring method using this configuration is also called the Moody method after the author that first explicitly gave an evaluation method [3] based on an earlier indication in a classical textbook [4].

The Union Jack configuration is well scalable, so that also for large surfaces measurements can be taken within a reasonable amount of time, and the calculation after Moody is reasonable straightforward. This evaluation method is currently in widespread use and implemented in software of most manufacturers of autocollimators, electronic levels, and displacement laser interferometer systems and is prescribed in ASME B89.3.7 [5]. The evaluation and uncertainty of rectangular grid configurations has been studied rather extensively [6–8], the sources of uncertainty in the evaluation according to the Moody method were evaluated by Drescher [9], taking the established method as the basis.

In this paper the original Moody evaluation is reviewed as a solution of the more general measurement problem that is presented as a matrix calculation. Based on this presentation the measurements of the central lines that are omitted in the calculation of the node points according to Moody, are integrated in a least-squares sense. For measurement using electronic levels the gravity vector can be used as a common reference for measurement lines having a same direction. This calculation, that potentially gives a further error reduction, is shown and evaluated as well. An

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appealing feature of the Moody evaluation is that the reference plane, defined by equaling the endpoints of the respective diagonals, is integrated in the calculation. However this reference plane may be different to the minimum-zone plane that should be the basis for a flatness evaluation.

The uncertainty of these three evaluation methods with an additional with a minimum-zone evaluation regarding the reference plane, is illustrated for a measurement example, supplemented with Monte-Carlo simulations.

2. Measurement Configuration and Basic Evaluation

In the basic measurement configuration, the straightness of three lines is measured in both the x- and y- direction, and two diagonal lines. This is illustrated in Figure 1, where the rectangular horizontal surface to be measured is viewed from above.

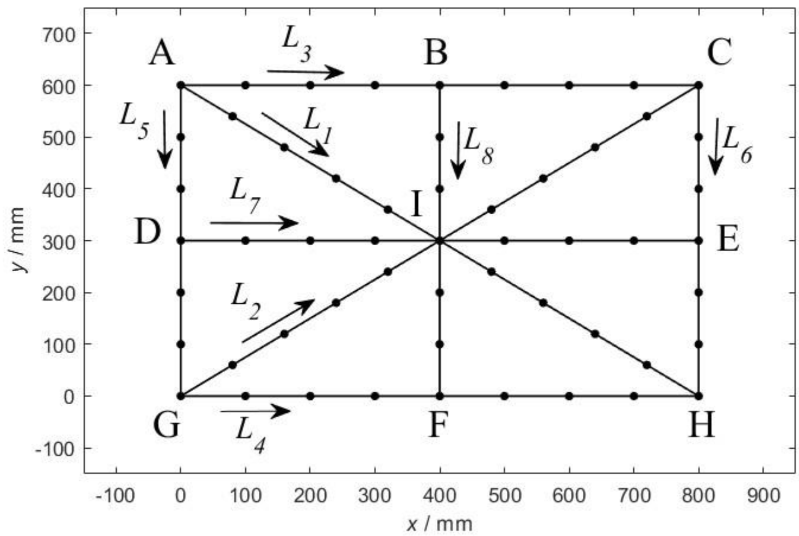

Figure 1. Basic Union Jack configuration where the surface plate is viewed from above, with the measurement lines indicated as $L_{1..8}$ and the node points as A..I.

Indicated are the measured lines $L_{1..8}$ and the node points A..I. The indication and orientation of these lines and points may vary between various publications and software implementations. The heights of the node points are denoted by *A..I*. The number of measured points taken at each line may vary depending on the measurement base and the size of the object. In the example sketched in Figure 1, N_x=8 height differences are measured along the lines in the x-direction, N_y=6 height differences along the lines in the y-direction and N_d =10 height differences are measured along the diagonal lines. The individual measurements $m_{k,i}$ indicate the height difference of the measured points along line L_k. These measurements will have a bias depending on the setting of the measuring device, and a bias may be subtracted from the measurements of each line as basically the variations in the gradients are relevant rather than the gradients themselves. For example if points X and Y have a defined height *X* and *Y*, the heights $h_{k,j}$, j=0..*N* along a line *L* between these points are given by:

$$h_{i,j} = X + \sum_{i=1}^{j} m_{k,i} - \frac{j}{N}\sum_{i=1}^{N} m_{k,i} + (Y - X)\frac{j}{N}. \quad (1)$$

If the points X and Y have the same height and the height of the point in the center is defined as zero, the heights along a line *L* are given by:

$$h_{1,j} = \sum_{i=1}^{j} m_{k,i} - \frac{j}{N}\sum_{i=1}^{N} m_{k,i} - \sum_{i=1}^{\frac{N}{2}} m_{k,i} + \frac{1}{2}\sum_{i=1}^{N} m_{k,i}. \quad (2)$$

The basic measurements used for the calculations of the heights of the nodes *A..I* are the height differences measured between these points, denoted by $M_{1..16}$ and indicated in Figure 2, for example:

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$$M_6 - M_5 = A + C - 2B = \sum_{i=\frac{1+N_x}{2}}^{N_x} m_{1,i} - \sum_{i=1}^{N_x/2} m_{1,i}. \tag{3}$$

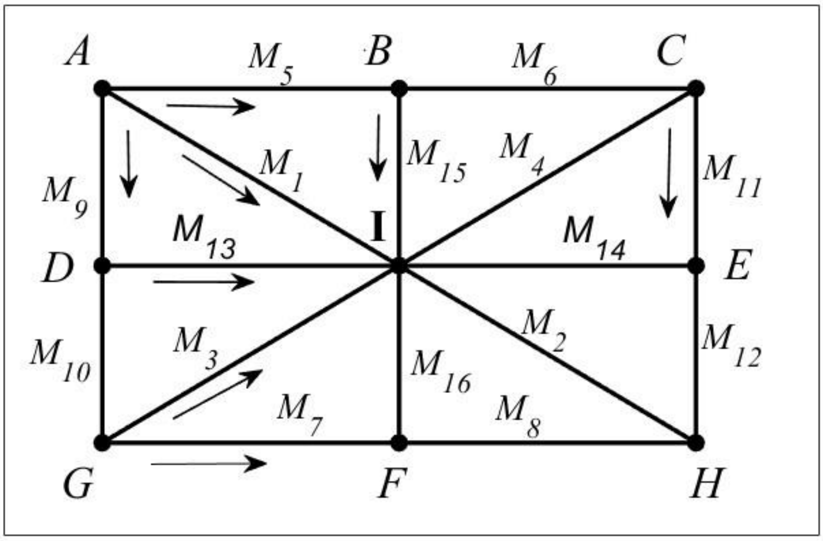

Figure 2. Basic Union Jack configuration indication the measurements of the height differences $M_{1..16}$ from which the heights of the nodes $A..I$ must be evaluated.

Assuming that the reference directions of the lines are not related, as is commonly the case when the measurements are taken using an autocollimator or a laser interferometer, with a mirror or angular optics respectively on a measurement base, the measurement situation for calculating the heights $A..I$ can be summarized in matrix-notation as given in Equation 4:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} M_2 - M_1 \\ M_4 - M_3 \\ M_6 - M_5 \\ M_8 - M_7 \\ M_{10} - M_9 \\ M_{12} - M_{11} \\ M_{14} - M_{13} \\ M_{16} - M_{15} \end{pmatrix}$$

. (4)

This set of equations cannot be solved as the height and orientation of the reference plane is not defined. In the next sections it is shown how this is solved using the Moody-method and alternatives are investigated.

3. Measurement Evaluation

3.1. Evaluation According to Moody

In the original Moody method, measurements $M_{13}..M_{16}$ are not used in the calculation of the heights of points $A..I$, however at the end of the evaluations the lines L_7 and L_8 are added to the

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already calculated heights *B*-*E* and *D*-*F* respectively, using Equation 1. In the evaluation, the point I is defined as having a zero height (*I*=0), and the endpoints of the diagonals are defined to have a same value; this means that *H*=*A* and *G*=*C*. Substituting this in (4) gives as characteristic matrix Equation 5:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -2 \\ 1 & 0 & 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} M_2 - M_1 \\ M_4 - M_3 \\ M_6 - M_5 \\ M_8 - M_7 \\ M_{10} - M_9 \\ M_{12} - M_{11} \end{pmatrix} . \qquad (5)$$

This equation involves a square matrix that can readily be inverted, giving the solution given in Equation 6:

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0.25 & -0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0 & 0 & -0.5 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0 & -0.5 \\ 0.25 & 0.25 & 0 & -0.5 & 0 & 0 \end{pmatrix} \begin{pmatrix} M_2 - M_1 \\ M_4 - M_3 \\ M_6 - M_5 \\ M_8 - M_7 \\ M_{10} - M_9 \\ M_{12} - M_{11} \end{pmatrix} . \qquad (6)$$

For example: A=H=0.5(M_2-M_1) and B=0.5(M_5-M_6)+0.25*(M_2+M_4-M_1-M_3). After these heights are determined the points in-between are calculated using Equation 1. Then the height of the lowest point is added to all other measured points and the surface is plotted using these values. The height of the highest point is the flatness deviation FLTt(EDPL) with the diagonal endpoints (ED) defining the reference plane (PL).

This is equivalent to the original sequence of calculations given by Moody:

1. The diagonals A-H and G-C are evaluated using Equation 2, giving *A*,*C*, *G* and *H* and all other points on the lines.
2. The perimeter lines A-C, C-H, A-G and G-H are evaluated using Equation 1, giving *B*, *D*, *E* and *F* and all other points on the line
3. The inner lines B-E and D-F are evaluated using Equation 1 giving all points on the line including additional values for *I* that are called the *closure errors*.
4. The height of the lowest point is added to all other measured points and the surface is plotted using these values. The height of the highest point is the flatness deviation FLTt(EDPL).

Equation 6 can be used to evaluate the propagation of uncertainties. Assuming a same uncertainty $u(M_i)$ in all measurements M_i and taking these measurements as independent, we find for the uncertainties in the corners *A*, *C*, *G* and *H*: $u(A,C,G,H) = u(M_i)/\sqrt{2}$ and for the node points along the edges $u(A,C,G,H) = u(M_i)\sqrt{3/4}$.

This result can be refined considering the amount of measurements taken along each line. Assuming the uncertainties in the individual measurements [illegible]([illegible]) are equal and independent, assuming a same uncertainty in the measured angle and a same length of the measurement base, , it can be derived from Equation 3 that $u\ (M_2 - M_1) = u\ (M_4 - M_3) = u(m\ )\sqrt{N_d}$; $u\ (M_6 - M_5) = u\ (M_8 - M_7) = u(M_{14} - M_{13}) = u(m\ )\sqrt{N_x}$ and $u\ (M_{12} - M_{11}) = u\ (M_{10} - M_9) = u(M_{16} - M_{15}) = u(m\ )\sqrt{N_y}$.

This gives for the uncertainty in the heights of the corner points $u(A)$ = u(C)=u(G)=$u(H)$ = 0.5·$\sqrt{[illegible]}$·$u(m)$, for the uncertainty in the heights of points B and F along the perimeter $u(B) =$

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$u(F) = u(m\ )\sqrt{0.125 \cdot N_d + 0.25 \cdot N_x}$ and for the uncertainty in the heights of points D and E along the edges $u(D) = u(E) = u(m\ )\sqrt{0.125 \cdot N_d + 0.25 \cdot N_y}$. For example: for N_d=10, N_x=8 and N_y=6 we find $u(A,C,G,H)$=1.58·$u(m)$, $u(B,F)$=1.80·$u(m)$ and $u(D,E)$=1.66·$u(m)$.

The lines L_7 and L_8 may not give a zero value in point I. In the original Moody calculation method, these discrepancies are taken as a measure for the measurement error, and are called 'closure errors'. The closure errors can be denoted as E_{BF} and E_{DE} and can be written as the measured height of I along the lines L_7 and L_8 respectively as following:

$$E_{BF} = 0.5 \cdot (B + F + M_{16} - M_{15}) = 0.25\big((M_1 - M_2) + (M_3 - M_4) + (M_6 - M_5) + (M_8 - M_7)\big) + 0.5(M_{16} - M_{15}) \quad (7)$$

and

$$E_{DE} = 0.5 \cdot (D + E + M_{14} - M_{13}) = 0.25\big((M_1 - M_2) + (M_3 - M_4) + (M_{10} - M_9) + (M_{12} - M_{11})\big) + 0.5(M_{14} - M_{13}) \quad (8)$$

For perfect measurements with zero uncertainty, $u(m_i) = 0$, these closure errors are zero. The expected deviations from zero of the closure errors for real measurements can be expressed as the uncertainty in their value as:

$$\sqrt{\langle E_{BF}\rangle^2} = u(m\ ) \cdot \sqrt{0.125 \cdot N_d + 0.125 \cdot N_x + 0.25 \cdot N_y}, \quad (9)$$

and

$$\sqrt{\langle E_{DE}\rangle^2} = u(m\ ) \cdot \sqrt{0.125 \cdot N_d + 0.125 \cdot N_y + 0.25 \cdot N_x} \quad (10)$$

For example: for N_d=10, N_x=8 and N_y=6 we find for the expected closure error in the y-direction $\sqrt{\langle E_{BF}\rangle^2} = 1.93 \cdot u(m\ )$ and for the expected closure error in the x-direction $\sqrt{\langle E_{DE}\rangle^2} = 2 \cdot u(m\ )$. Comparing this with the expected errors in the points *A..H* this means that the closure errors give a reasonable estimate of the errors in the points *A..H*.

From Equations 7 and 8 an estimate of the uncertainty can be made; however both equations are dependent. Using the Gram-Schmidt orthogonalization principle the closure error E_{DE} can be modified to give a measure of the uncertainty $u(m\ )$ that is independent of E_{BF}, giving the following equation:

$$E_{DEM} = E_{DE} - 0.25 * E_{BF} = \frac{3}{16}\big((M_1 - M_2) + (M_3 - M_4)\big) + \frac{1}{16}\big((M_5 - M_6) + (M_7 - M_8)\big) + \frac{1}{4}\big((M_{10} - M_9) + (M_{12} - M_{11})\big) + 0.5(M_{14} - M_{13}) + \frac{1}{8}(M_{15} - M_{16}). \quad (11)$$

The expected value can be expressed as:

$$\sqrt{\langle E_{DEM}\rangle^2} = u(m\ ) \cdot \sqrt{0.07 \cdot N_d + 0.258 \cdot N_x + 0.141 \cdot N_y}. \quad (12)$$

The uncertainty $u(m\ )$ can be estimated from the root mean square of the $u(m\ )$ values derived from Equations 9 and 12:

$$u(m\ ) = \sqrt{\frac{E_{DEM}^{\ 2}}{2(0.07 \cdot N_d + 0.258 \cdot N_x + 0.141 \cdot N_y)} + \frac{E_{BF}^2}{2(0.125 \cdot N_d + 0.125 \cdot N_x + 0.25 \cdot N_y)}}. \quad (13)$$

This means that that the standard uncertainty $u(m\ )$ can be estimated from the measurements with two degrees of freedom.

3.2. Least-Squares Evaluation Based on Straightness Measurements

Measurements $M_{13..16}$ can be involved in the evaluation rather than giving closure errors in point I. This extends the matrix given in Equation 5 to:

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$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -2 \\ 1 & 0 & 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} M_2 - M_1 \\ M_4 - M_3 \\ M_6 - M_5 \\ M_8 - M_7 \\ M_{10} - M_9 \\ M_{12} - M_{11} \\ M_{14} - M_{13} \\ M_{16} - M_{15} \end{pmatrix} \quad (14)$$

Assuming the measurements $M_{1..16}$ have equal uncertainties, Equation 14 can be written as $\boldsymbol{\alpha}\bar{a} = \bar{y}$, with as the least-squares solution $\bar{a} = (\boldsymbol{\alpha}' \cdot \boldsymbol{\alpha})^{-1}\boldsymbol{\alpha}\ \bar{y}$, where the '-sign denotes the transposed matrix and $(\alpha' \cdot \alpha)^{-1}\alpha$ is the pseudo-inverse. This gives as a solution:

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \frac{1}{60} \begin{pmatrix} 24 & -6 & 3 & 3 & 3 & 3 & 6 & 6 \\ 6 & 6 & -23 & 7 & 2 & 2 & 4 & 14 \\ -6 & 24 & 3 & 3 & 3 & 3 & 6 & 6 \\ 6 & 6 & 2 & 2 & -23 & 7 & 14 & 4 \\ 6 & 6 & 2 & 2 & 7 & -23 & 14 & 4 \\ 6 & 6 & 7 & -23 & 2 & 2 & 4 & 14 \end{pmatrix} \begin{pmatrix} M_2 - M_1 \\ M_4 - M_3 \\ M_6 - M_5 \\ M_8 - M_7 \\ M_{10} - M_9 \\ M_{12} - M_{11} \\ M_{14} - M_{13} \\ M_{16} - M_{15} \end{pmatrix} \quad (15)$$

This gives I=0 for all lines through the center. The uncertainty in the corner points is given by $u(A,C,G,H) = u(m) \cdot \sqrt{0.17 \cdot N_d + 0.015 \cdot N_x + 0.015 \cdot N_y}$, for the uncertainty in the points B and F along the perimeter $u(B,F) = u(m)\sqrt{0.02 \cdot N_d + 0.165 \cdot N_x + 0.0567 \cdot N_y}$ and for the uncertainty in the points D and E along the perimeter $u(D,E) = u(m)\sqrt{0.02 \cdot N_d + 0.0567 \cdot N_x + 0.165 \cdot N_y}$. For example: for N_d=10, N_x=8 and N_y=6 we find $u(A,C,G,H)$=1.382·$u(m)$, $u(B,F)$=1.364·$u(m)$ and $u(D)$=1.282·$u(m)$. Obviously, the uncertainty in the points along the perimeters benefits most from incorporating the lines L_7 and L_8 in the evaluation. Overall, the uncertainty in the height of the nodes is reduced by 18% by applying the least-squares solution instead of the traditional 'Moody' evaluation, and the height of the center point I is fixed unequivocally.

The solution can be refined by weighing the measurements by the inverse square of their uncertainty, replacing Equation 14 by Equation 16:

$$\begin{pmatrix} 2/N_d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/N_d & 0 & 0 & 0 \\ 1/N_x & -2/N_x & 1/N_x & 0 & 0 & 0 \\ 1/N_x & 0 & 1/N_x & 0 & 0 & -2/N_x \\ 1/N_y & 0 & 1/N_y & -2/N_y & 0 & 0 \\ 1/N_y & 0 & 1/N_y & 0 & -2/N_y & 0 \\ 0 & 0 & 0 & 1/N_x & 1/N_x & 0 \\ 0 & 1/N_y & 0 & 0 & 0 & 1/N_y \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} (M_2 - M_1)/N_d \\ (M_4 - M_3)/N_d \\ (M_6 - M_5)/N_x \\ (M_8 - M_7)/N_x \\ (M_{10} - M_9)/N_y \\ (M_{12} - M_{11})/N_y \\ (M_{14} - M_{13})/N_x \\ (M_{16} - M_{15})/N_y \end{pmatrix} \quad (16)$$

and proceed as above. However we found no significant further decrease in uncertainties for this and other grid sizes so we conclude that Equation 15 can well be applied generally for reasonably square measurement configurations with no major differences in the number of measured points in the lines.

The heights of the points in-between the nodes may be calculated using Equation (1). However the adjustment of the points means that in fact the straightness measurements along lines $L_1..L_8$ are adjusted halfway the lines and a kink in a line may appear in a measurement that gave an apparent straight line. A more smooth adjustment can be obtained by making a parabolic correction to the measured line, for example for line L_3:

$$h_{3,j} = A + \sum_{i=1}^{j} m_{1,i} - (1 + C - A)\frac{j}{N_x}\sum_{i=1}^{N_x} m_{1,i} + 4\cdot(A + C - 2B - M_6 + M_5)\cdot\left(\left(\frac{j}{N_x}\right) - \left(\frac{j}{N_x}\right)^2\right). \quad (17)$$

This may especially be useful if many points are taken along the lines, for example by a profilometer.

The uncertainty [illegible]([illegible]) may still be estimated from Equations 7,8,11 and 13. An alternative is the calculation of the Chi-square value given by:

$$\text{[illegible]}^2 = (2\text{[illegible]} - \text{[illegible]}_2 + \text{[illegible]}_1)^2 + (2\text{[illegible]} - \text{[illegible]}_4 + \text{[illegible]}_3)^2 + (\text{[illegible]} - 2\text{[illegible]} + \text{[illegible]} - \text{[illegible]}_6 + \text{[illegible]}_5)^2 + (\text{[illegible]} + \text{[illegible]} - 2\text{[illegible]} - \text{[illegible]}_8 + \text{[illegible]}_7)^2 + (\text{[illegible]} + \text{[illegible]} - 2\text{[illegible]} - \text{[illegible]}_{10} + \text{[illegible]}_9)^2 + (\text{[illegible]} + \text{[illegible]} - 2\text{[illegible]} - \text{[illegible]}_{12} + \text{[illegible]}_{11})^2 + (\text{[illegible]} + \text{[illegible]} - \text{[illegible]}_{14} + \text{[illegible]}_{13})^2 + (\text{[illegible]} + \text{[illegible]} - \text{[illegible]}_{16} + \text{[illegible]}_{15})^2. \quad (18)$$

Because of the evaluation method the terms in Equation 18 are very correlated, because there are only two degrees of freedom (6 heights are calculated from 8 independent measurements), for example the first and the second term are qual, as are the third and fourth term. From Equation 18 the measurement uncertainty [illegible]([illegible]) can be estimated as:

$$u(m\ ) = \sqrt{\frac{\chi^2}{0.5\cdot N_d + 0.75\cdot N_x + 0.75\cdot N_y}} \quad (19)$$

In practice this gives the same result as Equation 13.

3.3. Least-squares Evaluation Based on Lines Having a Common Reference

When using electronic levels for the gradient measurements, the parallelism between the lines can be measured as well by keeping the same reference direction, for example by keeping the reference level at a same position on the object during the measurements in the x- and y-directions respectively. In this case, instead of the second derivative of each line, the height differences between the node points can be considered directly. Unless the absolute gradient to the gravity vector is measured, the reference directions for the diagonals are still undetermined, therefore for these lines the second derivative is considered. The reference height is defined again by defining I=0.

This gives as equations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{pmatrix} = \begin{pmatrix} M_2 - M_1 \\ M_4 - M_3 \\ M_5 - \alpha \\ M_6 - \alpha \\ M_7 - \alpha \\ M_8 - \alpha \\ M_9 - \beta \\ M_{10} - \beta \\ M_{11} - \beta \\ M_{12} - \beta \\ M_{13} - \alpha \\ M_{14} - \alpha \\ M_{15} - \beta \\ M_{16} - \beta \end{pmatrix}. \quad (20)$$

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Here the constants α and β represent the tilt of the reference plane, expressed as the height difference between the points measured by M_i in the x- and y- direction respectively. The least-squares solution of Equation 20 is:

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{pmatrix} = \frac{1}{96}\begin{pmatrix} 30 & 6 & -33 & -15 & -9 & -15 & -33 & -15 & -9 & -15 & -18 & -6 & -18 & -6 \\ 12 & 12 & 26 & -26 & 2 & -2 & -14 & -10 & -14 & -10 & -4 & 4 & -44 & -4 \\ 6 & 30 & 15 & 33 & 15 & 9 & -9 & -15 & -33 & -15 & 6 & 18 & -18 & -6 \\ 12 & 12 & -14 & -10 & -14 & -10 & 26 & -26 & 2 & -2 & -44 & -4 & -4 & 4 \\ 12 & 12 & 10 & 14 & 10 & 14 & 2 & -2 & 26 & -26 & 4 & 44 & -4 & 4 \\ 12 & 12 & 2 & -2 & 26 & -26 & 10 & 14 & 10 & 14 & -4 & 4 & 4 & 44 \\ 6 & 30 & -9 & -15 & -33 & -15 & 15 & 33 & 15 & 9 & -18 & -6 & 6 & 18 \\ 30 & 6 & 15 & 9 & 15 & 33 & 15 & 9 & 15 & 33 & 6 & 18 & 6 & 18 \end{pmatrix}\begin{pmatrix} M_2-M_1 \\ M_4-M_3 \\ M_5-\alpha \\ M_6-\alpha \\ M_7-\alpha \\ M_8-\alpha \\ M_9-\beta \\ M_{10}-\beta \\ M_{11}-\beta \\ M_{12}-\beta \\ M_{13}-\alpha \\ M_{14}-\alpha \\ M_{15}-\beta \\ M_{16}-\beta \end{pmatrix} \quad (21)$$

This gives as a solution the heights $A..H$ in a tilted plane, that can be levelled separately to the diagonal endpoints, the least-squares plane or the minimum-zone plane. In order to enable a comparison of the uncertainties to the solutions given in section 3.1 and 3.2, a levelling to equal the diagonal endpoints must be integrated in the solution. This is achieved as following:

Taking A-H = 0 from Equation 14 gives an equation for $\alpha + \beta$; and taking C-G = 0 gives an equation for α - β. Solving these two equations gives:

$$\alpha = \frac{3}{16}(M_5 + M_6 + M_7 + M_8) + \frac{1}{16}(M_9 - M_{10} - M_{11} + M_{12}) + \frac{1}{8}(M_{13} + M_{14}), \quad (22)$$

and

$$\beta = \frac{1}{16}(M_5 - M_6 - M_7 + M_8) + \frac{3}{16}(M_9 + M_{10} + M_{11} + M_{12}) + \frac{1}{8}(M_{15} + M_{16}). \quad (23)$$

Using Equations 22 and 23, the variables in the right-hand side of Equation 21 can be expressed in the measurements M_i by the following transformation:

$$\begin{pmatrix} M_2-M_1 \\ M_4-M_3 \\ M_5-\alpha \\ M_6-\alpha \\ M_7-\alpha \\ M_8-\alpha \\ M_9-\beta \\ M_{10}-\beta \\ M_{11}-\beta \\ M_{12}-\beta \\ M_{13}-\alpha \\ M_{14}-\alpha \\ M_{15}-\beta \\ M_{16}-\beta \end{pmatrix} = \frac{1}{16}\begin{pmatrix} -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 13 & -3 & -3 & -3 & -1 & 1 & 1 & -1 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 13 & -3 & -3 & -1 & 1 & 1 & -1 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -3 & 13 & -3 & -1 & 1 & 1 & -1 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -3 & -3 & 13 & -1 & 1 & 1 & -1 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 13 & -3 & -3 & -3 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -3 & 13 & -3 & -3 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -3 & -3 & 13 & -3 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -3 & -3 & -3 & 13 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & -3 & -3 & -3 & -3 & -1 & 1 & 1 & -1 & 14 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -3 & -3 & -3 & -1 & 1 & 1 & -1 & -2 & 14 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -3 & -3 & -3 & -3 & 0 & 0 & 14 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -3 & -3 & -3 & -3 & 0 & 0 & -2 & 14 \end{pmatrix}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_8 \\ M_9 \\ M_{10} \\ M_{11} \\ M_{12} \\ M_{13} \\ M_{14} \\ M_{15} \\ M_{16} \end{pmatrix} \quad (24)$$

Combining Equations 21 and 24 gives finally:

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \frac{1}{96}\begin{pmatrix} -30 & 30 & -6 & 6 & -9 & -3 & 3 & 9 & -9 & -3 & 3 & 9 & -6 & 6 & -6 & 6 \\ -12 & 12 & -12 & 12 & 32 & -32 & -4 & 4 & 4 & 8 & 4 & 8 & -4 & 4 & -32 & 8 \\ -6 & 6 & -30 & 30 & 3 & 9 & -9 & -3 & 3 & 9 & -9 & -3 & -6 & 6 & -6 & 6 \\ -12 & 12 & -12 & 12 & 4 & 8 & 4 & 8 & 32 & -32 & -4 & 4 & -32 & 8 & -4 & 4 \\ -12 & 12 & -12 & 12 & -8 & -4 & -8 & -4 & -4 & 4 & 32 & -32 & -8 & 32 & -4 & 4 \\ -12 & 12 & -12 & 12 & -4 & 4 & 32 & -32 & -8 & -4 & -8 & -4 & -4 & 4 & -8 & 32 \end{pmatrix}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_8 \\ M_9 \\ M_{10} \\ M_{11} \\ M_{12} \\ M_{13} \\ M_{14} \\ M_{15} \\ M_{16} \end{pmatrix}. \quad (25)$$

In Equation 20 the solutions for G and H are omitted, as they are equal to C and A respectively. Assuming a same uncertainty $u(m)$ in every measured height difference m_{ij}, it can be derived from Equation 3 that $u\,(M_{1..}M_4) = u(m\,)\sqrt{N_d/2}$; $u\,(M_{5..8}) = u\,\left(M_{13,14}\right) = u(m\,)\sqrt{N_x/2}$ and $u\,(M_{9..12}) = u\,\left(M_{15,16}\right) = u(m\,)\sqrt{N_{y/2}}$. This gives for the uncertainty in the corner points $u(A,C,G,H) = \sqrt{0.1 \cdot N_d + 0.014 \cdot N_x + 0.014 \cdot N_y} \cdot u(m)$, for the uncertainty in the points B and F along the perimeter $u(B,F) = \sqrt{0.031 \cdot N_d + 0.115 \cdot N_x + 0.065 \cdot N_y} \cdot u(m)$ and for the uncertainty in the points D and E along the edges $[illegible](D,E) = \sqrt{0.031 \cdot N_d + 0.065 \cdot N_x + 0.115 \cdot N_y} \cdot u(m)$. For example: for N_d=10, N_x=8 and N_y=6 we find $u(A,C,G,H)$=1.09·$u(m)$, $u(B,F)$=1.27·$u(m)$ and $u(D,E)$=1.23·$u(m)$. This means an overall error reduction of 29% compared to the traditional Moody method and 13% reduction compared to the least-squares solution given in section 3.1. This improved uncertainty of level measurements using a constant reference to gravity, for example by a differential level on a fixed position, must be outweighed against the additional care needed to keeping this reference constant.

A measure for the measurement uncertainty can be obtained by defining the Chi-square function based on Equation 20 using the values of α and β calculated in Equations 22 and 23:

$$\chi^2 = (2A - M_2 + M_1)^2 + (2C - M_4 + M_3)^2 + (B - A - M_5 + \alpha)^2 + (C - B - M_6 + \alpha)^2 + (F - G - M_7 + \alpha)^2 + (H - F - M_8 + \alpha)^2 + (D - A - M_9 + \beta)^2 + (G - D - M_{10} + \beta)^2 + (E - C - M_{11} + \beta)^2 + ([illegible] - [illegible] - M_{12} + [illegible])^2 + ([illegible] - [illegible] - M_{13})^2 + ([illegible] + [illegible] - M_{14})^2 + ([illegible] - [illegible] - M_{15})^2 + ([illegible] + [illegible] - M_{16})^2. \quad (26)$$

From Equation 26 the measurement uncertainty $u(m)$ can be estimated as:

$$u(m) = \sqrt{\frac{\chi^2}{0.875 \cdot N_d + 1.3125 \cdot N_x + 1.3125 \cdot N_y}} \quad (27)$$

This uncertainty is calculated with 10 degrees of freedom because 16 independent measurements are taken from which 6 heights are determined.

4. Reference Plane Definition

The definition of the reference plane by initially putting I=0 and equaling the diagonal endpoints has its advantages as it is uniquely defined and, as it is shown in this paper, can be directly integrated in the measurement evaluation and uncertainty evaluation. When considering the node points only, the least-squares plane can be used as well, giving as boundary conditions $A+B+C=G+F+H$; $E+D+G=C+E+H$ and $A+B+C+D+E+F+G+H+I=0$. This can be incorporated in Equation 5 instead of putting $A=H$, $C=G$ and I=0. However for determining a proper least-square plane all measured heights must be taken into account, including the points between the nodes, needing an additional calculation. In form deviations in the Geometric Product Specification system it is common to take the minimum-zone planes as the reference, according to which the flatness is defined as the minimum distance between two parallel planes within which the entire surface resides. This can be calculated by rotating the reference plane around the x- and y-axis in such a way that the difference between the minimum and maximum height is minimal. Writing the measured coordinates as $h_{ij}(x_i,y_i)$ this gives:

$$h'_{ij}(x_{ij},y_{ij}) = min_{\alpha,\beta}(range((h_{ij}(x_{ij},y_{ij}) - \alpha \cdot x_{ij} - \beta \cdot y_{ij})) \quad (28)$$

This can be evaluated by standard mathematical routines for minimizing a function depending on two variables: the two angles α and β. An alternative is an exchange algorithm that determines the critical points that define the minimum zone plane [10]. After this operation, the z-value of the lowest point can be subtracted from all measured points. Note that at the time the Moody method was designed (1955) this calculation was virtually impossible, computers being hardy available. The orientation of the minimum-zone plane depends on the actually measured topography and is task-specific. Therefore general expressions for the uncertainties cannot be made and Monte-Carlo simulations must be carried out to evaluate the propagation of uncertainties.

5. Monte Carlo Simulations

In section 3 the uncertainties in the node points could be analyzed; however for the uncertainties in the points in-between an analytical calculation is virtually impossible. This is also the case for the

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flatness deviation FLTt(EDPL) where it depends on the actual topography which points determine the flatness deviations. Therefore, in order to evaluate the uncertainty for actual measurements using both the minimum-zone plane as a reference as well as equaling the diagonal endpoints, Monte Carlo simulations were carried out using the following steps:

1. The measurements are evaluated according to one of the methods given in sections 3.1-3.3, giving the heights $h_{i,j}$ that are used as a reference in the simulations.
2. The flatness deviation FLTt is evaluated for both the diagonal-endpoints reference, giving FLTt(EDPL), and the minimum zone plane, giving FLTt(MZPL). These values are taken as a reference in the simulations.
3. The measurement uncertainty $u(m)$ is estimated from the measurements using the appropriate method given in section 3.1-3.3. It may be adjusted for unrealistic values or for research purposes.
4. A set of simulated measurements m_{ij} is generated using the calculated values of h_{ij} and $u(m)$ as following:

$$m_{ij} = h_{i,j+1} - h_{i,j} + r \cdot u(m), \quad (29)$$

were r is a Gaussian distributed random number with zero mean and a standard deviation of unity

5. The simulated measurements are evaluated and the flatness deviations FLTt(EDPL) and FLTt(MZPL) are noted
6. Step 4 and 5 are repeated some thousand times where the deviation of the mean simulated flatness deviation from the reference is noted as the bias and the standard deviation in the flatness value of the simulated measurements is noted as the standard deviation. This bias is noted as it is recognized that non-repeatable measurements tends to giver higher rather than lower form deviation values [11].

As an example we take the measurement of a perfect flat, giving h_{ij}=0, measured with a a measuring instrument giving a standard deviation of 1 µm, for example an electronic level with a standard deviation of 10 µrad and a measurement base of 100 mm.

Table 1 gives the result of the simulations for a 2 x 2 measurement grid ($N_d = N_x = N_y = 2$) and the 8 x 6 point configuration that was taken as an example: $N_d = 10$ $N_x = 8$ $N_y = 6$.

Table 1. Result of Monte Carlo simulations of measurements on a perfectly flat surface.

Bias and standard deviation in µm $u(m) = 1$ µm	Classical Moody evaluation (section 3.1)	Least squares Moody evaluation (section 3.2)	Least squares level evaluation (section 3.3)
2x2 ,FLTt(EDPL)	1.99 ± 0.59	1.59 ± 0.54	1.37 ± 0.46
2x2, FLTt(MZPL)	1.77 ± 0.56	1.34 ± 0.47	1.11 ± 0.36
8x6, FLT(EDPL)	5.59 ± 1.08	5.08 ± 1.00	4.77 ± 0.87
8x6, FLT(EDPL)	4.89 ± 0.97	4.43 ± 0.87	4.15 ± 0.70

The table shows that both the least-squares evaluations and taking the minimum-zone as the reference plane leads to lower bias and lower uncertainties compared to the classical Moody evaluation. In this case the bias is far lager than the standard deviation (repeatability) this means that both the bias and standard deviations must be considered in an uncertainty budget, especially for surface plates with a flatness deviation in the range of the measurement limit of the used instrument. The bias and standard deviation can be treated as type B and type A uncertainties respectively in further uncertainty evaluations.

6 Measurement Example

A surface plate of 900 mm x 700 mm x 150 mm was measured using electronic levels with a resolution of 1 µrad on a 80 mm measurement base, where an area of 640 mm x 480 mm was measured giving N_d= 10, N_x = 8 and N_y = 6. The measured topography is not essentially different between the methods and is displayed in Figure 3.

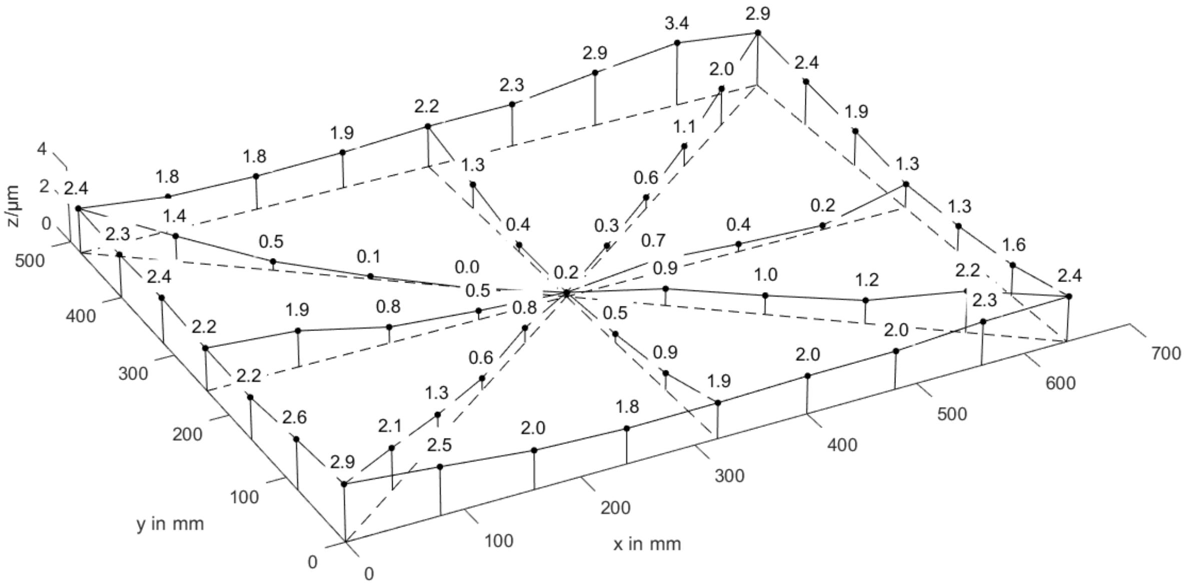

Figure 3. Example of a measured surface topography of a granite surface plate.

In Table 2 the results are summarized regarding the measured topography, the estimated uncertainty in the measurements and the bias and standard deviation based on $u(m)$ = 0.21 µm

Table 2. Results of the measurement evaluations according to the methods described in section 3.1, 3.2 and 3.3.

Results in µm	Classical Moody evaluation (section 3.1)	Least squares Moody evaluation (section 3.2)	Least squares level evaluation (section 3.3)
FLTt(EDPL)	3.50	3.46	3.26
FLTt(MZPL)	3.20	3.18	2.98
u(m)	0.15	0.15	0.21
Monte Carlo results; bias and standard deviation for $u(m)$ = 0.21 µm			
FLTt(EDPL)	0.23 ± 0.36	0.08 ± 0.34	0.08 ± 0.30
FLTt(MZPL)	0.24 ± 0.31	0.08 ± 0.26	0.09 ± 0.24

The results in Table 2 confirm the general trends analyzed in section 3. The higher uncertainty predicted by the least-square level evaluation may be due to the higher degrees of freedom and the involvement of the parallelism of the lines in the evaluation. The significant bias observed in the Monte Carlo simulations may be due to the lines DE and BF that cause the closure error and cross point I which may also be the lowest point defining the FLTt value in the simulations. Most of the values depend on the actual topography and measurements taken, but the general trend is that the least squares evaluations result in lower uncertainties than the classical Moody evaluation.

7. Traceability and Uncertainty Considerations

The traceability to the perfect plane is established by the straight travelling of light in air for autocollimator- and laser interferometer measurements, and the gravity vector for level measurements. Effects of air temperature- and pressure gradients and the earths' curvature are negligible for surfaces op to 2 m x 2 m. The values of the flatness deviations depend on the length of the measurement base and the calibration of the angular measurement device. Uncertainties in these

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values are commonly below the percentage level and are mostly not significant for reasonable flat surfaces. Other aspects are the coverage of the surface plate by these measurements, humidity temperature gradients of the surface plate and size of the measurement feet. These aspects have extensively been treated by Drescher [9] and are not repeated here. This paper is focused on the measurement evaluation and its uncertainty.

5. Conclusions

The classical Moody method can be improved by applying least-squares evaluation using the measurement redundancies. This improves both the repeatability of the results as well as the bias in flatness measurements by typically 20% and it gives a single value for the surface height in the center. Taking the minimum-zone reference plane may give a significantly lower FLTt-value, depending on the topography. Without these improvements, the Moody method tends to give too high FLTt-values, increasing the risk of false rejection of the plate. An additionally improved uncertainty of level measurements using a constant reference to gravity, for example by a differential level on a fixed position, must be outweighed against the additional care of keeping this reference constant.

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