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The Brooks-Chacon Biting Lemma and the Baum-Katz Theorem Along Subsequences

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Abstract

We prove Baum-Katz type theorems along subsequences of random variables under Komlós-Saks and Mazur-Orlicz type boundedness hypotheses.

Keywords: complete convergence; boundedness hypotheses; law of large numbers

MSC: 60F15; 60E15

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1. Introduction

The following result (see [7]) quantifies the rate of convergence in the strong law of large numbers, through a complete convergent numerical Baum-Katz-type series *along subsequences* of norm bounded random variables, without any supplementary probabilistic hypothesis on their (in)dependence or on their distributions:

Theorem 0. *On a complete probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *we consider a sequence of random variables* $(X_n)_{n\geq 1}$ *that is uniformly bounded in* L^p *for some* $0 < p < 2$, *i.e.,*

$$\sup_{n\geq 1} ||X_n||_p \leq C \textit{ for some } C > 0.$$

Then there exists a subsequence $(Y_n)_{n\geq 1}$ *of* $(X_n)_{n\geq 1}$ *such that, for all* $1 < r \leq p$, *we have*

$$\sum_{n=1}^{\infty} n^{p/r-2}\mathbf{P}\left(\left\{\omega \in \Omega : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \textit{ for any } \varepsilon > 0. \tag{1}$$

Remarks. The complete convergence of the series in formula (1) implies that the subsequence $(Y_n)_{n\geq 1}$ satisfies the strong law of large numbers, i.e., $Y_n/n^{1/p} \to 0$ **P**-a.s. The parameter $r \in (1,2)$ is keeping Theorems 0 within the realm of laws of large numbers; indeed, if $p \geq r \geq 2$ then by the central limit theorem for subsequences, the series in formula (1) diverges for all $\varepsilon > 0$ even if $(X_n)_{n\geq 1}$ is an i.i.d. sequence with mean zero and finite variance. Also note that formula (1) trivially holds if $0 < p < r < 2$.

There are situations (see, e.g., the recent papers [4] and [5] and their references) when the sequence of random variables in question is not (uniformly) bounded in L^p and yet satisfies a law of large numbers. In this case, Theorem 0 is not appropriate to quantify the rate of convergence in that law of large numbers. In addition, the examples in [8], [6] and [3] show that Theorem 0 fails if one drops the L^p-uniform boundedness hypothesis, for any $0 < p < 2$. Inspired by the celebrated Komlós-Saks-type extension of the law of large numbers (cf. [4], [5]), in Section 2 we shall prove a version of the Baum-Katz theorem under a special boundedness hypothesis, different from the L^p-boundedness condition required in Theorem 0. The idea is to construct a rich family of uniformly integrable subsequences of $(X_n)_{n\geq 1}$ as in [2], for which condition (1) holds; note that the hypotheses in [8] and [3] cannot produce Baum-Katz type theorems, as the families of subsequences therein are no longer uniformly integrable.

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Instead, we shall employ the methodology given by the celebrated Bitting Lemma of Brooks and Chacon (cf. [1]). A modification of this methodology is presented in Section 3; it will produce a second version of the Baum-Katz theorem under a Mazur-Orlicz-type hypothesis (cf. [4], [5]).

2. Main Result

Theorem 1. *Let $0 < p < 2$. On a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we consider a sequence of random variables $(X_n)_{n\geq 1}$ such that*

$$\limsup_{n\geq 1} |X_n(\omega)|^p < \infty \text{ for all } \omega \in \Omega.$$

Then there exists a subsequence $(Y_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that equation (1) holds for all $1 < r \leq p$. In particular, the subsequence $(Y_n)_{n\geq 1}$ satisfies the strong law of large numbers, i.e., $Y_n/n^{1/p} \to 0$ $\mathbf{P}$-a.s.

Examples. (i) Note that the working hypothesis in Theorem 1 is equivalent to

$$\sup_{n\geq 1} |X_n(\omega)|^p < \infty \text{ for all } \omega \in \Omega.$$

In particular, one can see that Theorems 0 and 1 do not overlap and do not imply each other. Indeed, uniformly L^p-bounded sequences of functions can still have an infinite limsup, and vice-versa: there are sequences of functions, with finite limsup, that are not (uniformly) bounded in L^p (see also [9]).

(ii) The hypotheses in Theorem 1 are satisfied, e.g., by the working condition in the motivational papers [4] and [5], namely:

$$\lim_{N\to\infty} N \cdot \sup_{n\geq 1} \mathbf{P}(|X_n| > N) = 0.$$

This condition implies uniform boundedness in L^0 (tightness), i.e.,

$$\lim_{N\to\infty} \sup_{n\geq 1} \mathbf{P}(|X_n| > N) = 0$$

and is implied by uniform integrability, i.e.,

$$\lim_{N\to\infty} \sup_{n\geq 1} \mathbf{E}(X_n \cdot \mathbf{1}_{\{|X_n|>N\}}) = 0,$$

where $\mathbf{E}$ denotes the expectation with respect to $\mathbf{P}$. Also note that L^p-boundedness condition in Theorem 0 is stronger than the last three conditions, provided $p \in [1, 2)$ (see, e.g., Example 4.2 in [4]).

Proof of Theorem 1. For any natural number $m \geq 1$, let us define the $\mathcal{F}$-measurable sets

$$A_m := \left\{\omega \in \Omega : \sup_{n\geq 1} |X_n(\omega)|^p \leq m\right\}.$$

Assume that $r < p$ and fix $a > p/r - 1$. As $\mathbf{P}(A_m) \to 1$ as $m \to \infty$ by hypothesis, we can choose an index $m_1 \geq 1$ such that $\mathbf{P}(A_{m_1}) > 1 - 2^{-a}$. By Fatou's lemma we obtain:

$$\sup_{n\geq 1} \int_{A_{m_1}} |X_n(\omega)|^p d\mathbf{P}(\omega) \leq m_1. \tag{2}$$

We now apply the Bitting Lemma (cf. [1]) to the sequence $(X_n)_{n\geq 1}$ and the subset A_{m_1} above, to obtain: a non-decreasing sequence $(B^1_k)_{k\geq 1}$ of subsets in $\mathcal{F}$ with $\mathbf{P}(B^1_k) \to 1$ as $k \to \infty$, and a subsequence $(X^1_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that $(X^1_n)_{n\geq 1}$ is uniformly integrable on each of the subsets $A_{m_1} \cap B^1_k$, $k \geq 1$.

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This latter fact together with estimate (2) show that Theorem 0 applies to the sequence $(X_n^1)_{n\geq 1}$ and gives:

$$\sum_{n=1}^{\infty} n^{p/r-2}\mathbf{P}\left(\left\{\omega \in A_{m_1} \cap B_k^1 : \left|\sum_{j=1}^{n} X_j^1(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for any } \varepsilon > 0 \text{ and } k \geq 1.$$

Next, we choose a natural number $m_2 \geq 1$ such that $\mathbf{P}(A_{m_2}) > 1 - 3^{-a}$, and another application of the Bitting Lemma, this time to $(X_n^1)_{n\geq 1}$, produces: a non-decreasing sequence $(B_k^2)_{k\geq 1}$ of subsets in $\mathcal{F}$ with $\mathbf{P}(B_k^2) \to 1$ as $k \to \infty$, and a subsequence $(X_n^2)_{n\geq 1}$ of $(X_n^1)_{n\geq 1}$, therefore a subsequence of $(X_n)_{n\geq 1}$ as well, such that $(X_n^2)_{n\geq 1}$ is uniformly integrable on each of the subsets $A_{m_2} \cap B_k^2$, $k \geq 1$, and

$$\sum_{n=1}^{\infty} n^{p/r-2}\mathbf{P}\left(\left\{\omega \in A_{m_2} \cap B_k^2 : \left|\sum_{j=1}^{n} X_j^2(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for any } \varepsilon > 0 \text{ and } k \geq 1.$$

Thus, by induction, we constructed for each $i \geq 1$: an $\mathcal{F}$-measurable set A_{m_i} with $\mathbf{P}(A_{m_i}) > 1 - (i+1)^{-a}$, a non-decreasing sequence $(B_k^i)_{k\geq 1}$ of subsets in $\mathcal{F}$ with $\mathbf{P}(B_k^i) \to 1$ as $k \to \infty$, and a subsequence $(X_n^i)_{n\geq 1}$ of $(X_n^{i-1})_{n\geq 1}$ such that $(X_n^i)_{n\geq 1}$ is uniformly integrable on each of the subsets $A_{m_i} \cap B_k^i$, $k \geq 1$, and

$$\sum_{n=1}^{\infty} n^{p/r-2}\mathbf{P}\left(\left\{\omega \in A_{m_i} \cap B_k^i : \left|\sum_{j=1}^{n} X_j^i(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for any } \varepsilon > 0 \text{ and } k, i \geq 1.$$

(the convention is that $(X_n^0)_{n\geq 1}$ is precisely $(X_n)_{n\geq 1}$).

Now define $Y_n := X_n^n$, $n \geq 1$; it follows that $(Y_n)_{n\geq 1}$ is a subsequence of $(X_n)_{n\geq 1}$ and, using a diagonal argument in the previous formula, we obtain that

$$\sum_{n=1}^{\infty} n^{p/r-2}\mathbf{P}\left(\left\{\omega \in A_{m_n} \cap B_k^n : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for any } \varepsilon > 0 \text{ and } k \geq 1. \tag{3}$$

As $\mathbf{P}(B_k^n) \to 1$ as $k \to \infty$ for all $n \geq 1$, formula (3) and the dominated convergence theorem imply that

$$\sum_{n=1}^{\infty} n^{p/r-2}\mathbf{P}\left(\left\{\omega \in A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for any } \varepsilon > 0. \tag{4}$$

Therefore, to prove that our series (1) converges for this particular subsequence $(Y_n)_{n\geq 1}$, it suffices to prove formula (4) with A_{m_n} replaced by its set complement, i.e.,

$$\sum_{n=1}^{\infty} n^{p/r-2}\mathbf{P}\left(\left\{\omega \in \Omega \setminus A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for any } \varepsilon > 0. \tag{5}$$

Indeed, as $\mathbf{P}(A_{m_n}) > 1 - (n+1)^{-a} > 1 - n^{-a}$ and $a > p/r - 1 > 0$, the series in (5) is

$$\leq \sum_{n=1}^{\infty} n^{p/r-2}\mathbf{P}\left(\left\{\omega \in \Omega \setminus A_{m_n}\right\}\right) \leq \sum_{n=1}^{\infty} n^{p/r-2-a} < \infty,$$

so the proof is achieved in the case $r < p$.

If $r = p$, then we modify the above methodology as follows: by induction we now choose $\mathcal{F}$-measurable sets A_{m_i} with $\mathbf{P}(A_{m_i}) > i/(i+1)$ for all $i \geq 1$; as such, the Biting Lemma and the diagonal argument produce the subsequence $(Y_n)_{n\geq 1}$ and the following replacement of equation (4):

$$\sum_{n=1}^{\infty} \frac{1}{n}\mathbf{P}\left(\left\{\omega \in A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for any } \varepsilon > 0.$$

To show that the series (1) converges for along the subsequence $(Y_n)_{n\geq 1}$, it suffices to prove the following replacement of equation (5):

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbf{P}\left(\left\{\omega \in \Omega \setminus A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for any } \varepsilon > 0.$$

Indeed, by the new choice of A_{m_n}, the latter series is

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} \mathbf{P}\left(\left\{\omega \in \Omega \setminus A_{m_n}\right\}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty,$$

and this achieves the proof in the case $r = p$.

3. A Variant of the Main Result

Proposition 2. *Let $0 < p < 2$. On a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we consider a sequence of random variables $(X_n)_{n\geq 1}$ satisfying the following condition: for every subsequence $(\tilde{X}_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ and $n \geq 1$, there exists a convex combination Z_n of $\{|\tilde{X}_n|^p, |\tilde{X}_{n+1}|^p, \ldots\}$, such that $\sup_{n\geq 1} |Z_n(\omega)|^p < \infty$ for all $\omega \in \Omega$. Then there exists a subsequence $(Y_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that equation (1) holds for all $1 < r \leq p$ and $Y_n/n^{1/p} \to 0$ $\mathbf{P}$-a.s.*

Examples. On $[0, 1]$ endowed with the Lebesgue measure, the sequence $X_n(\omega) = n^2$ if $0 < \omega < 1/n$ and 0 otherwise, satisfies Theorem 2 because $X_n \to 0$ Lebesgue-a.s., yet it does not satisfy Theorem 1 with $p = 1$ because it is not bounded in $L^1[0, 1]$. As a matter of fact, both Theorems 1 and 2 may fail for unbounded sequences, e.g., $X_n(\omega) = n$.

Proof of Proposition 2. By hypothesis we can write

$$Z_n = \sum_{i\in I_n} \lambda_i^n |\tilde{X}_{n+i}|^p \text{ for some } \lambda_i^n \geq 0 \text{ with } \sum_{i\in I_n} \lambda_i^n = 1,$$

where I_n are finite subsets of $\{0, 1, 2 \ldots\}$. In addition, the sequence $(Z_n)_{n\geq 1}$ satisfies the condition $\sup_{n\geq 1} |Z_n(\omega)|^p < \infty$ for all $\omega \in \Omega$. For any natural number $m \geq 1$, let us define the $\mathcal{F}$-measurable sets

$$A_m := \left\{\omega \in \Omega : \sup_{n\geq 1} |Z_n(\omega)|^p \leq m\right\}.$$

We have $\mathbf{P}(A_m) \to 1$ as $m \to \infty$, so we can choose an index $m_1 \geq 1$ such that $\mathbf{P}(A_{m_1}) > 1 - 2^{-a}$ for some $a > p/r - 1$ fixed, or $1/2$, according to $p > r$ or $p = r$, respectively. In both cases, by applying Fatou's lemma we obtain:

$$\sup_{n\geq 1} \sum_{i\in I_n} \lambda_i^n \int_{A_{m_1}} |\tilde{X}_{n+i}(\omega)|^p d\mathbf{P}(\omega) \leq m_1.$$

Hence there is a subsequence $(\bar{X}_n)_{n\geq 1}$ of $(\tilde{X}_n)_{n\geq 1}$, therefore a subsequence of $(X_n)_{n\geq 1}$ as well, such that

$$\sup_{n\geq 1} \int_{A_{m_1}} |\bar{X}_n(\omega)|^p d\mathbf{P}(\omega) \leq m_1.$$

The remainder of the proof goes exactly as in the proof of Theorem 1, with equation (2) replaced by the equation above, and applied to the subsequence $(\bar{X}_n)_{n\geq 1}$.

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