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Fibonacci Identities via Calculus

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Abstract: We present a differential-calculus-based method which allows one to derive more identities from *any* given Fibonacci-Lucas identity containing a finite number of terms and having at least one free index. The strength of our method is that no additional information is required about the given original identity. The method readily extends to a generalized Fibonacci sequence.

Keywords: Fibonacci number; Lucas number; gibonacci sequence; generalized Fibonacci sequence; summation identity; Gelin-Cesàro identity; Candido's identity

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1. Introduction

Let F_j and L_j be the jth Fibonacci and Lucas numbers, defined for all integers by

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}, \quad L_j = \alpha^j + \beta^j, \tag{1.1}$$

where $\alpha = (1+\sqrt{5})/2$, the golden ratio, and $\beta = (1-\sqrt{5})/2 = -1/\alpha$. Of course, $\alpha + \beta = 1$, $\alpha\beta = -1$, $\alpha - \beta = \sqrt{5}$.

Our goal in this paper is to present a method which allows the discovery of more identities from any known Fibonacci-Lucas identity having at least one free index, that is an index that is not being summed over.

To illustrate what we mean, consider the identity

$$\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} F_{j+k}^4 = 25^n \left(F_{2n+k+1}^4 - F_{2n+k}^4 \right), \tag{1.2}$$

derived, among other similar results, by Hoggatt and Bicknell [2]. This identity has a free index, k. Working only with the knowledge of (1.2), our method allows us to derive the following presumably new identity:

$$\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} F_{j+k}^3 L_{j+k} = 25^n \left(F_{2n+k+1}^3 L_{2n+k+1} - F_{2n+k}^3 L_{2n+k} \right); \tag{1.3}$$

which, in turn, implies the identity

$$\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} F_{2j+k}^2 = 25^n F_{2(4n+k+1)}. \tag{1.4}$$

We are not done yet, as (1.4) implies

$$\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} F_{4j+2k} = 25^n L_{2(4n+k+1)}; \tag{1.5}$$

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which finally implies

$$\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} L_{4j+2k} = 5^{2n+1} F_{2(4n+k+1)}. \tag{1.6}$$

Thus, the four identities (1.3), (1.4), (1.5) and (1.6) all follow from a knowledge of (1.2).

As another example, consider the following well-known identity (see, for example, Hoggatt and Ruggles [3, Theorem 4]):

$$\tan^{-1} \frac{1}{F_{2k+1}} = \tan^{-1} \frac{1}{F_{2k}} - \tan^{-1} \frac{1}{F_{2k+2}}. \tag{1.7}$$

Our method shows that (1.7) implies the following apparently new identity:

$$\frac{L_{2k+1}}{F_{2k+1}^2 + 1} = \frac{L_{2k}}{F_{2k}^2 + 1} - \frac{L_{2k+2}}{F_{2k+2}^2 + 1}.$$

Let $(G_j)_{j\in\mathbb{Z}}$ be the gibonacci sequence having the same recurrence relation as the Fibonacci and Lucas sequences but starting with arbitrary initial values; that is, let

$$G_0 = a,\, G_1 = b;\, G_j = G_{j-1} + G_{j-2}, \quad (j \geq 2), \tag{1.8}$$

where a and b are arbitrary numbers (usually integers) not both zero.

The method to be developed in this paper also applies to the gibonacci sequence; so that more gibonacci identities can be discovered from any known gibonacci identity containing at least one free index. For example, our method shows that the following identity of Howard [6, Corollary 3.5]:

$$F_s G_{k+r} + (-1)^{r-1} F_{s-r} G_k = F_r G_{k+s}, \tag{1.9}$$

containing three free indices r, s and k, implies the following identities:

$$L_s G_{k+r} + (-1)^{r-1} L_{s-r} G_k = F_r (G_{k+s+1} + G_{k+s-1}), \tag{1.10}$$

$$F_s \left(G_{k+r+1} + G_{k+r-1}\right) + (-1)^r L_{s-r} G_k = L_r G_{k+s}, \tag{1.11}$$

$$L_s \left(G_{k+r+1} + G_{k+r-1}\right) + (-1)^r 5 F_{s-r} G_k = L_r \left(G_{k+s+1} + G_{k+s-1}\right). \tag{1.12}$$

Consider the generalized Fibonacci sequence $(W_j) = \big(W_j(a,b;p)\big)$ defined, for all integers and arbitrary real numbers a, b and $p \neq 0$, by the recurrence relation

$$W_0 = a,\, W_1 = b, \quad W_j = pW_{j-1} + W_{j-2}, \quad j \geq 2, \tag{1.13}$$

with $W_{-j} = W_{-j+2} - pW_{-j+1}$.

Note that the (W_j) sequence studied here is a special case of the Horadam sequence [4], corresponding to setting $q = -1$ in that sequence.

Two important cases of (W_j) are the special Lucas sequences of the first kind, $(U_j(p)) = (W_j(0,1;p))$, and of the second kind, $(V_j(p)) = (W_j(2,p;p))$; so that

$$U_0 = 0,\, U_1 = 1, \quad U_j = pU_{j-1} + U_{j-2}, \quad j \geq 2, \tag{1.14}$$

and

$$V_0 = 2,\, V_1 = p, \quad V_j = pV_{j-1} + V_{j-2}, \quad j \geq 2, \tag{1.15}$$

with $U_{-j} = U_{-j+2} - pU_{-j+1}$ and $V_{-j} = V_{-j+2} - pV_{-j+1}$.

We will show that the new method also applies to the generalized Fibonacci sequence. For example, we will see that the identity [4, Equation (3.14)]:

$$U_r W_{k+1} + U_{r-1} W_k = W_{k+r} \tag{1.16}$$

implies

$$V_r W_{k+1} + V_{r-1} W_k = W_{k+r+1} + W_{k+r-1}. \tag{1.17}$$

The new method presented in this paper provides some illumination on some observations noted by researchers (see, for example, Long [9], Dresel [1] and Melham [11]).

2. The method

Delaying rigorous justification to section 4, we present the method and give examples.

Here then is how to obtain more identities from any given Fibonacci-Lucas identity having a free index:

1. Let k be a free index in the known identity. Replace each Fibonacci number, say $F_{h(k,\cdots)}$, with a certain differentiable function of k, namely, $f(h(k,\cdots))$, with k now considered a variable; and replace each Lucas number, say $L_{h(k,\cdots)}$, with a certain differentiable function $l(h(k,\cdots))$. The subscript h will be considered a function of several variables; that is variable k and other parameters (if any) indicated by ellipses $\cdots$. The explicit form of $f(h(k,\cdots))$ or $l(h(k,\cdots))$ will not enter into play.
2. By applying the usual rules of calculus, differentiate, with respect to k, through the identity obtained in step 1.
3. Simplify the equation obtained in step 2 and take the real part of the whole expression/equation, using also the following prescription:

$$\Re f(h(k,\cdots)) = F_{h(k,\cdots)}, \tag{2.1}$$

$$\Re l(h(k,\cdots)) = L_{h(k,\cdots)}, \tag{2.2}$$

$$\Re \frac{\partial}{\partial k} f(h(k,\cdots)) = \frac{L_{h(k,\cdots)}}{\sqrt{5}} \ln \alpha, \tag{2.3}$$

$$\Re \frac{\partial}{\partial k} l(h(k,\cdots)) = F_{h(k,\cdots)} \sqrt{5} \ln \alpha; \tag{2.4}$$

where $\Re X$ denotes the real part of X.

Remark 2.1. Formally, the method described in this section proceeds in two easy steps:

i Treat the subscripts of Fibonacci and Lucas numbers as variables and differentiate through the given identity, with respect to the index of interest, using the rules of differential calculus.
ii Make the following replacements:

$$\frac{\partial}{\partial k} F_{h(k,\cdots)} \to \frac{L_{h(k,\cdots)}}{\sqrt{5}} \frac{\partial}{\partial k} h(k,\cdots), \tag{2.5}$$

$$\frac{\partial}{\partial k} L_{h(k,\cdots)} \to F_{h(k,\cdots)} \sqrt{5} \frac{\partial}{\partial k} h(k,\cdots). \tag{2.6}$$

For example, given the fundamental identity:

$$F_{2k} = L_k F_k,$$

we have, by step i,

$$\frac{d}{dk}F_{2k} = \frac{d}{dk}(L_kF_k) = L_k\frac{d}{dk}F_k + F_k\frac{d}{dk}L_k;$$

so that, by step ii, using (2.5) and (2.6), we get

$$\frac{L_{2k}}{\sqrt{5}} \times \frac{d}{dk}(2k) = L_k \times \frac{L_k}{\sqrt{5}} + F_k \times F_k\sqrt{5};$$

and hence,

$$2L_{2k} = L_k^2 + 5F_k^2. \tag{2.7}$$

Note that in using this two-step version of the method; if imaginary quantities appear in the final identity (such as would happen when one differentiates $(-1)^m$ with respect to m), terms containing such quantities must be dropped. Such a situation is automatically handled in the full implementation of the method as described in steps 1 to 3 above.

2.1. Examples

We illustrate the method with a couple of examples from familiar results.

2.2. Example from a connecting formula between Fibonacci and Lucas numbers

In this example we show that

$$L_k = F_{k+1} + F_{k-1} \implies 5F_k = L_{k+1} + L_{k-1}.$$

Following step 1 we write

$$l(k) = f(k+1) + f(k-1) \tag{2.8}$$

and (step 2) differentiate with respect to k, obtaining

$$\frac{d}{dk}l(k) = \frac{d}{dk}f(k+1) + \frac{d}{dk}f(k-1).$$

Step 3 now gives

$$\Re\frac{d}{dk}l(k) = \Re\frac{d}{dk}f(k+1) + \Re\frac{d}{dk}f(k-1);$$

and by (2.3) and (2.4),

$$F_k\sqrt{5}\ln\alpha = \frac{L_{k+1}}{\sqrt{5}}\ln\alpha + \frac{L_{k-1}}{\sqrt{5}}\ln\alpha;$$

that is

$$5F_k = L_{k+1} + L_{k-1}.$$

2.2.1. Example from the fundamental identity of Fibonacci and Lucas numbers

In this example we demonstrate that:

$$F_{2k} = L_kF_k \implies 2L_{2k} = L_k^2 + 5F_k^2. \tag{2.9}$$

For the identity $F_{2k} = L_kF_k$, step 1 is

$$f(2k) = l(k)f(k);$$

where k is now considered a variable.

Following step 2, we differentiate with respect to k to obtain

$$2\frac{d}{dk}f(2k) = l(k)\frac{d}{dk}f(k) + f(k)\frac{d}{dk}l(k).$$

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Step 3 gives

$$2\Re\frac{d}{dk}f(2k) = L_k\Re\frac{d}{dk}f(k) + F_k\Re\frac{d}{dk}l(k).$$

Thus, using (2.3) and (2.4), we have

$$2\frac{L_{2k}}{\sqrt{5}}\ln\alpha = L_k\frac{L_k}{\sqrt{5}}\ln\alpha + F_k\sqrt{5}F_k\ln\alpha;$$

which, dropping $\ln\alpha$ and multiplying through by $\sqrt{5}$, is

$$2L_{2k} = L_k^2 + 5F_k^2.$$

The interested reader may wish to verify that

$$2L_{2k} = L_k^2 + 5F_k^2 \implies F_{2k} = L_kF_k.$$

2.2.2. Example from the multiplication formula of Fibonacci and Lucas numbers

Here we show that the multiplication formula

$$F_{k+m} + (-1)^m F_{k-m} = L_m F_k$$

implies

$$L_{k+m} + (-1)^m L_{k-m} = L_m L_k \tag{2.10}$$

and

$$L_{k+m} - (-1)^m L_{k-m} = 5F_m F_k. \tag{2.11}$$

We write

$$f(k+m) + (-1)^m f(k-m) = l(m)f(k); \tag{2.12}$$

so that, treating k as the free index of interest gives

$$\frac{\partial}{\partial k}f(k+m) + (-1)^m\frac{\partial}{\partial k}f(k-m) = l(m)\frac{\partial}{\partial k}f(k).$$

Thus,

$$\Re\frac{\partial}{\partial k}f(k+m) + (-1)^m\Re\frac{\partial}{\partial k}f(k-m) = l(m)\Re\frac{\partial}{\partial k}f(k);$$

and hence, by step 3,

$$\frac{L_{k+m}}{\sqrt{5}}\ln\alpha + (-1)^m\frac{L_{k-m}}{\sqrt{5}}\ln\alpha = L_m\frac{L_k}{\sqrt{5}}\ln\alpha;$$

from which we get (2.10).

Taking m as the index of interest and differentiating (2.12) with respect to m yields

$$\frac{\partial}{\partial m}f(k+m) - (-1)^m\frac{\partial}{\partial m}f(k-m) + (-1)^m i\pi f(k-m) = f(k)\frac{\partial}{\partial m}l(m),$$

so that

$$\Re\frac{\partial}{\partial m}f(k+m) - (-1)^m\Re\frac{\partial}{\partial m}f(k-m) = F_k\Re\frac{\partial}{\partial m}l(m);$$

and hence

$$\frac{L_{k+m}}{\sqrt{5}}\ln\alpha - (-1)^m\frac{L_{k-m}}{\sqrt{5}}\ln\alpha = F_kF_m\sqrt{5}\ln\alpha;$$

from which (2.11) follows.

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The reader may verify that the remaining multiplication formula can be discovered by differentiating (2.10) with respect to m.

2.2.3. Example from an inverse tangent Fibonacci number identity

Consider the following identity:

$$\tan^{-1}\frac{F_{2m}}{F_{2k+2m-1}}=\tan^{-1}\frac{L_m}{L_{2k+m-1}}-\tan^{-1}\frac{L_m}{L_{2k+3m-1}},\quad m \text{ even}, \tag{2.13}$$

which can be derived using the inverse tangent addition formula and basic Fibonacci-Lucas identities.

We now demonstrate that (2.13) implies

$$\frac{1}{5}\frac{F_{2m}L_{2k+2m-1}}{F_{2k+2m-1}^2+F_{2m}^2}=\frac{L_mF_{2k+m-1}}{L_{2k+m-1}^2+L_m^2}-\frac{L_mF_{2k+3m-1}}{L_{2k+3m-1}^2+L_m^2},\quad m \text{ even}. \tag{2.14}$$

We treat k as the free index of interest. Step 1 gives

$$\tan^{-1}\frac{f(2m)}{f(2k+2m-1)}=\tan^{-1}\frac{l(m)}{l(2k+m-1)}-\tan^{-1}\frac{l(m)}{l(2k+3m-1)}; \tag{2.15}$$

so that step 2 yields

$$\begin{aligned}&\frac{2f(2m)}{f(2k+2m-1)^2+f(2m)^2}\frac{\partial}{\partial k}f(2k+2m-1)\\&=\frac{2l(m)}{l(2k+m-1)^2+l(m)^2}\frac{\partial}{\partial k}l(2k+m-1)\\&\quad-\frac{2l(m)}{l(2k+3m-1)^2+l(m)^2}\frac{\partial}{\partial k}l(2k+3m-1),\end{aligned} \tag{2.16}$$

whence taking real part and replacing the derivatives using (2.3) and (2.4) gives (2.14).
By treating m as the free index, the interested reader can verify, using our method, that (2.13) also implies

$$\frac{2}{5}\frac{F_{2k-1}}{F_{2k+2m-1}^2+F_{2m}^2}=-\frac{F_{2k-1}}{L_{2k+m-1}^2+L_m^2}+\frac{F_{2k+3m-1}L_m+F_{2k+2m-1}}{L_{2k+3m-1}^2+L_m^2},\quad m \text{ even}. \tag{2.17}$$

2.3. Extension to a generalized Fibonacci sequence

We now describe how the method for obtaining new identities from existing ones works for the generalized Fibonacci sequence $\left(W_j(a,b;p)\right)$ whose terms are given in (1.13). The scheme is the following:

1. Let k be a free index in the known identity. Replace each generalized Fibonacci number, say $W_{h(k,\cdots)}$, with a certain differentiable function of k, namely, $w(h(k,\cdots))$, with k now considered a variable.
2. By applying the usual rules of calculus, differentiate, with respect to k, through the identity obtained in step 1.
3. Simplify the equation obtained in step 2 and take the real part, using also the following prescription:

$$\Re w(h(k,\cdots))=W_{h(k,\cdots)}, \tag{2.18}$$

$$\Re\frac{\partial}{\partial k}w(h(k,\cdots))=\frac{W_{h(k+1,\cdots)}+W_{h(k-1,\cdots)}}{\Delta}\ln\tau; \tag{2.19}$$

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where $\tau = (p+\Delta)/2$ and $\Delta = \sqrt{p^2+4}$.

Note that, on account of (4.7) and (4.9), for the special Lucas sequences, (2.18) and (2.19) reduce to

$$\Re u(h(k,\cdots)) = U_{h(k,\cdots)}, \tag{2.20}$$

$$\Re \frac{\partial}{\partial k} u(h(k,\cdots)) = \frac{V_{h(k,\cdots)}}{\Delta} \ln \tau \tag{2.21}$$

and

$$\Re v(h(k,\cdots)) = V_{h(k,\cdots)}, \tag{2.22}$$

$$\Re \frac{\partial}{\partial k} v(h(k,\cdots)) = U_{h(k,\cdots)} \Delta \ln \tau; \tag{2.23}$$

of which the Fibonacci and Lucas relations (2.1)–(2.4) are particular cases.

For the gibonacci sequence, (2.18) and (2.19) reduce to

$$\Re g(h(k,\cdots)) = G_{h(k,\cdots)}, \tag{2.24}$$

$$\Re \frac{\partial}{\partial k} g(h(k,\cdots)) = \frac{G_{h(k+1,\cdots)} + G_{h(k-1,\cdots)}}{\sqrt{5}} \ln \alpha. \tag{2.25}$$

2.4. More examples

We give further examples involving the gibonacci sequence and the generalized Fibonacci sequence.

2.4.1. Examples from an identity of Howard

Consider the following identity, derived by Howard [6, Corollary 3.5]:

$$F_s G_{k+r} + (-1)^{r-1} F_{s-r} G_k = F_r G_{k+s}, \tag{2.26}$$

Identity (2.26) has three free indices r, s and k.

We write

$$f(s)g(k+r) + (-1)^{r-1} f(s-r)g(k) = f(r)g(k+s). \tag{2.27}$$

Treating s as the index of interest and differentiating (2.27) with respect to s gives

$$g(k+r)\frac{d}{ds}f(s) + (-1)^{r-1}g(k)\frac{\partial}{\partial s}f(s-r) = f(r)\frac{\partial}{\partial s}g(k+s);$$

so that, taking the real part, we get

$$G_{k+r}\Re\frac{d}{ds}f(s) + (-1)^{r-1}G_k\Re\frac{\partial}{\partial s}f(s-r) = F_r\Re\frac{\partial}{\partial s}g(k+s).$$

We now use (2.3) to replace the derivatives on the left hand side and (2.25) to replace the derivative on the right hand side, obtaining

$$L_s G_{k+r} + (-1)^{r-1} L_{s-r} G_k = F_r (G_{k+s+1} + G_{k+s-1}). \tag{1.10}$$

On the other hand, treating r as the index of interest and differentiating (2.27) with respect to r yields

$$f(s)\frac{\partial}{\partial r}g(k+r) + (-1)^{r-1} i\pi f(s-r)g(k) - (-1)^{r-1}g(k)\frac{\partial}{\partial r}f(s-r)$$
$$= g(k+s)\frac{d}{dr}f(r);$$

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so that, taking real part,

$$F_s \Re \frac{\partial}{\partial r} g(k+r) - (-1)^{r-1} G_k \Re \frac{\partial}{\partial r} f(s-r) = G_{k+s} \Re \frac{d}{dr} f(r).$$

Use of (2.25) and (2.3) finally gives

$$F_s\left(G_{k+r+1} + G_{k+r-1}\right) + (-1)^r L_{s-r} G_k = L_r G_{k+s}. \tag{1.11}$$

The interested reader is invited to discover, by differentiating with respect to s, that (1.11) implies

$$L_s\left(G_{k+r+1} + G_{k+r-1}\right) + (-1)^r 5F_{s-r} G_k = L_r\left(G_{k+s+1} + G_{k+s-1}\right); \tag{2.28}$$

and that differentiating (2.26) with respect to k does not produce a new result.

2.4.2. Example from a general recurrence relation

Consider the following identity of Horadam [4, Equation (3.14)]:

$$U_r W_{k+1} + U_{r-1} W_k = W_{k+r}. \tag{2.29}$$

We write

$$u(r)w(k+1) + u(r-1)w(k) = w(k+r);$$

and differentiate with respect to r, obtaining

$$\frac{d}{dr} u(r) \times w(k+1) + \frac{d}{dr} u(r-1) \times w(k) = \frac{\partial}{\partial r} w(k+r);$$

so that, taking real part, we find

$$\Re \frac{d}{dr} u(r) \times W_{k+1} + \Re \frac{d}{dr} u(r-1) \times W_k = \Re \frac{\partial}{\partial r} w(k+r);$$

and hence, upon using (2.21) and (2.19) to replace the derivatives, we obtain

$$V_r W_{k+1} + V_{r-1} W_k = W_{k+r+1} + W_{k+r-1}. \tag{1.17}$$

In particular,

$$V_r U_{k+1} + V_{r-1} U_k = V_{k+r}, \tag{2.30}$$

$$V_r V_{k+1} + V_{r-1} V_k = (p^2+4) U_{k+r}. \tag{2.31}$$

2.4.3. Example from a multiplication formula

Here we will demonstrate that the identity [4, Equation (3.16)]:

$$W_{k+r} + (-1)^r W_{k-r} = V_r W_k$$

implies the identity

$$(W_{k+r+1} + W_{k+r-1}) - (-1)^r (W_{k-r+1} + W_{k-r-1}) = U_r W_k \Delta^2. \tag{2.32}$$

We write

$$w(k+r) + (-1)^r w(k-r) = v(r) w(k)$$

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and differentiate through with respect to r to obtain

$$\frac{\partial}{\partial r}w(k+r) + (-1)^r \pi i w(k-r) - (-1)^r \frac{\partial}{\partial r}w(k-r) = w(k)\frac{d}{dr}v(r);$$

so that

$$\Re\frac{\partial}{\partial r}w(k+r) - (-1)^r \Re\frac{\partial}{\partial r}w(k-r) = w(k)\Re\frac{d}{dr}v(r).$$

Using (2.19) and (2.23), we get

$$\frac{W_{k+r+1} + W_{k+r-1}}{\Delta} - (-1)^r \frac{(W_{k-r+1} + W_{k-r-1})}{\Delta} = W_k U_r \Delta; \tag{2.33}$$

and hence (2.32).

Identities

$$V_{k+r} - (-1)^r V_{k-r} = U_k U_r \Delta^2 \tag{2.34}$$

and

$$U_{k+r} - (-1)^r U_{k-r} = U_r V_k \tag{2.35}$$

are special cases of (2.32).

3. Applications

In this section, we pick various known results from the literature and apply our method to discover new identities.

3.1. New identities from an identity of Long

Long [10, Equation (44)] showed that, for a non-negative integer n and any integers k and r,

$$\sum_{j=0}^{n} \binom{n}{j} F_{r+2kj} = L_k^n F_{r+nk}, \quad \text{if } k \text{ is even.} \tag{3.1}$$

Based on the knowledge of (3.1) alone, we will derive the results stated in the proposition.

Proposition 1. *If n is a non-negative integer, k is an even integer and r is any integer, then*

$$2\sum_{j=0}^{n} j\binom{n}{j} L_{r+2kj} = 5nL_k^{n-1}F_{r+nk}F_k + nL_k^n L_{r+nk}, \tag{3.2}$$

$$2\sum_{j=0}^{n} j\binom{n}{j} F_{r+2kj} = nL_k^{n-1}L_{r+nk}F_k + nL_k^n F_{r+nk}. \tag{3.3}$$

Identity (3.1) contains two free indices r and k. Treating r as the index of interest immediately gives the Lucas version of (3.1), namely,

$$\sum_{j=0}^{n} \binom{n}{j} L_{r+2kj} = L_k^n L_{r+nk}, \quad \text{if } k \text{ is even;}$$

coming from

$$\sum_{j=0}^{n} \binom{n}{j} \Re\frac{\partial}{\partial r} f(r+2kj) = l(k)^n \Re\frac{\partial}{\partial r} f(r+nk)$$

and prescription (2.3).

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To derive (3.2), write (3.1) as

$$\sum_{j=0}^{n}\binom{n}{j}f(r+2kj)=l(k)^n f(r+nk);$$

treat k as the index of interest and differentiate with respect to k (step 2) to obtain

$$\sum_{j=0}^{n}2j\binom{n}{j}\frac{\partial}{\partial k}f(r+2kj)=nl(k)^{n-1}f(r+nk)\frac{\partial}{\partial k}l(k)+nl(k)^n\frac{\partial}{\partial k}f(r+nk),$$

and, taking real part,

$$\sum_{j=0}^{n}2j\binom{n}{j}\Re\frac{\partial}{\partial k}f(r+2kj)=nL_k^{n-1}F_{r+nk}\Re\frac{\partial}{\partial k}l(k)+nL_k^n\Re\frac{\partial}{\partial k}f(r+nk). \tag{3.4}$$

Thus (3.2) follows from step 3 of section 2, after using (2.3) and (2.4) to replace the derivatives in (3.4).

To derive (3.3) treat r as the free index of interest in (3.2) and write

$$2\sum_{j=0}^{n}j\binom{n}{j}\frac{\partial}{\partial r}l(r+2kj)=5nL_k^{n-1}\frac{\partial}{\partial r}f(r+nk)f(k)+nL_k^n\frac{\partial}{\partial r}l(r+nk).$$

3.2. New identities arising from an identity of Hoggatt and Bicknell

Based on Hoggatt and Bicknell's result [2, Identity 2′]:

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}F_{j+k}^4=25^n\left(F_{2n+k+1}^4-F_{2n+k}^4\right), \tag{1.2}$$

we wish to derive the four identities (1.3), (1.4), (1.5) and (1.6) stated in the Introduction section.

Write (1.2) as

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}f(j+k)^4=25^n\left(f(2n+k+1)^4-f(2n+k)^4\right);$$

and differentiate through, with respect to k, to obtain

$$\begin{aligned}&\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}4f(j+k)^3\frac{\partial}{\partial k}f(j+k)\\&\quad=25^n\left(4f(2n+k+1)^3\frac{\partial}{\partial k}f(2n+k+1)-4f(2n+k)^3\frac{\partial}{\partial k}f(2n+k)\right);\end{aligned}$$

and taking real parts:

$$\begin{aligned}&\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}4F_{j+k}^3\Re\frac{\partial}{\partial k}f(j+k)\\&\quad=25^n\left(4F_{2n+k+1}^3\Re\frac{\partial}{\partial k}f(2n+k+1)-4F_{2n+k}^3\Re\frac{\partial}{\partial k}f(2n+k)\right).\end{aligned}$$

Thus,

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}F_{j+k}^3\frac{L_{j+k}}{\sqrt{5}}$$
$$=25^n\left(F_{2n+k+1}^3\frac{L_{2n+k+1}}{\sqrt{5}}-F_{2n+k}^3\frac{L_{2n+k}}{\sqrt{5}}\right);$$

and hence (1.3). Identities (1.4), (1.5) and (1.6) are derived in the same manner; (1.4) is obtained from (1.3), etc.

3.3. New identities from an inverse tangent identity

Proposition 2. *If k is any integer, then*

$$\frac{L_{2k+1}}{F_{2k+1}^2+1}=\frac{L_{2k}}{F_{2k}^2+1}-\frac{L_{2k+2}}{F_{2k+2}^2+1}, \tag{3.5}$$

$$\frac{L_{2k+1}}{L_{2k}L_{2k+2}}\frac{(F_{2k}^2+1)(F_{2k+2}^2+1)}{(F_{2k+1}^2+1)}=\frac{(F_{2k+2}^2+1)}{L_{2k+2}}-\frac{(F_{2k}^2+1)}{L_{2k}}. \tag{3.6}$$

Recall

$$\tan^{-1}\frac{1}{F_{2k+1}}=\tan^{-1}\frac{1}{F_{2k}}-\tan^{-1}\frac{1}{F_{2k+2}}. \tag{1.7}$$

To derive (3.5), write (1.7) as

$$\tan^{-1}\frac{1}{f(2k+1)}=\tan^{-1}\frac{1}{f(2k)}-\tan^{-1}\frac{1}{f(2k+2)}, \tag{3.7}$$

and differentiate with respect to k to obtain

$$\frac{1}{f(2k+1)^2+1}\frac{d}{dk}f(2k+1)$$
$$=\frac{1}{f(2k)^2+1}\frac{d}{dk}f(2k)+\frac{1}{f(2k+2)^2+1}\frac{d}{dk}f(2k+2),$$

and, taking real part,

$$\frac{1}{F_{2k+1}^2+1}\Re\frac{d}{dk}f(2k+1)$$
$$=\frac{1}{F_{2k}^2+1}\Re\frac{d}{dk}f(2k)+\frac{1}{F_{2k+2}^2+1}\Re\frac{d}{dk}f(2k+2),$$

and hence (3.5), upon using (2.3). Identity (3.6) is a rearrangement of (3.5).

Simple telescoping of (3.5) and (3.6) produces the results stated in the next proposition.

Proposition 3. *If n is any integer, then*

$$\sum_{k=1}^{n}\frac{L_{2k+1}}{F_{2k+1}^2+1}=\frac{3}{2}-\frac{L_{2(n+1)}}{F_{2(n+1)}^2+1},$$

$$\sum_{k=1}^{n}\frac{L_{2k+1}}{L_{2k}L_{2k+2}}\frac{\left(F_{2k}^2+1\right)\left(F_{2k+2}^2+1\right)}{\left(F_{2k+1}^2+1\right)}=\frac{F_{2(n+1)}^2+1}{L_{2n+2}}-\frac{2}{3};$$

with the limiting case:

$$\sum_{k=1}^{\infty} \frac{L_{2k+1}}{F_{2k+1}^2+1} = \frac{3}{2}.$$

3.4. New identities from an identity of Jennings

Jennings [7, Theorem 2] showed, among results of a similar nature, that

$$F_k \sum_{j=0}^{n} (-1)^{(k+1)(n+j)} \binom{n+j}{2j} L_k^{2j} = F_{(2n+1)k}.$$

Writing

$$\sum_{j=0}^{n} (-1)^{(k+1)(n+j)} \binom{n+j}{2j} l(k)^{2j} = \frac{f((2n+1)k)}{f(k)}$$

and differentiating with respect to k gives

$$\begin{aligned} &\sum_{j=0}^{n} (-1)^{(k+1)(n+j)} (n+j)\pi i \binom{n+j}{2j} l(k)^{2j} + \sum_{j=0}^{n} (-1)^{(k+1)(n+j)} 2j \binom{n+j}{2j} l(k)^{2j-1} \frac{d}{dk} l(k) \\ &\quad = \frac{2n+1}{f(k)} \frac{d}{dk} f((2n+1)k) - \frac{f((2n+1)k)}{f(k)^2} \frac{d}{dk} f(k), \end{aligned}$$

and taking real parts,

$$\begin{aligned} &\sum_{j=0}^{n} (-1)^{(k+1)(n+j)} 2j \binom{n+j}{2j} L_k^{2j-1} \Re \frac{d}{dk} l(k) \\ &\quad = \frac{2n+1}{F_k} \Re \frac{d}{dk} f((2n+1)k) - \frac{F_{(2n+1)k}}{F_k^2} \Re \frac{d}{dk} f(k), \end{aligned}$$

which, by (2.3) and (2.4) gives

$$\sum_{j=0}^{n} (-1)^{(k+1)(n+j)} 2j \binom{n+j}{2j} L_k^{2j-1} F_k \sqrt{5} = \frac{2n+1}{F_k} \frac{L_{(2n+1)k}}{\sqrt{5}} - \frac{F_{(2n+1)k}}{F_k^2} \frac{L_k}{\sqrt{5}};$$

and hence the result stated in the next proposition.

Proposition 4. *For non-negative integers k and n, we have*

$$F_k^3 \sum_{j=0}^{n} (-1)^{(k+1)(n+j)} j \binom{n+j}{2j} L_k^{2j} = \frac{1}{10} \left((2n+1) F_{2k} L_{(2n+1)k} - F_{(2n+1)k} L_k^2 \right).$$

We also have the following divisibility property.

Proposition 5. *If n and k are non-negative integers, then*

$$10 \text{ divides } (2n+1) F_{2k} L_{(2n+1)k} - F_{(2n+1)k} L_k^2.$$

3.5. New identities from Candido's identity

Setting $x = G_k$, $y = G_{k+1}$ in the algebraic identity

$$2\left(x^4 + y^4 + (x+y)^4\right) = \left(x^2 + y^2 + (x+y)^2\right)^2 \tag{3.8}$$

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gives the following generalization of Candido's identity:

$$2\left(G_k^4+G_{k+1}^4+G_{k+2}^4\right)=\left(G_k^2+G_{k+1}^2+G_{k+2}^2\right)^2. \tag{3.9}$$

Writing

$$2(g(k)^4+g(k+1)^4+g(k+2)^4)=\left(g(k)^2+g(k+1)^2+g(k+2)^2\right)^2,$$

differentiating with respect to k and applying the prescription (2.24) and (2.25) gives

$$\begin{aligned}&2\left(G_k^3(G_{k+1}+G_{k-1})+G_{k+1}^3(G_{k+2}+G_k)+G_{k+2}^3(G_{k+3}+G_{k+1})\right)\\&\quad=(G_k^2+G_{k+1}^2+G_{k+2}^2)\,(G_k(G_{k+1}+G_{k-1})+\\&\qquad G_{k+1}(G_{k+2}+G_k)+G_{k+2}(G_{k+3}+G_{k+1}))\,,\end{aligned} \tag{3.10}$$

which can be arranged as stated in the next proposition.

Proposition 6. *For every integer k,*

$$\begin{aligned}&G_k^2\left(G_{k+1}(G_{k+2}+G_k)+G_{k+2}(G_{k+3}+G_{k+1})-G_k(G_{k+1}+G_{k-1})\right)\\&\quad+G_{k+1}^2\left(G_k(G_{k+1}+G_{k-1})+G_{k+2}(G_{k+3}+G_{k+1})-G_{k+1}(G_{k+2}+G_k)\right)\\&\quad+G_{k+2}^2\left(G_k(G_{k+1}+G_{k-1})+G_{k+1}(G_{k+2}+G_k)-G_{k+2}(G_{k+3}+G_{k+1})\right)\\&\quad=0.\end{aligned} \tag{3.11}$$

In particular,

$$F_k^2F_{2k+3}+F_{k+1}^2F_{2k+2}=F_{k+2}^2F_{2k+1}, \tag{3.12}$$

$$L_k^2F_{2k+3}+L_{k+1}^2F_{2k+2}=L_{k+2}^2F_{2k+1}. \tag{3.13}$$

Subtraction of (3.12) from (3.13) gives

$$F_{k-1}F_{k+1}F_{2k+3}+F_kF_{k+2}F_{2k+2}=F_{k+1}F_{k+3}F_{2k+1}, \tag{3.14}$$

while their addition yields

$$(F_{k+1}^2+F_{k-1}^2)F_{2k+3}+(F_{k+2}^2+F_k^2)F_{2k+2}=(F_{k+3}^2+F_{k+1}^2)F_{2k+1}. \tag{3.15}$$

Before closing this section, we bring forth a Candido-type identity of R. S. Melham and discover new identities from it. Melham [12, Theorem 1] has shown that:

$$6\left(\sum_{j=0}^{2n-1}G_{k+j}^2\right)^2=F_{2n}^2\left(G_{k+n-2}^4+4G_{k+n-1}^4+4G_{k+n}^4+G_{k+n+1}^4\right);$$

from which, writing $f(2n)$ for F_{2n}, $g(k+n-2)$ for G_{k+n-2}, etc. , and differentiating with respect to k, we have

$$\begin{aligned}&\left(12\sum_{j=0}^{2n-1}g(k+j)^2\right)\sum_{j=0}^{2n-1}2g(k+j)\frac{\partial}{\partial k}g(k+j)\\&=f(2n)^2\left(4g(k+n-2)^3\frac{\partial}{\partial k}g(k+n-2)+16g(k+n-1)^3\frac{\partial}{\partial k}g(k+n-1)\right.\\&\left.+16g(k+n)^3\frac{\partial}{\partial k}g(k+n)+4g(k+n+1)^3\frac{\partial}{\partial k}g(k+n+1)\right).\end{aligned} \tag{3.16}$$

Taking the real part according to the prescription of steps 2 and 3, using (2.24) and (2.25) to replace the derivatives, we obtain the result stated in the next proposition.

Proposition 7. *If n is a non-negative integer and k is any integer, then*

$$\begin{aligned}6\sum_{j=0}^{2n-1} G_{j+k}^2 \sum_{j=0}^{2n-1} G_{j+k}(G_{j+k+1}+G_{j+k-1})\\ =F_{2n}^2\Big(G_{k+n-2}^3(G_{k+n-1}+G_{k+n-3})+4G_{k+n-1}^3(G_{k+n}+G_{k+n-2})\\ +4G_{k+n}^3(G_{k+n+1}+G_{k+n-1})+G_{k+n+1}^3(G_{k+n+2}+G_{k+n})\Big)\,.\end{aligned} \tag{3.17}$$

In particular,

$$\begin{aligned}6\sum_{j=0}^{2n-1} F_{j+k}^2 \sum_{j=0}^{2n-1} F_{2j+2k} = F_{2n}^2\Big(F_{k+n-2}^2F_{2(k+n-2)}+4F_{k+n-1}^2F_{2(k+n-1)}\\ +4F_{k+n}^2F_{2(k+n)}+F_{k+n+1}^2F_{2(k+n+1)}\Big)\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}6\sum_{j=0}^{2n-1} L_{j+k}^2 \sum_{j=0}^{2n-1} F_{2j+2k} = F_{2n}^2\Big(L_{k+n-2}^2F_{2(k+n-2)}+4L_{k+n-1}^2F_{2(k+n-1)}\\ +4L_{k+n}^2F_{2(k+n)}+L_{k+n+1}^2F_{2(k+n+1)}\Big)\,.\end{aligned} \tag{3.19}$$

Subtraction of (3.18) from (3.19) gives

$$\begin{aligned}6\sum_{j=0}^{2n-1} F_{j+k+1}F_{j+k-1} \sum_{j=0}^{2n-1} F_{2j+2k}\\ =F_{2n}^2\Big(F_{k+n-1}F_{k+n-3}F_{2(k+n-2)}+4F_{k+n}F_{k+n-2}F_{2(k+n-1)}\\ +4F_{k+n+1}F_{k+n-1}F_{2(k+n)}+F_{k+n+2}F_{k+n}F_{2(k+n+1)}\Big)\,.\end{aligned} \tag{3.20}$$

3.6. New identities from the Gelin Cesàro identity

The Gelin Cesàro identity

$$F_{j-2}F_{j-1}F_{j+1}F_{j+2}=F_j^4-1 \tag{3.21}$$

has the following generalization (Horadam and Shannon [5, Identity (2.5), $q=-1$]):

$$W_{k-2}W_{k-1}W_{k+1}W_{k+2}=W_k^4+(-1)^k\gamma W_k^2-\delta^2; \tag{3.22}$$

where $\gamma=e(p^2-1)$, $\delta=ep$ and $e=pW_0W_1+W_0^2-W_1^2$.

For the sequence of Lucas numbers, $\gamma=0$ and $e=5=\delta$, so that

$$L_{k-2}L_{k-1}L_{k+1}L_{k+2}=L_k^4-25; \tag{3.23}$$

while for the gibonacci sequence, $\gamma=0$, $\delta=e=G_0G_1+G_0^2-G_1^2$ and

$$G_{k-2}G_{k-1}G_{k+1}G_{k+2}=G_k^4-e^2. \tag{3.24}$$

Writing

$$w(k-2)w(k-1)w(k+1)w(k+2)=w(k)^4+(-1)^k\gamma w(k)^2-\delta^2$$

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and differentiating with respect to k and making use of (2.18) and (2.19) from section 2.3 yields the result stated in the next proposition.

Proposition 8. *For every integer k,*

$$\begin{aligned}&(W_{k-1}+W_{k-3})W_{k-1}W_{k+1}W_{k+2}\\&\quad+W_{k-2}(W_k+W_{k-2})W_{k+1}W_{k+2}\\&\quad+W_{k-2}W_{k-1}(W_k+W_{k+2})W_{k+2}\\&\qquad+W_{k-2}W_{k-1}W_{k+1}(W_{k+3}+W_{k+1})\\&=2W_k(W_{k+1}+W_{k-1})(2W_k^2+(-1)^k\gamma).\end{aligned}\tag{3.25}$$

In particular,

$$\begin{aligned}&(G_{k-1}+G_{k-3})G_{k-1}G_{k+1}G_{k+2}\\&\quad+G_{k-2}(G_k+G_{k-2})G_{k+1}G_{k+2}\\&\quad+G_{k-2}G_{k-1}(G_k+G_{k+2})G_{k+2}\\&\qquad+G_{k-2}G_{k-1}G_{k+1}(G_{k+3}+G_{k+1})\\&=4G_k^3(G_{k+1}+G_{k-1});\end{aligned}\tag{3.26}$$

with the special cases

$$F_{k+1}F_{k+2}F_{2k-3}+F_{k-1}F_{k-2}F_{2k+3}=2F_k^3L_k=2F_k^2F_{2k}\tag{3.27}$$

and

$$L_{k+1}L_{k+2}F_{2k-3}+L_{k-1}L_{k-2}F_{2k+3}=2L_k^3F_k=2L_k^2F_{2k};\tag{3.28}$$

where, to arrive at (3.27) and (3.28), we used

$$F_{k+1}+F_{k-1}=L_k,\quad L_{k+1}+L_{k-1}=5F_k,$$

and [14, Identity (16a)]

$$L_mF_n+L_nF_m=2F_{m+n}.$$

Substituting $k+2$ for k and arranging (3.27) and (3.28) as

$$\frac{F_{2k+1}}{F_kF_{k+1}}+\frac{F_{2k+7}}{F_{k+3}F_{k+4}}=\frac{2F_{k+2}^2F_{2k+4}}{F_{k+2}^4-1}$$

and

$$\frac{F_{2k+1}}{L_kL_{k+1}}+\frac{F_{2k+7}}{L_{k+3}L_{k+4}}=\frac{2L_{k+2}^2F_{2k+4}}{L_{k+2}^4-25};$$

and the use of the telescoping summation formula

$$\sum_{k=1}^n(-1)^{k-1}\left(f_k+(-1)^{m-1}f_{k+m}\right)=\sum_{k=1}^m(-1)^{k-1}f_k+(-1)^{n-1}\sum_{k=1}^m(-1)^{k-1}f_{k+n}$$

yields the summation identities stated in the next proposition.

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Proposition 9. *If n is a non-negative integer, then*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1} F_{k+2}^2 F_{2k+4}}{F_{k+2}^4 - 1} = \frac{5}{6} + \frac{(-1)^{n-1}}{2} \left(\frac{F_{2n+3}}{F_{n+1}F_{n+2}} - \frac{F_{2n+5}}{F_{n+2}F_{n+3}} + \frac{F_{2n+7}}{F_{n+3}F_{n+4}} \right) \tag{3.29}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k-1} L_{k+2}^2 F_{2k+4}}{L_{k+2}^4 - 25} = \frac{5}{14} + \frac{(-1)^{n-1}}{2} \left(\frac{F_{2n+3}}{L_{n+1}L_{n+2}} - \frac{F_{2n+5}}{L_{n+2}L_{n+3}} + \frac{F_{2n+7}}{L_{n+3}L_{n+4}} \right); \tag{3.30}$$

with

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{k+2}^2 F_{2k+4}}{F_{k+2}^4 - 1} = \frac{5}{6}, \tag{3.31}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{k+2}^2 F_{2k+4}}{L_{k+2}^4 - 25} = \frac{5}{14}. \tag{3.32}$$

Arranging (3.25) as

$$\begin{aligned} &\frac{W_{j-1} + W_{j-3}}{W_{j-2}} + \frac{W_j + W_{j-2}}{W_{j-1}} + \frac{W_{j+2} + W_j}{W_{j+1}} + \frac{W_{j+3} + W_{j+1}}{W_{j+2}} \\ &= \frac{2W_j(W_{j+1} + W_{j-1})(2W_j^3 + (-1)^j \gamma)}{W_{j-2}W_{j-1}W_{j+1}W_{j+2}} \end{aligned} \tag{3.33}$$

and summing produces the next result.

Proposition 10. *If n and k are integers then,*

$$\begin{aligned} &\sum_{j=1}^{n} \frac{(-1)^{j-1} 2W_{j+k}(W_{j+k+1} + W_{j+k-1})(2W_{j+k}^2 + (-1)^{j+k}\gamma)}{W_{j+k-2}W_{j+k-1}W_{j+k+1}W_{j+k+2}} \\ &= (-1)^{n+1} \frac{W_{n+k} + W_{n+k-2}}{W_{n+k-1}} + \frac{W_k + W_{k-2}}{W_{k-1}} \\ &\quad + (-1)^{n+1} \frac{W_{n+k+3} + W_{n+k+1}}{W_{n+k+2}} + \frac{W_{k+3} + W_{k+1}}{W_{k+2}}; \end{aligned} \tag{3.34}$$

provided none of the denominators vanishes.

3.7. New identities from a reciprocal series of Fibonacci numbers with subscripts $k2^j$

In this section we apply our method to discover new results associated with the following identity of Rabinowitz [13, Equation (4)]:

$$\sum_{j=0}^{n} \frac{1}{U_{k2^j}} = \frac{1 + U_{k-1}}{U_k} + \frac{1 - (-1)^k}{U_{2k}} - \frac{U_{k2^n-1}}{U_{k2^n}}. \tag{3.35}$$

Writing

$$\sum_{j=0}^{n} \frac{1}{u(k2^j)} = \frac{1 + u(k-1)}{u(k)} + \frac{1 - (-1)^k}{u(2k)} - \frac{u(k2^n - 1)}{u(k2^n)}, \tag{3.36}$$

and differentiating with respect to k gives

$$\begin{aligned}\sum_{j=0}^{n}\frac{-2^j v(k2^j)}{u(k2^j)^2} &= \frac{1}{u(k)}\frac{d}{dk}u(k-1) - \frac{(1+u(k-1))}{u(k)^2}\frac{d}{dk}u(k)\\ &\quad - \frac{(-1)^k\pi i}{u(2k)} - \frac{2(1-(-1)^k)}{u(2k)^2}\frac{d}{dk}u(2k)\\ &\quad - \frac{2^n}{u(k2^n)}\frac{\partial}{\partial k}u(k2^n-1) + \frac{2^n u(k2^n-1)}{u(k2^n)^2}\frac{\partial}{\partial k}u(k2^n).\end{aligned}$$

Taking the real part while using (2.20)–(2.23), we have the next result.

Proposition 11. *If n and k are positive integers, then*

$$\sum_{j=0}^{n}\frac{2^j V_{k2^j}}{U_{k2^j}^2} = \frac{(-1)^k 2 + V_k}{U_k^2} + \frac{2(1-(-1)^k)V_{2k}}{U_{2k}^2} - \frac{2^{n+1}}{U_{k2^n}^2},$$

$$\sum_{j=0}^{\infty}\frac{2^j V_{k2^j}}{U_{k2^j}^2} = \frac{(-1)^k 2 + V_k}{U_k^2} + \frac{2(1-(-1)^k)V_{2k}}{U_{2k}^2}.$$

Note that in arriving at the final form of the first expression in Proposition 11, we used

$$U_r V_s - V_r U_s = (-1)^s 2U_{r-s}.$$

In particular, we have

$$\sum_{j=0}^{n}\frac{2^j V_{2^j}}{U_{2^j}^2} = p + \frac{2\Delta^2}{p^2} - \frac{2^{n+1}}{U_{2^n}^2}$$

and

$$\sum_{j=0}^{n}\frac{2^j V_{2^{j+1}}}{U_{2^{j+1}}^2} = \frac{\Delta^2}{p^2} - \frac{2^{n+1}}{U_{2^{n+1}}^2};$$

with the special cases

$$\sum_{j=0}^{n}\frac{2^j L_{2^j}}{F_{2^j}^2} = 11 - \frac{2^{n+1}}{F_{2^n}^2}$$

and

$$\sum_{j=0}^{n}\frac{2^j L_{2^{j+1}}}{F_{2^{j+1}}^2} = 5 - \frac{2^{n+1}}{F_{2^{n+1}}^2}.$$

4. Justification of the method

In this section we provide the rationale behind the method that was described in section 2. To facilitate the discussion, we need the closed formula for the generalized Fibonacci sequence (W_j).

4.1. Closed formula for the generalized Fibonacci sequence

Standard methods for solving difference equations give the closed (Binet) formula of the generalized Fibonacci sequence (W_j) defined by the recurrence relation (1.13), in the non-degenerated case, $p^2+4>0$, as

$$W_j = A\tau^j + B\sigma^j, \tag{4.1}$$

where

$$A = \frac{W_1 - W_0\sigma}{\tau-\sigma}, \quad B = \frac{W_0\tau - W_1}{\tau-\sigma}, \tag{4.2}$$

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with

$$\tau = \frac{p + \sqrt{p^2+4}}{2}, \qquad \sigma = \frac{p - \sqrt{p^2+4}}{2}; \tag{4.3}$$

so that

$$\tau + \sigma = p, \quad \tau - \sigma = \sqrt{p^2+4} = \Delta, \quad \text{and } \tau\sigma = -1. \tag{4.4}$$

In particular,

$$U_j = \frac{\tau^j - \sigma^j}{\tau - \sigma}, \quad V_j = \tau^j + \sigma^j. \tag{4.5}$$

Using the Binet formulas, it is readily established that

$$U_{-j} = (-1)^{j-1} U_j, \quad V_{-j} = (-1)^j V_j. \tag{4.6}$$

It is also straightforward to establish the following:

$$U_{j+1} + U_{j-1} = V_j, \tag{4.7}$$
$$U_{j+1} - U_{j-1} = pU_j, \tag{4.8}$$
$$V_{j+1} + V_{j-1} = U_j \Delta^2, \tag{4.9}$$

and

$$V_{j+1} - V_{j-1} = pV_j. \tag{4.10}$$

Lemma 1. *For any integer j,*

$$A\tau^j - B\sigma^j = \frac{W_{j+1} + W_{j-1}}{\Delta}.$$

Proof. Let $Q = A\tau^j - B\sigma^j$. Then,

$$\tau Q = A\tau^{j+1} + B\sigma^{j-1},$$
$$\sigma Q = -A\tau^{j-1} - B\sigma^{j+1}.$$

Thus,

$$Q \times (\tau - \sigma) = \left(A\tau^{j+1} + B\sigma^{j+1}\right) + \left(A\tau^{j-1} + B\sigma^{j-1}\right);$$

that is

$$Q\Delta = W_{j+1} + W_{j-1}.$$

□

Lemma 1 is at the heart of the justification of the calculus-based method of obtaining Fibonacci identities.

4.2. *Justification of the method*

Consider a generalized Fibonacci function $w(x)$ defined by

$$w(x) = A\tau^x + B\sigma^x, \quad x \in \mathbb{R}, \tag{4.11}$$

where A and B are as defined in (4.2) and τ and σ are as given in (4.3).

Clearly,

$$w(j) = W_j, \quad j \in \mathbb{Z}. \tag{4.12}$$

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Theorem. *The following identity holds:*

$$\Re\left(\frac{d}{dx}w(x)\Big|_{x=j\in\mathbb{Z}}\right)=\frac{W_{j+1}+W_{j-1}}{\Delta}\ln\tau, \tag{4.13}$$

where, as usual, $\Re(X)$ denotes the real part of X.

Proof. We have

$$\begin{aligned}\frac{d}{dx}w(x)&=A\tau^x\ln\tau+B\sigma^x\ln\sigma\\&=A\tau^x\ln\tau-B\sigma^x\ln\tau+B\sigma^x\ln\tau+B\sigma^x\ln\sigma\\&=(A\tau^x-B\sigma^x)\ln\tau+B\sigma^x\ln(\tau\sigma);\end{aligned}$$

that is,

$$\frac{d}{dx}w(x)=(A\tau^x-B\sigma^x)\ln\tau+B\sigma^x\ln(-1). \tag{4.14}$$

Evaluating (4.14) at $x=j\in\mathbb{Z}$, we have

$$\begin{aligned}\frac{d}{dx}w(x)\Big|_{x=j\in\mathbb{Z}}&=\left(A\tau^j-B\sigma^j\right)\ln\tau+B\sigma^j\ln(-1)\\&=\frac{W_{j+1}+W_{j-1}}{\Delta}\ln\tau+B\sigma^j\ln(-1),\text{by Lemma 1},\end{aligned}$$

from which, on taking real parts, (4.13) follows, since σ, Δ, B, W_{j+1} and W_{j-1} are real quantities and τ is a positive number. □

Of course (2.3) and (2.4) are particular cases of (4.13) with $\Delta=\sqrt{5}$, $F_{j+1}+F_{j-1}=L_j$ and $L_{j+1}+L_{j-1}=5F_j$. Similarly, (2.19), (2.21), (2.23) and (2.25) are all particular cases of (4.13).

Thus, given a (generalized) Fibonacci identity having a free index, on account of (4.11), (4.12) and (4.13), we can replace (generalized) Fibonacci numbers with (generalized) Fibonacci functions, perform differentiation and evaluate at integer values to obtain a new (generalized) Fibonacci identity.

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