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Kazuhito Owada *

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Collatz Trees: Trunk–Branch Indexing and Affine Cycle-Freeness

Kazuhito Owada

Independent Researcher; kazu55.owada@gmail.com

Abstract

The Collatz Conjecture remains one of the most enduring unsolved problems in mathematics, despite being based on an extraordinarily simple rule. Given any natural number n, the conjecture posits that repeatedly applying the operation—dividing by 2 if even, or multiplying by 3 and adding 1 if odd—will eventually result in the number 1. This paper develops a structural perspective by proposing the *Collatz Tree* as a framework to organize and visualize natural numbers. Each *branch* is the geometric ray $\{k \cdot 2^b\}_{b \geq 0}$ for an odd *odd core* k, and the *trunk* is the ray from 1. We introduce a trunk–branch indexing that bijects $\mathbb{N}$ with $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Algebraically, we encode Collatz steps as affine maps and prove *absence of nontrivial finite cycles* for a three-way map T; via a *bridge*, this implies the same for the standard accelerated map $A(n) = (3n+1)/2^{\nu_2(3n+1)}$ on odd integers. Thus the global Collatz convergence reduces to an independent pillar: coverage (reachability) of the inverse tree rooted at 1, *isolating cycle-freeness from coverage and reducing the conjecture to the remaining reachability pillar.* Prior work (e.g., Kosobutskyy) studied reverse-oriented trees via Jacobsthal sequences, emphasizing periodic and statistical aspects. Our approach differs in both formulation and aim: we build a tree rooted at 1 and give a constructive, graph-theoretic route toward cycle-freeness and reduction to coverage.

Keywords: Collatz conjecture; directed tree; geometric sequence; reverse computation; natural numbers

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1. Decomposing All Natural Numbers into Geometric Sequences

1.1. Background and Objective

We express $\mathbb{N}$ as a collection of rays parameterized by odd cores and powers of two, providing a structural stage for Collatz dynamics.

1.2. Definitions and Goals

Let

$$S = \{(2a+1) \cdot 2^b \mid a, b \in \mathbb{Z}_{\geq 0}\}.$$

We show $S = \mathbb{N}$ and the representation is unique.

1.3. Prime Factorization and Classification

Every $n \in \mathbb{N}$ decomposes uniquely as

$$n = 2^b \cdot k, \qquad b \in \mathbb{Z}_{\geq 0},\ k \text{ odd}.$$

1.4. Exhaustion of Odd Numbers

Any odd k is $k = 2a+1$ with $a \geq 0$, giving $1, 3, 5, 7, \ldots$.

1.5. Exhaustion of Even Parts

For each odd k, the ray $k, 2k, 4k, \ldots$ exhausts the even multiples of k.

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1.6. Construction of S and Uniqueness

By the above, every $n = (2a+1)2^b$ with $a, b \geq 0$. If

$$(2a+1)2^b = (2a'+1)2^{b'},$$

then $(2a+1)/(2a'+1) = 2^{b'-b}$, forcing $a = a'$ and $b = b'$ since the left side is odd rational and the right is a power of two. Hence $S = \mathbb{N}$ bijectively.

1.7. Remarks from the Collatz Perspective

For odd k, $3k+1$ is even and belongs to some ray $(2a'+1)2^{b'}$. This exhibits *inter-branch* connections. However, the assertion that *every* number lies on a finite *forward* path to 1 (global convergence) is a separate issue (coverage) made precise by Theorem 4; it is not implied by the mere classification $S = \mathbb{N}$.

Takeaway of Chapter 1. We obtain a clean, bijective *indexing* of $\mathbb{N}$ by odd core and 2-adic height, furnishing a coordinate system on which later structural/affine arguments are staged.

2. The Structure of the Collatz Tree

2.1. Definition (Branches and Trunk)

Define the *trunk* $T_0 = \{1 \cdot 2^b : b \geq 0\}$ and for each odd $k \geq 3$ the *branch* $B_k = \{k \cdot 2^b : b \geq 0\}$. These rays partition $\mathbb{N}$ disjointly.

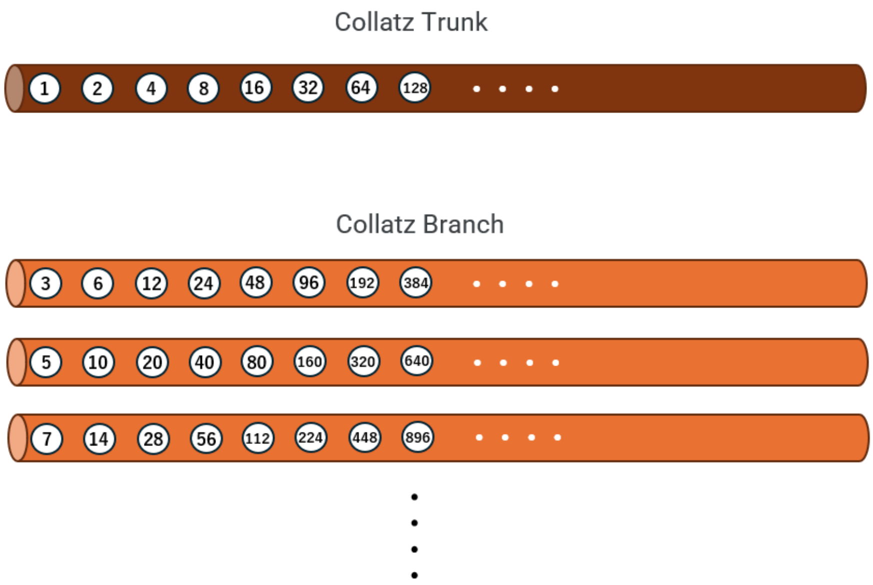

Figure 1. Trunk and branches (schematic; reverse orientation when embedded into the inverse graph: edges point to preimages).

2.2. Branch–Branch Links via $3k+1$

Given odd k, $3k+1$ is even and decomposes as $(2a'+1)2^{b'}$, indicating where the branch from k can *merge* into another branch/trunk in forward dynamics. This shows linkage patterns but *does not* by itself prove global coverage of the tree by reverse generation.

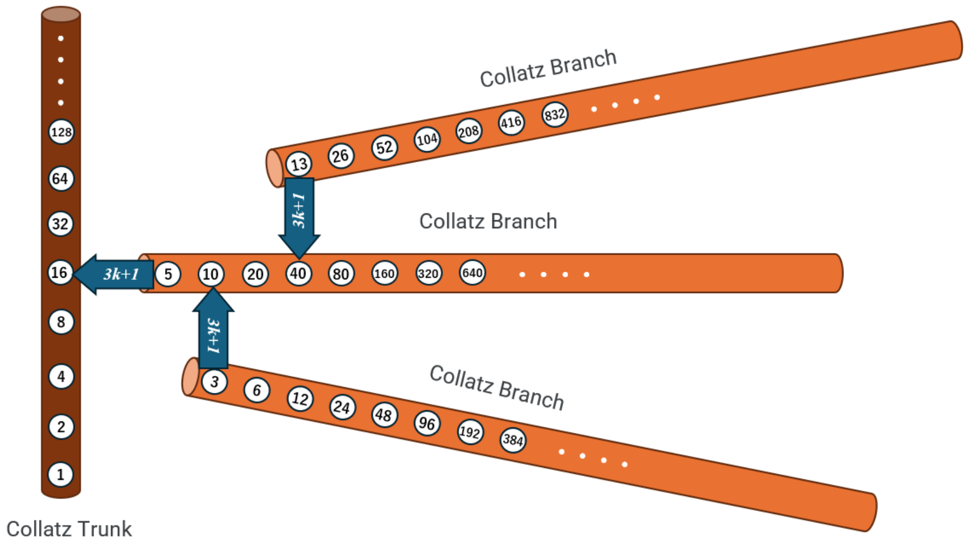

Figure 2. Branch connections (schematic; reverse orientation: edges point to preimages).

2.3. Forward vs. Reverse Orientation

Let the standard forward map be

$$f(n) = \begin{cases} n/2, & n \text{ even}, \\ 3n+1, & n \text{ odd}. \end{cases}$$

The *forward* graph (edges $n \to f(n)$) is a functional digraph (outdegree 1). We do *not* call it a DAG because it contains the trivial $1 \leftrightarrow 2 \leftrightarrow 4$ 3-cycle; nontrivial finite cycles are excluded later (Section 4).

The *reverse* (preimage) graph rooted at 1, with edges to preimages under f, is a true DAG: levels increase with each application of a reverse step.

2.4. Tree Language

When drawing a *reverse* BFS tree rooted at 1, each node is assigned a unique *parent by construction* (though a number may have up to two preimages as graph children). Connectivity of *every* node to 1 in the forward sense is equivalent to *coverage* of the reverse tree, which is equivalent to the Collatz convergence; see Theorem 4.

3. Trunk–Branch Indexing of the Natural Numbers

Definition 1 (Odd core, 2-adic valuation)**.** *For $n \in \mathbb{N}$, write uniquely $n = \mathrm{odd}(n) \cdot 2^{\nu_2(n)}$ where $\mathrm{odd}(n)$ is odd and $\nu_2(n) \in \mathbb{Z}_{\geq 0}$ is the exponent of 2 in n.*

Definition 2 (Trunk and branches)**.** *The trunk is $T_0 = \{1 \cdot 2^b : b = 0, 1, 2, \dots\} = 1, 2, 4, 8, \dots$. For any odd $k \geq 3$, $B_k = \{k \cdot 2^b : b = 0, 1, 2, \dots\}$. Then $\{T_0\} \cup \{B_k : k \text{ odd} \geq 3\}$ is a disjoint partition of $\mathbb{N}$.*

Definition 3 (Indices)**.** *Order the odd numbers as $1, 3, 5, 7, \dots$. Assign the branch index $\mathrm{br}(odd) = (odd - 1)/2 \in \mathbb{Z}_{\geq 0}$, so that $\mathrm{br}(1) = 0$ and $\mathrm{br}(3) = 1$, $\mathrm{br}(5) = 2$, etc. Define the height $\mathrm{ht}(n) = \nu_2(n)$. Set*

$$\mathrm{Idx} : \mathbb{N} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}, \qquad \mathrm{Idx}(n) = \big(\mathrm{br}(\mathrm{odd}(n)), \mathrm{ht}(n)\big), \quad \mathrm{Idx}^{-1}(i, b) = (2i+1) \cdot 2^b.$$

Theorem 1 (Complete classification)**.** *The map $n \mapsto \mathrm{Idx}(n)$ is a bijection from $\mathbb{N}$ onto $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.*

Proof. Uniqueness of $\mathrm{odd}(n)$ and $\nu_2(n)$ is immediate; disjointness/exhaustiveness of rays follows. $\square$

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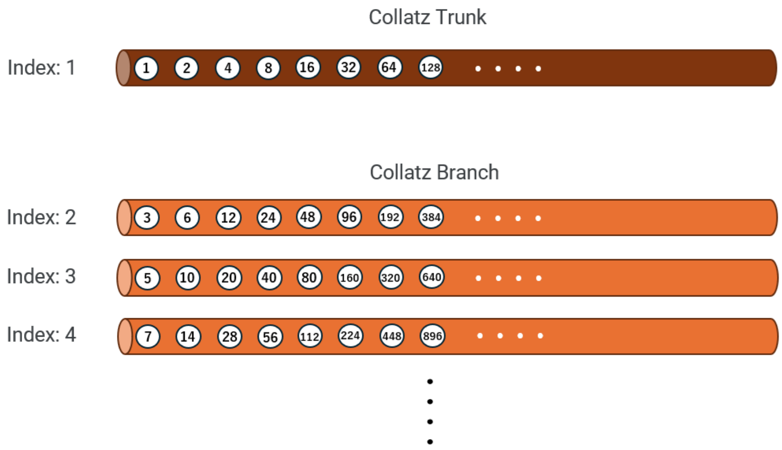

Figure 3. Trunk–branch indexing (schematic; reverse orientation in the inverse graph).

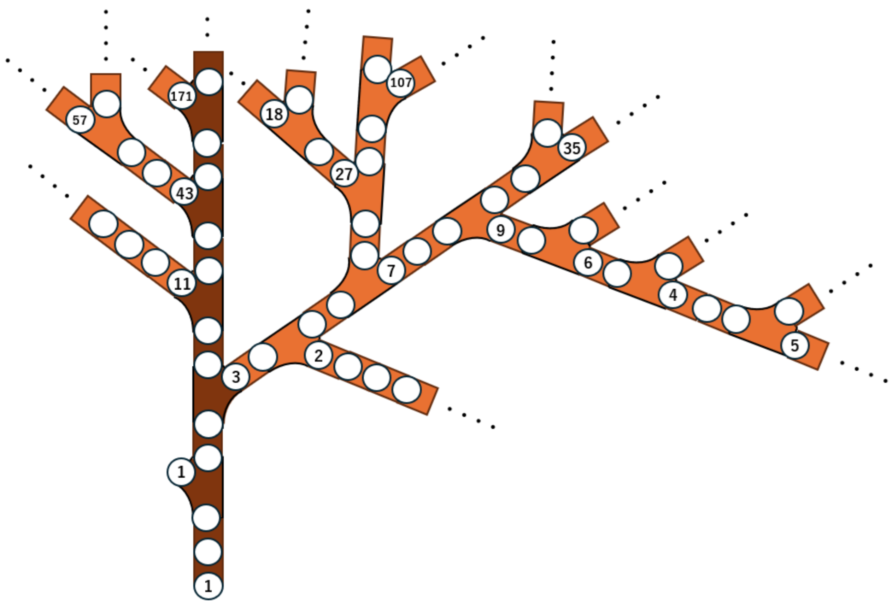

Figure 4. Indexed reverse tree (schematic; edges point to preimages).

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Table: Trunk–Branch Indexing (sample)

Odd k	Index	Next Index (rule)	parity-based trend	transition factor
1	1	–	–	–
3	2	3	increase	3/2
5	3	1	decrease	1/3
7	4	6	increase	3/2
9	5	4	decrease	4/5
11	6	9	increase	3/2
13	7	3	decrease	3/7
15	8	12	increase	3/2
17	9	7	decrease	7/9
19	10	15	increase	3/2

4. Affine Word Method: Absence of Nontrivial Finite Cycles

We encode even/odd steps as affine maps and use finite-word compositions $F_W(x) = a_W x + c_W$ to prove **absence of nontrivial finite cycles**.

Definition of the three-way map T.

Define $T : \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}$ by

$$T(n) = \begin{cases} \frac{3}{2}\,n, & n \text{ even}, \\ \frac{n-1}{2}, & n \text{ odd and } \nu_2(3n+1) = 1, \\ \frac{3n+1}{4}, & n \text{ odd and } \nu_2(3n+1) \geq 2. \end{cases}$$

Elementary steps.

$$E(x) = \frac{3}{2}x, \qquad O_1(x) = \frac{x-1}{2}, \qquad O_2(x) = \frac{3x+1}{4}.$$

(For odd n, choose O_1 if $\nu_2(3n+1) = 1$, and O_2 if $\nu_2(3n+1) \geq 2$.)

Words and composition.

For any finite word W over $\{E, O_1, O_2\}$, the composition is affine:

$$F_W(x) = a_W x + c_W.$$

Writing $m = \#E$, $o_1 = \#O_1$, $o_2 = \#O_2$, we have

$$a_W = \left(\frac{3}{2}\right)^m \left(\frac{1}{2}\right)^{o_1} \left(\frac{3}{4}\right)^{o_2} = \frac{3^{\,m+o_2}}{2^{\,m+o_1+2o_2}},$$

and the denominator of c_W divides $2^{o_1+2o_2}$ (each O_1 contributes $-1/2$, each O_2 contributes $+1/4$; E does not increase the power-of-two denominator).

Lemma 1 (No $a_W = 1$ for nonempty words). *If W is nonempty then $3^m = 2^{\,m+o_1+2o_2}$ cannot hold.*

Lemma 2 (Odd numerator for $1 - a_W$).

$$1 - a_W = \frac{2^{\,m+o_1+2o_2} - 3^{\,m+o_2}}{2^{\,m+o_1+2o_2}},$$

so the numerator is odd (even minus odd).

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Lemma 3 (If $m \geq 1$, a periodic solution cannot be integral)**.** *Since* $\text{den}(c_W) \mid 2^{o_1+2o_2}$*, we have* $c_W \cdot 2^{m+o_1+2o_2} = (\text{integer}) \cdot 2^m$*. By Lemma 2,* $1 - a_W = (\text{odd})/2^{m+o_1+2o_2}$*. Hence*

$$x = \frac{c_W}{1 - a_W} = \frac{(\text{integer}) \cdot 2^m}{\text{odd}},$$

and the odd denominator cannot cancel 2^m*. Thus* x *is not an integer.*

Lemma 4 (If $m = 0$, contraction; the only integer fixed point is 1)**.** *When* $m = 0$*,* $a_W = 2^{-(o_1+2o_2)} < 1$*, so any integer periodic point must be a fixed point. Solving* $x = O_1(x)$ *and* $x = O_2(x)$ *yields* $x = 1$ *as the only integer fixed point.*

Theorem 2 (Loop-freeness for T)**.** *Under the three-way rule above, the map* T *admits no finite cycle other than the trivial* 1*-cycle.*

Proof. By Lemma 1, consider $x = c_W/(1 - a_W)$. If $m \geq 1$, Lemma 3 rules out integer $x > 1$. If $m = 0$, Lemma 4 leaves only $x = 1$ as a fixed point. □

Interpretation. If an even step E appears at least once, a factor 2^m remains in the numerator of $x = c_W/(1 - a_W)$ and cannot be canceled by the *odd* denominator, so no integer fixed point arises. If E never appears, the composition is a contraction and the only integer fixed point is 1. This algebraic loop-elimination aligns with the inverse-generation intuition.

5. Bridge to the Accelerated Collatz Map

Let the standard accelerated map on odd integers be

$$A(n) = \frac{3n+1}{2^{\nu_2(3n+1)}}.$$

We show: *If* A *had a nontrivial finite cycle, then* T *would have one as well.*

Coefficient matching.

If one tour of the A-cycle multiplies by $3^e/2^E$, take on the T-side a word with $m = e$ and $o_1 + 2o_2 = E - m$ so that $a_W = 3^e/2^E$.

Constant congruence by adjacent swaps.

For adjacent swaps $EO \leftrightarrow OE$ ($O \in \{O_1, O_2\}$), the normalized constant changes by $\Delta C \equiv \pm u \cdot 3^t \pmod D$ where $D = 2^E - 3^e$ and u is an odd unit modulo D. Using the order of 3 modulo prime powers of D and CRT, one can realize any residue class mod D.

Realizability (intermediate consistency).

Parity and ν_2 constraints at each step reduce to a finite system of linear congruences $\alpha_j x + \beta_j \equiv r_j \pmod{2^{m_j}}$ for the initial value x. The same swap operations adjust β_j by controlled powers of two, allowing a simultaneous solution by CRT.

Theorem 3 (Bridge, complete version)**.** *If the accelerated map* A *has a nontrivial finite cycle, then the map* T *has a nontrivial finite cycle.*

Corollary 1 (No nontrivial finite cycle for A)**.** *Combining Theorem 2 with Theorem 3, the accelerated Collatz map* A *has no nontrivial finite cycle.*

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6. Reduction to Convergence

A functional digraph with outdegree 1 decomposes into a directed cycle with an in-tree (basin). By Corollary 1, the only allowed cycle is 1. Thus global convergence splits into two independent pillars: **(i) cycle-freeness** (proved here) and **(ii) coverage**.

Theorem 4 (Reduction to convergence)**.** *For the accelerated map A, the following are equivalent:*

1. *Every positive integer reaches* 1 *in finitely many steps (global convergence).*
2. *The inverse generation tree rooted at* 1 *covers all positive integers (reachability/coverage).*

Proof. Each connected component of a functional digraph is a directed cycle with an in-tree. Since only the 1-cycle is permitted (Corollary 1), global convergence holds iff every node is in the basin of 1, i.e. iff the inverse tree covers $\mathbb{N}$. □

7. Related Work

The affine-composition viewpoint with coefficient $3^m/2^E$ is classical in studies of cycles and their lengths *(in our three-way encoding,* $a_W = 3^{m+o_2}/2^{m+o_1+2o_2}$*)*. Our novelty is to integrate (i) the trunk–branch indexing and inverse-tree structure, with (ii) a complete loop-elimination for the three-way map T, and (iii) a bridge transporting hypothetical cycles of A into T, thereby isolating cycle-freeness as a stand-alone pillar and reducing full convergence to coverage.

Appendix: Python Code for Reverse Collatz Tree Visualization

This script visualizes structure up to a finite cutoff and is *not* a proof of coverage.

The following script *visualizes* the reverse graph from 1 up to a finite cutoff `limit`.

```python
import networkx as nx
import matplotlib.pyplot as plt

def generate_tree(limit=250):
    G = nx.DiGraph()
    G.add_node(1, level=0)
    queue = [(1, 0)]  # (node, level)
    visited = set([1])

    while queue:
        n, level = queue.pop(0)

        # Rule 1: multiply by 2 (always a preimage)
        child1 = 2 * n
        if child1 <= limit and child1 not in visited:
            G.add_edge(n, child1)  # reverse edge: parent -> preimage
            G.nodes[child1]['level'] = level + 1
            queue.append((child1, level + 1))
            visited.add(child1)

        # Rule 2: inverse of 3n+1; require odd preimage
        if n % 2 == 0 and (n - 1) % 3 == 0:
            child2 = (n - 1) // 3
            if child2 % 2 == 1 and child2 > 0 and child2 not in visited:
                G.add_edge(n, child2)
                G.nodes[child2]['level'] = level + 1
                queue.append((child2, level + 1))
                visited.add(child2)

```

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```
    return G

def draw_tree(G):
    levels = nx.get_node_attributes(G, 'level')
    pos = {}
    level_widths = {}

    for node, level in levels.items():
        level_widths.setdefault(level, []).append(node)

    for level, nodes in level_widths.items():
        xs = range(-len(nodes) + 1, len(nodes), 2)
        for x, node in zip(xs, nodes):
            pos[node] = (x, level)

    plt.figure(figsize=(12, 10))
    nx.draw(G, pos, with_labels=True, node_size=700,
            node_color='peru', edge_color='black',
            font_size=10, font_weight='bold', alpha=0.85)
    plt.show()

if __name__ == "__main__":
    G = generate_tree(limit=250)
    draw_tree(G)
```

Listing 1: Reverse Collatz Tree (visualization only)

Appendix: Python-Generated Tree Visualizations (Illustrative Only)

Note. These are finite-cutoff visualizations generated programmatically. They aid intuition but are *not* proofs of coverage.

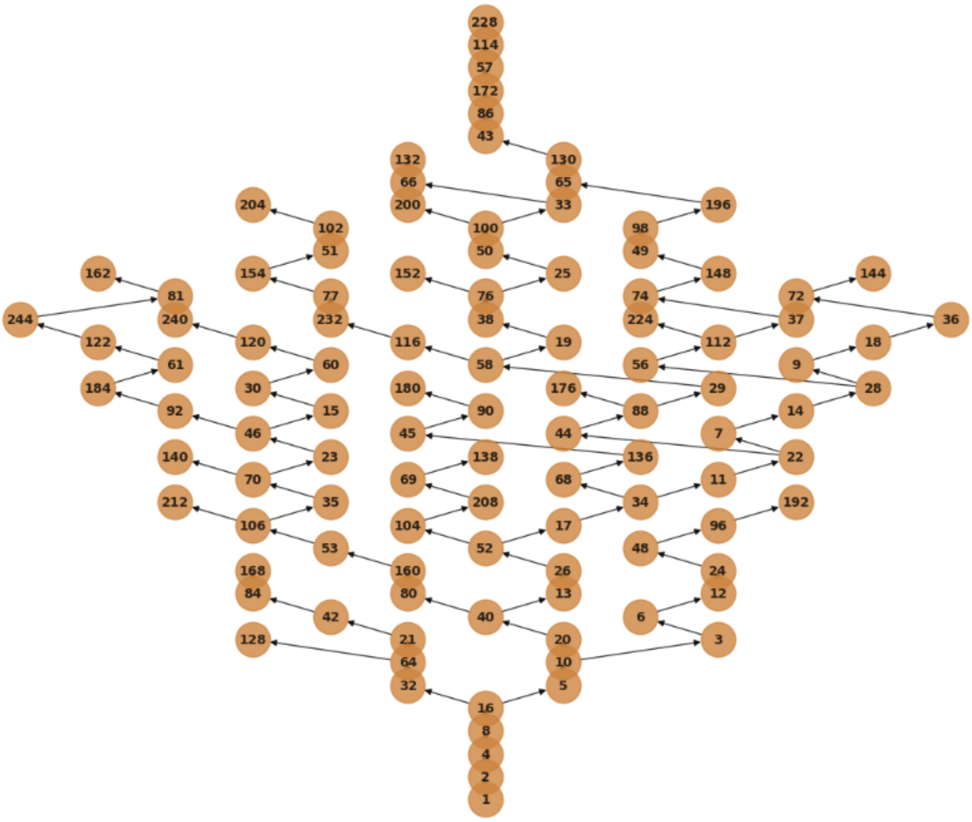

Figure A5. Reverse Collatz tree generated programmatically (limit = 250) (finite cutoff; visualization, not a proof)

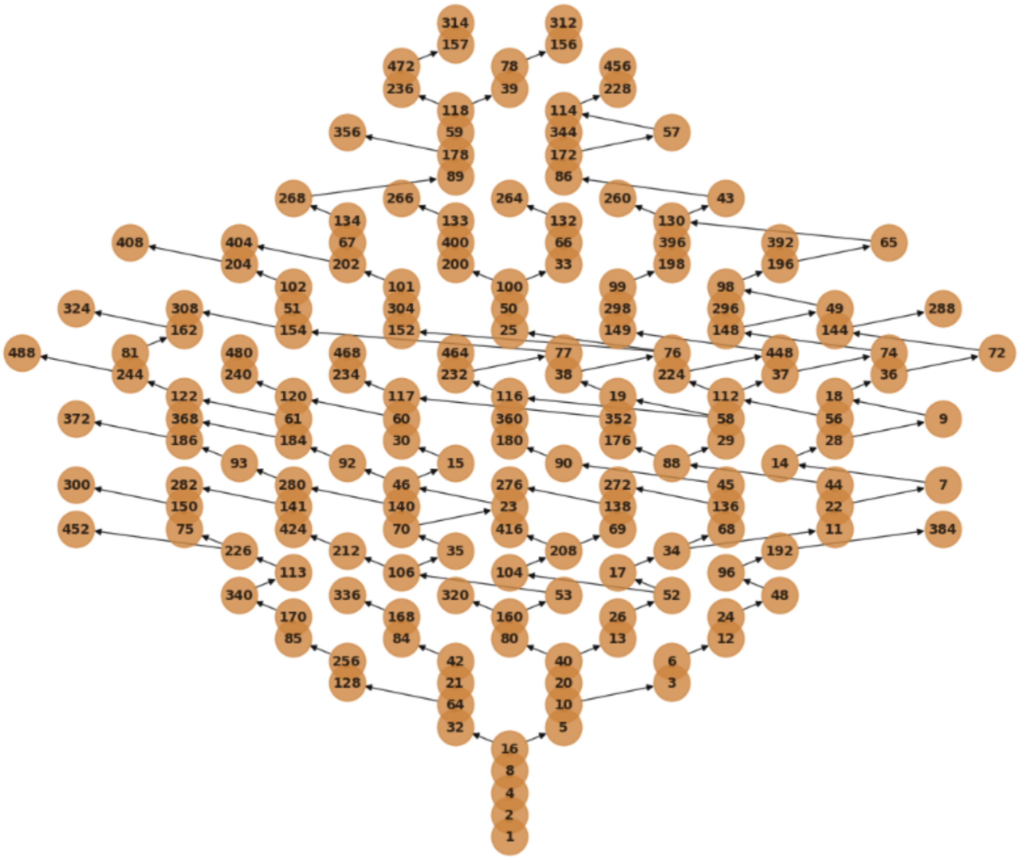

Figure A6. Reverse Collatz tree generated programmatically (limit = 500) (finite cutoff; visualization, not a proof)

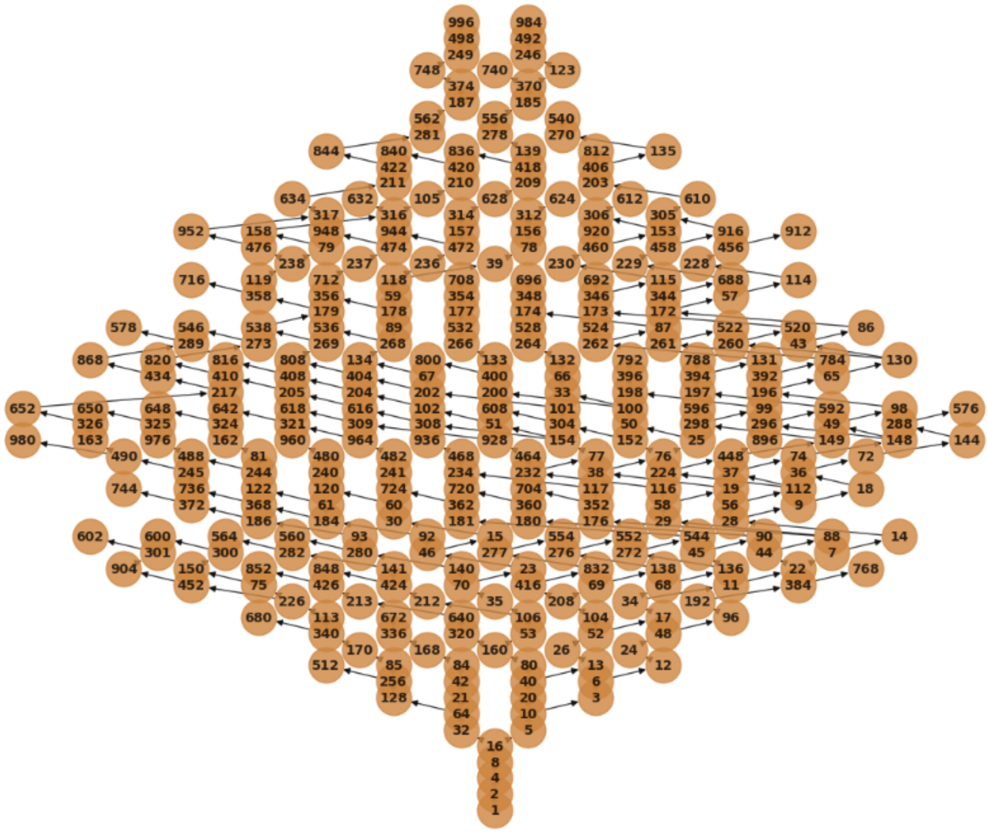

Figure A7. Reverse Collatz tree generated programmatically (limit = 1000) (finite cutoff; visualization, not a proof)

Data Availability Statement: Figures can be regenerated using the Python in the Appendix (for visualization; not a proof of coverage).

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4. Petro Kosobutskyy, *The Collatz problem* $(a \cdot q \pm 1, a = 1, 3, 5, \ldots)$ *from the point of view of transformations of Jacobsthal numbers*, arXiv preprint, arXiv:2306.14635, 2023.