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Article

One Class of H_∞ Cheap Control Problems: Asymptotic Solution

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Abstract

An infinite-horizon H_∞ linear-quadratic control problem is considered. This problem has the following features: (i) the quadratic form of the control in the integrand of the cost functional is with a positive small multiplier (small parameter), meaning that the control cost is much smaller than the state cost; (ii) the current cost of the fast state variable in the cost functional is a positive semi-definite (but non-zero) quadratic form. These features require developing a significantly novel approach to asymptotic solution of the matrix Riccati algebraic equation associated with the considered H_∞ problem by solvability conditions. Using this solution, an asymptotic analysis of the H_∞ problem is carried out. This analysis yields parameter-free solvability conditions for this problem and a simplified controller solving this problem. An illustrative example is presented.

Keywords: infinite-horizon H_∞ control problem; cheap control problem; perturbed matrix Riccati algebraic equation; asymptotic solution; simplified controller

MSC: 93C05; 93C15; 93C73; 93D15; 93D20; 15A24; 65F45

1. Introduction

Uncertain dynamics systems are widely studied in the literature (see, e.g., [1–14] and references therein). The two most considered types of uncertainties are the following: (I) uncertainties which belong to a specified bounded set of an Euclidean space; (II) uncertainties which are quadratically integrable in a given interval (finite or infinite). For controlled systems with the second type of uncertainties (additive disturbances), the H_∞ control problem is frequently considered (see, e.g., [1,3–5,8,13] and references therein).

In the present paper, we consider an infinite-horizon H_∞ linear-quadratic control problem. The features of this problem are the following: (i) the current quadratic cost of the control in the cost functional is with a small multiplier ε^2 ($\varepsilon > 0$ is a small parameter), meaning that the control cost is much smaller than the state cost; (ii) the current quadratic cost of the fast state variable in the cost functional is a positive semi-definite (but non-zero) quadratic form. The first feature means that the considered H_∞ problem is a cheap control problem.

Infinite-horizon H_∞ cheap control problems and closed to these problems infinite-horizon zero-sum cheap control differential games have been studied in a number of works in the open literature. In these works, the following two types of the current quadratic cost of the fast state variable in the cost functional are considered: (a) the cost is a positive definite quadratic form (see, e.g., [13,15–19] and references therein); (b) the cost is zero (see, [16]). As it is aforementioned, in the present paper, we study an infinite-horizon H_∞ cheap control problem in the case where the current quadratic cost of the fast state variable in the cost functional is a positive semi-definite (but non-zero) quadratic form. To the best of our knowledge, this case is completely novel.

The motivation of this *purely theoretical paper* is three-fold. The first motivation is to study asymptotically a considerably new class of H_∞ cheap control problems. The second motivation is

to carry out an asymptotic analysis of a new type of ε -dependent Riccati matrix algebraic equation associated with the original H_∞ cheap control problem by solvability conditions. This analysis requires developing an essentially novel approach to the derivation of the asymptotic solution of the Riccati equation with respect to ε . The third motivation is to establish ε -free solvability conditions for the considered H_∞ cheap control problem and to design a simplified controller, solving this problem.

The paper is organized as follows. In the next section, the problem is rigorously formulated. Objectives of the paper are stated. In Section 3, ε -dependent solvability conditions for the formulated infinite-horizon H_∞ cheap control problem are presented. Asymptotic solution of the ε -dependent Riccati matrix algebraic equation, appearing in these solvability conditions, is obtained in Section 4. Based on this asymptotic solution, ε -free conditions of solvability of the H_∞ cheap control problem, valid for all sufficiently small $\varepsilon > 0$, are established. In Section 5, the simplified controller, solving this H_∞ cheap control problem for all sufficiently small $\varepsilon > 0$, is designed. An academic illustrative example is presented in Section 6. Conclusions are placed in Section 7. Appendix A is devoted to the proof of the theorem justifying the validity of the simplified controller.

The following main notations are applied in the paper.

1. $\mathbb{R}^n$ denotes the n -dimensional real Euclidean space;
2. $\|\cdot\|$ denotes the Euclidean norm either of a vector ($\|z\|$) or of a matrix ($\|A\|$);
3. $L_2[0, +\infty; \mathbb{R}^n]$ denotes the linear space of the functions $f(t) : [0, +\infty) \rightarrow \mathbb{R}^n$ square integrable in the interval $[0, +\infty)$;
4. $\|\cdot\|_{L_2[0, +\infty)}$ denotes the norm in the space $L_2[0, +\infty; \mathbb{R}^n]$;
5. the superscript " T " denotes the transposition either of a vector (z^T) or of a matrix (A^T);
6. I_n denotes the identity matrix of dimension n ;
7. $\text{col}(x_1, x_2, \dots, x_k)$, where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, ..., $x_k \in \mathbb{R}^{n_k}$, denotes the column block-vector of the dimension $n_1 + n_2 + \dots + n_k$ with the upper block x_1 , the next block x_2 and so on, and the lower block x_k ;
8. $\text{Re}(\mu)$ denotes the real part of a complex number μ .

2. Problem Formulation

Consider the system

$$dx(t)/dt = A_1x(t) + A_2y(t) + F_1w(t), \quad t \geq 0, \quad x(0) = 0, \quad (1)$$

$$dy(t)/dt = A_3x(t) + A_4y(t) + u(t) + F_2w(t), \quad t \geq 0, \quad y(0) = 0, \quad (2)$$

$$v(t) = \text{col}(C_1x(t), C_2y(t), \varepsilon u(t)), \quad t \geq 0, \quad (3)$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$, ($m > 1$) are the state vectors; $u(t) \in \mathbb{R}^m$ is the control; $w(t) \in \mathbb{R}^q$ is a disturbance; $v(t) \in \mathbb{R}^{p_1+p_2+m}$ is an output; $\varepsilon > 0$ is a small parameter ($\varepsilon \ll 1$); A_i , ($i = 1, \dots, 4$) and F_j , C_j , ($j = 1, 2$) are given constant matrices of corresponding dimensions.

Assuming the fulfillment of the inclusions $u(t) \in L_2[0, +\infty; \mathbb{R}^m]$ and $w(t) \in L_2[0, +\infty; \mathbb{R}^q]$, we consider the cost functional

$$J(u, w) = \|v(t)\|_{L_2[0, +\infty)}^2 - \gamma^2 \|w(t)\|_{L_2[0, +\infty)}^2, \quad (4)$$

where $\gamma > 0$ is a given constant called a performance level.

Using the notations

$$D_j = C_j^T C_j, \quad j = 1, 2, \quad (5)$$

we can rewrite the functional (4) in the following equivalent form:

$$J(u, w) = \int_0^{+\infty} [x^T(t)D_1x(t) + y^T(t)D_2y(t) + \varepsilon^2 u^T(t)u(t) - \gamma^2 w^T(t)w(t)] dt. \quad (6)$$

Let us consider the block vector

$$z \triangleq \text{col}(x, y) \in \mathbb{R}^{n+m}, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m, \quad (7)$$

and let $\varepsilon > 0$ be a given number.

Definition 1. The infinite horizon H_∞ control problem with the dynamics (1)-(2) and the cost functional (6) is to design a controller $u_\varepsilon^*(z)$ such that, for any $w(\cdot) \in L_2[0, +\infty; \mathbb{R}^q]$: (i) there exists the unique locally absolutely continuous solution $z_\varepsilon^*(t; w(\cdot)) = \text{col}(x_\varepsilon^*(t; w(\cdot)), y_\varepsilon^*(t; w(\cdot)))$, $(x_\varepsilon^*(t; w(\cdot)) \in \mathbb{R}^n, y_\varepsilon^*(t; w(\cdot)) \in \mathbb{R}^m)$ of the initial-value problem (1)-(2) with $u(t) = u_\varepsilon^*(z)$ in the entire interval $[0, +\infty)$; (ii) the following inequality is valid

$$J(u_\varepsilon^*(z_\varepsilon^*(t; w(\cdot))), w(t)) \leq 0. \quad (8)$$

Remark 1. Due to the smallness of the parameter $\varepsilon > 0$, the H_∞ problem, formulated in Definition 1, is a cheap control problem, i.e., the problem where a control cost in the cost functional is much smaller than a state cost. In what follows, we call the H_∞ problem with the dynamics (1)-(2) and the cost functional (6) the H_∞ Cheap Control Problem (HCCP).

Remark 2. Due to the equation (5), the matrices D_1 and D_2 are symmetric and at least positive semi-definite. In what follows, for the sake of simplicity (but without loss of the generality), we assume that the matrix D_2 has the form

$$D_2 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m), \quad \lambda_k \geq 0, \quad k = 1, \dots, m. \quad (9)$$

Remark 3. Using the non-singular control transformation $\hat{u}(t) = \varepsilon u(t)$, ($\hat{u}(t)$ is a new control), we convert the HCCP into the equivalent H_∞ control problem, which consists of the differential equations (1) and

$$\varepsilon \frac{dy(t)}{dt} = \varepsilon [A_3 x(t) + A_4 y(t) + F_2 w(t)] + \hat{u}(t), \quad y(0) = y_0, \quad (10)$$

and of the cost functional

$$\hat{J}(\hat{u}, w) = \int_0^{+\infty} [x^T(t) D_1 x(t) + y^T(t) D_2 y(t) + \hat{u}^T(t) \hat{u}(t) - \gamma^2 w^T(t) w(t)] dt. \quad (11)$$

The dynamic system of the new H_∞ control problem is singularly perturbed. In this dynamic system $x(t)$ is a slow state variable and $y(t)$ is a fast state variable (see, e.g., [20]). Therefore, we call the state variables $x(t)$ and $y(t)$ of the HCCP a slow state variable and a fast state variable, respectively.

Objectives of this paper are:

- (I) to analyze asymptotically with respect to $\varepsilon > 0$ the HCCP using the method of its solution based on the game-theoretic matrix Riccati algebraic equation;
- (II) to derive an asymptotic solution to this Riccati equation;
- (III) to obtain ε -independent conditions for the existence of a controller $u_\varepsilon^*(z)$ solving the HCCP for all sufficiently small $\varepsilon > 0$;
- (IV) to design an asymptotically simplified controller $u_{0,\varepsilon}(z)$ solving the HCCP for all sufficiently small $\varepsilon > 0$.

3. Solvability Conditions of the HCCP

First, we introduce the following block-form matrices:

$$A \triangleq \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, B \triangleq \begin{bmatrix} 0 \\ I_m \end{bmatrix}, F \triangleq \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, D \triangleq \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad (12)$$

where A and D are of the dimension $(n+m) \times (n+m)$; B is of the dimension $(n+m) \times m$; F is of the dimension $(n+m) \times q$.

Based on these matrices, let us construct the following matrix Riccati algebraic equation with respect to $(n+m) \times (n+m)$ -matrix P :

$$PA + A^T P - P[S_u(\varepsilon) - S_w]P + D = 0, \quad (13)$$

where

$$S_u(\varepsilon) = \varepsilon^{-2} B B^T = \varepsilon^{-2} \begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix}, \quad S_w = \gamma^{-2} F F^T = \gamma^{-2} \begin{bmatrix} F_1 F_1^T & F_1 F_2^T \\ F_2 F_1^T & F_2 F_2^T \end{bmatrix}. \quad (14)$$

We assume:

A1. For a given $\varepsilon > 0$, the equation (13) has a real solution $P = P^*(\varepsilon)$ such that:

- (a) $(P^*(\varepsilon))^T = P^*(\varepsilon)$;
- (b) all roots $\mu = \mu_l(\varepsilon)$, ($l = 1, \dots, n+m$) of the polynomial equation

$$\det(\mu I_{n+m} - A + S_u(\varepsilon)P^*(\varepsilon)) = 0 \quad (15)$$

lie strictly inside the left-hand half-plane.

Lemma 1. Let the assumption A1 be valid. Then:

- (i) $P^*(\varepsilon)$ is a positive semi-definite matrix;
- (ii) the controller

$$u_\varepsilon^*(z) = -\frac{1}{\varepsilon^2} B^T P^*(\varepsilon) z \quad (16)$$

solves the HCCP.

Proof. We start with the item (i).

Consider the matrix

$$Q^*(\varepsilon) = D + P^*(\varepsilon)S_u(\varepsilon)P^*(\varepsilon). \quad (17)$$

Since the matrices D and $S_u(\varepsilon)$ are symmetric and positive semi-definite, and the matrix $P^*(\varepsilon)$ is real and symmetric, then the matrix $Q^*(\varepsilon)$ is symmetric and positive semi-definite.

Using the equation (17) and that $P^*(\varepsilon)$ is a solution of the equation (13), we obtain the following equality:

$$P^*(\varepsilon)(A - S_u(\varepsilon)P^*(\varepsilon)) + (A - S_u(\varepsilon)P^*(\varepsilon))^T P^*(\varepsilon) = -(Q^*(\varepsilon) + P^*(\varepsilon)S_w P^*(\varepsilon)). \quad (18)$$

Let us consider this equality as a matrix Lyapunov algebraic equation (see, e.g., [21]) for $P^*(\varepsilon)$ with the matrix-valued coefficients $(A - S_u(\varepsilon)P^*(\varepsilon))^T$ and $(A - S_u(\varepsilon)P^*(\varepsilon))$, and the matrix-valued non-homogeneity $-(Q^*(\varepsilon) + P^*(\varepsilon)S_w P^*(\varepsilon))$. Thus, using the item (b) of the assumption A1 and the results of [21], we can rewrite the equality (18) in the equivalent form

$$P^*(\varepsilon) = \int_0^{+\infty} \exp\left((A - S_u(\varepsilon)P^*(\varepsilon))^T \chi\right) (Q^*(\varepsilon) + P^*(\varepsilon)S_w P^*(\varepsilon)) \exp\left((A - S_u(\varepsilon)P^*(\varepsilon))\chi\right) d\chi. \quad (19)$$

Since the matrices $Q^*(\varepsilon)$ and $P^*(\varepsilon)S_wP^*(\varepsilon)$ are positive semi-definite, then the matrix in the right-hand side of the equality (19) is positive semi-definite. Hence, $P^*(\varepsilon)$ is positive semi-definite, which completes the proof of the item (i).

Proceed to the item (ii).

Consider the Lyapunov function

$$V[z] = z^T P^*(\varepsilon) z, \quad z \in \mathbb{R}^{n+m}. \quad (20)$$

Let

$$V^*(t; w(\cdot)) \triangleq V[z_\varepsilon^*(t; w(\cdot))], \quad (21)$$

where, for any given $w(\cdot) \in L_2[0, +\infty; \mathbb{R}^q]$, $z_\varepsilon^*(t; w(\cdot))$ is the solution of the initial-value problem (1)-(2) generated by the controller $u(t) = u_\varepsilon^*(z)$ (see the equation (16)). Due to the assumption A1 and the form of $u_\varepsilon^*(z)$, the aforementioned solution of the problem (1)-(2) exists, is unique and locally absolutely continuous in the entire interval $[0, +\infty)$.

Differentiating $V^*(t; w(\cdot))$ with respect to $t \in [0, +\infty)$ and using the assumption A1, we obtain the following expression:

$$\frac{dV^*(t; w(\cdot))}{dt} = 2 \left(\frac{dz_\varepsilon^*(t; w(\cdot))}{dt} \right)^T P^*(\varepsilon) z_\varepsilon^*(t; w(\cdot)), \quad t \geq 0. \quad (22)$$

Using the equations (1)-(2) with $u(t) = u_\varepsilon^*(z)$, as well as the equations (7),(12),(13),(14) and that $P^*(\varepsilon)$ is a solution of (13), we can convert (22) after a routine algebra to the following expression:

$$\begin{aligned} \frac{dV^*(t; w(\cdot))}{dt} = & - \left(z_\varepsilon^*(t; w(\cdot)) \right)^T (D + P^*(\varepsilon)S_u(\varepsilon)P^*(\varepsilon) + P^*(\varepsilon)S_wP^*(\varepsilon)) z_\varepsilon^*(t; w(\cdot)) \\ & + 2w^T(t)F^T P^*(\varepsilon) z_\varepsilon^*(t; w(\cdot)), \quad t \geq 0. \end{aligned} \quad (23)$$

Finally, using the equation (16) and the notation

$$w_\varepsilon^*(t) \triangleq \gamma^{-2} F^T P^*(\varepsilon) z_\varepsilon^*(t; w(\cdot)),$$

the expression (23) can be rewritten as:

$$\begin{aligned} \frac{dV^*(t; w(\cdot))}{dt} = & - \left(z_\varepsilon^*(t; w(\cdot)) \right)^T D z_\varepsilon^*(t; w(\cdot)) - \varepsilon^2 (u_\varepsilon^*(t))^T u_\varepsilon^*(t) + \gamma^2 w^T(t) w(t) \\ & - \gamma^2 (w(t) - w_\varepsilon^*(t))^T (w(t) - w_\varepsilon^*(t)), \quad t \geq 0, \end{aligned} \quad (24)$$

where $u_\varepsilon^*(t)$ is the time realization of the controller $u_\varepsilon^*(z)$ along $z = z_\varepsilon^*(t; w(\cdot))$.

Let $z_0 = \text{col}(x_0, y_0)$, ($x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$) be any given vector and t_0 be any non-negative number. Let $z_{\varepsilon,0}(t) = \text{col}(x_{\varepsilon,0}(t), y_{\varepsilon,0}(t))$, $t \geq t_0$ be the solution of the system of the differential equations in (1)-(2), generated by the controller $u(t) = u_\varepsilon^*(z)$ (see the equation (16)), the disturbance $w(t) \equiv 0$ and the initial condition $z(t_0) = z_0$. Due to the item (b) of the assumption A1,

$$\lim_{t \rightarrow +\infty} z_{\varepsilon,0}(t) = 0. \quad (25)$$

Let

$$V_0(t) \triangleq V[z_{\varepsilon,0}(t)], \quad t \geq t_0. \quad (26)$$

Then, quite similarly to the expression (24), we obtain

$$\frac{dV_0(t)}{dt} = -z_{\varepsilon,0}^T(t)Dz_{\varepsilon,0}(t) - \gamma^2 w_{\varepsilon,0}^T(t)w_{\varepsilon,0}(t) - \varepsilon^2 (u_{\varepsilon,0}^*(t))^T u_{\varepsilon,0}^*(t), \quad t \geq t_0, \quad (27)$$

where

$$w_{\varepsilon,0}(t) \triangleq \gamma^{-2}FP^*(\varepsilon)z_{\varepsilon,0}(t), \quad u_{\varepsilon,0}^*(t) \triangleq u_{\varepsilon}^*[z_{\varepsilon,0}(t)], \quad t \geq t_0. \quad (28)$$

The expression (27) implies the inequality

$$-\frac{dV_0(t)}{dt} \geq 0, \quad t \geq t_0. \quad (29)$$

Integrating this inequality from $t = t_0$ to $+\infty$ and using the equations (20),(25),(26) and that $z_{\varepsilon,0}(t_0) = z_0$, we obtain

$$z_0^T P^*(\varepsilon) z_0 \geq 0. \quad (30)$$

Remember that in this inequality z_0 is any $(n+m)$ -dimensional vector. This observation directly yields the validity of the item (i) of the lemma. Thus, the chain of the equations (20)-(30) represents an alternative proof of this item.

Based on the inequality (30), we continue to treat the expression (24). Since $\gamma^2(w(t) - w_{\varepsilon}^*(t))^T (w(t) - w_{\varepsilon}^*(t)) \geq 0$, this expression implies the following inequality for all $t \geq 0$:

$$\left(z_{\varepsilon}^*(t; w(\cdot))\right)^T Dz_{\varepsilon}^*(t; w(\cdot)) + \varepsilon^2 (u_{\varepsilon}^*(t))^T u_{\varepsilon}^*(t) - \gamma^2 w^T(t)w(t) \leq -\frac{dV^*(t; w(\cdot))}{dt}. \quad (31)$$

Integrating this inequality from $t = 0$ to any $t \geq 0$, using the equation (20) and taking into account that $z_{\varepsilon}^*(0; w(\cdot)) = 0$, we obtain

$$\begin{aligned} \int_0^t \left[\left(z_{\varepsilon}^*(\sigma; w(\cdot))\right)^T Dz_{\varepsilon}^*(\sigma; w(\cdot)) + \varepsilon^2 (u_{\varepsilon}^*(\sigma))^T u_{\varepsilon}^*(\sigma) - \gamma^2 w^T(\sigma)w(\sigma) \right] d\sigma \\ \leq -\left(z_{\varepsilon}^*(t; w(\cdot))\right)^T P^*(\varepsilon) z_{\varepsilon}^*(t; w(\cdot)), \quad t \geq 0. \end{aligned} \quad (32)$$

Since $P^*(\varepsilon)$ is a positive semi-definite matrix, the inequality (32) yields

$$\int_0^t \left[\left(z_{\varepsilon}^*(\sigma; w(\cdot))\right)^T Dz_{\varepsilon}^*(\sigma; w(\cdot)) + \varepsilon^2 (u_{\varepsilon}^*(\sigma))^T u_{\varepsilon}^*(\sigma) - \gamma^2 w^T(\sigma)w(\sigma) \right] d\sigma \leq 0, \quad t \geq 0, \quad (33)$$

which, along with the inclusion $w(t) \in L_2[0, +\infty; \mathbb{R}^q]$, implies

$$\int_0^t \left[\left(z_{\varepsilon}^*(\sigma; w(\cdot))\right)^T Dz_{\varepsilon}^*(\sigma; w(\cdot)) + \varepsilon^2 (u_{\varepsilon}^*(\sigma))^T u_{\varepsilon}^*(\sigma) \right] d\sigma \leq \gamma^2 \|w(\cdot)\|_{L_2(0, +\infty)}^2, \quad t \geq 0. \quad (34)$$

Since the matrix D is symmetric and positive semi-definite, the integral in the left-hand side of the inequality (34) is a non-decreasing function of $t \geq 0$ and this function is bounded from above. Therefore, this integral has a finite limit for $t \rightarrow +\infty$. This observation, along with the equations (6),(12) and the inequality (34), directly yields the inequality (8), where $u_{\varepsilon}^*[\cdot]$ is given by the equation (16). Thus, the item (ii) of the lemma is proven. This completes the proof of the lemma $\square$

Substituting $u(t) = u_{\varepsilon}^*(z)$ into the system (1)-(2) and into the cost functional (6), and using the equations (7) and (12), we obtain after some rearrangement the following system and cost functional:

$$dz(t)/dt = \mathcal{A}(\varepsilon)z(t) + Fw(t), \quad t \geq 0, \quad z(0) = 0, \quad (35)$$

$$J^*(w) = J(u_{\varepsilon}^*, w) = \int_0^{+\infty} [z^T(t)\mathcal{D}(\varepsilon)z(t) - \gamma^2 w^T(t)w(t)] dt, \quad (36)$$

where

$$A(\varepsilon) = A - S_u(\varepsilon)P^*(\varepsilon), \quad \mathcal{D}(\varepsilon) = D + P^*(\varepsilon)S_u(\varepsilon)P^*(\varepsilon). \quad (37)$$

For the differential system (35), we consider the output equation

$$\zeta(t) = (\mathcal{D}(\varepsilon))^{1/2}z(t), \quad t \geq 0, \quad (38)$$

where $(\mathcal{D}(\varepsilon))^{1/2}$ is the unique symmetric positive semi-definite square root of the symmetric positive semi-definite matrix $\mathcal{D}(\varepsilon)$.

Remark 4. By virtue of Lemma 1 and the results of [3], the assumption A1 guarantees that the H_∞ norm $\gamma_H(\varepsilon) > 0$ of the system (35),(38) satisfies the inequality

$$\gamma_H(\varepsilon) < \gamma. \quad (39)$$

Moreover, for all $w(\cdot) \in L_2[0, +\infty; \mathbb{R}^q]$ and any number $\tilde{\gamma} \in (\gamma_H(\varepsilon), \gamma]$, the following inequality is valid:

$$\int_0^{+\infty} \left[\left(z_\varepsilon^*(t; w(\cdot)) \right)^T \mathcal{D}(\varepsilon) z_\varepsilon^*(t; w(\cdot)) - \tilde{\gamma}^2 w^T(t) w(t) \right] dt \leq 0. \quad (40)$$

Remark 5. It should be noted the following. Lemma 1 is similar to the results of [13] (Theorem 3.4). However, this theorem is proven subject to the more restrictive assumptions than Lemma 1, while yields the stronger result than Lemma 1. Namely, in contrast to Lemma 1, Theorem 3.4 from [13] yields the positive definiteness of the solution to the matrix Riccati algebraic equation.

4. Asymptotic Solution of the Equation (13)

We derive the asymptotic solutions to the equation (13) (and, then, to the HCCP) subject to the condition

$$\begin{aligned} \lambda_p &> 0, \quad p = 1, \dots, m_1, \quad 1 \leq m_1 < m, \\ \lambda_r &= 0, \quad r = m_1 + 1, \dots, m, \end{aligned} \quad (41)$$

where λ_k , ($k = 1, 2, \dots, m$) are the entries of the matrix D_2 (see the equation (9)).

4.1. Transformation of the Equation (13)

Due to the form of the matrix $S_u(\varepsilon)$, the left-hand side of the equation (13) has the singularity for $\varepsilon = 0$. In order to remove this singularity, we are going to transform this equation in a proper way. Namely, assuming the symmetry of the solution to (13) and using the form of the matrices D_2 and $S_u(\varepsilon)$ (see the equations (9),(41) and (14)), we look for this solution in the following block form:

$$P = P(\varepsilon) = \begin{bmatrix} P_1(\varepsilon) & \varepsilon P_2(\varepsilon) & \varepsilon P_3(\varepsilon) \\ \varepsilon P_2^T(\varepsilon) & \varepsilon P_4(\varepsilon) & \varepsilon^2 P_5(\varepsilon) \\ \varepsilon P_3^T(\varepsilon) & \varepsilon^2 P_5^T(\varepsilon) & \varepsilon^2 P_6(\varepsilon) \end{bmatrix}, \quad (42)$$

where the matrices $P_1(\varepsilon)$, $P_4(\varepsilon)$ and $P_6(\varepsilon)$ are of the dimensions $n \times n$, $m_1 \times m_1$ and $(m - m_1) \times (m - m_1)$, respectively, and these matrices are symmetric.

Let us partition the matrices A and S_w into blocks as follows:

$$A = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ \bar{A}_4 & \bar{A}_5 & \bar{A}_6 \\ \bar{A}_7 & \bar{A}_8 & \bar{A}_9 \end{bmatrix}, \quad (43)$$

$$S_w = \begin{bmatrix} S_{w,1} & S_{w,2} & S_{w,3} \\ S_{w,2}^T & S_{w,4} & S_{w,5} \\ S_{w,3}^T & S_{w,5}^T & S_{w,6} \end{bmatrix}, \quad (44)$$

where

$$\bar{A}_1 = A_1, \quad [\bar{A}_2, \bar{A}_3] = A_2, \quad \begin{bmatrix} \bar{A}_4 \\ \bar{A}_7 \end{bmatrix} = A_3, \quad \begin{bmatrix} \bar{A}_5 & \bar{A}_6 \\ \bar{A}_8 & \bar{A}_9 \end{bmatrix} = A_4; \quad (45)$$

the blocks $\bar{A}_1, \bar{A}_5, \bar{A}_9$ and $S_{w,1}, S_{w,4}, S_{w,6}$ are of the same dimensions as the corresponding blocks in (42); $S_{w,j}^T = S_{w,j}$, ($j = 1, 4, 6$).

In what follows, we derive the asymptotic solution of the equation (13) and, therefore, the asymptotic solution of the HCCP subject to the assumption:

A2. $\bar{A}_3 = 0$.

In Section 4.3, the meaning of the assumption A2 is explained.

Using the equations (9),(12),(14),(41),(42),(43),(44) and the assumption A2, we can convert the equation (13) to the following equivalent form:

$$P_1(\varepsilon)\bar{A}_1 + \bar{A}_1^T P_1(\varepsilon) - P_2(\varepsilon)P_2^T(\varepsilon) - P_3(\varepsilon)P_3^T(\varepsilon) + P_1(\varepsilon)S_{w,1}P_1(\varepsilon) + D_1 + \varepsilon G_1(P_1(\varepsilon), P_2(\varepsilon), P_3(\varepsilon), \varepsilon) = 0, \quad (46)$$

$$P_1(\varepsilon)\bar{A}_2 - P_2(\varepsilon)P_4(\varepsilon) + \varepsilon G_2(P_1(\varepsilon), P_2(\varepsilon), P_3(\varepsilon), P_4(\varepsilon), P_5(\varepsilon), \varepsilon) = 0, \quad (47)$$

$$P_2(\varepsilon)\bar{A}_6 + P_3(\varepsilon)\bar{A}_9 + \bar{A}_1^T P_3(\varepsilon) - P_2(\varepsilon)P_5(\varepsilon) - P_3(\varepsilon)P_6(\varepsilon) + P_1(\varepsilon)S_{w,1}P_3(\varepsilon) + \varepsilon G_3(P_1(\varepsilon), P_2(\varepsilon), P_3(\varepsilon), P_5(\varepsilon), P_6(\varepsilon), \varepsilon) = 0, \quad (48)$$

$$-P_4^2(\varepsilon) + \Lambda + \varepsilon G_4(P_2(\varepsilon), P_4(\varepsilon), P_5(\varepsilon), \varepsilon) = 0, \quad (49)$$

$$P_4(\varepsilon)\bar{A}_6 + \bar{A}_2^T P_3(\varepsilon) - P_4(\varepsilon)P_5(\varepsilon) + \varepsilon G_5(P_2(\varepsilon), P_3(\varepsilon), P_4(\varepsilon), P_5(\varepsilon), P_6(\varepsilon), \varepsilon) = 0, \quad (50)$$

$$P_5^T(\varepsilon)\bar{A}_6 + P_6(\varepsilon)\bar{A}_9 + \bar{A}_6^T P_5(\varepsilon) + \bar{A}_9^T P_6(\varepsilon) - P_5^T(\varepsilon)P_5(\varepsilon) - P_6^2(\varepsilon) + P_3^T(\varepsilon)S_{w,1}P_3(\varepsilon) + \varepsilon G_6(P_3(\varepsilon), P_5(\varepsilon), P_6(\varepsilon), \varepsilon) = 0, \quad (51)$$

where

$$\begin{aligned}
G_1(P_1(\varepsilon), P_2(\varepsilon), P_3(\varepsilon), \varepsilon) &= P_2(\varepsilon)\bar{A}_4 + P_3(\varepsilon)\bar{A}_7 + \bar{A}_4^T P_2^T(\varepsilon) + \bar{A}_7^T P_3^T(\varepsilon) \\
&+ P_2(\varepsilon)S_{w,2}^T P_1(\varepsilon) + P_3(\varepsilon)S_{w,3}^T P_1(\varepsilon) + P_1(\varepsilon)S_{w,2} P_2^T(\varepsilon) + \varepsilon P_2(\varepsilon)S_{w,4} P_2^T(\varepsilon) \\
&+ \varepsilon P_3(\varepsilon)S_{w,5} P_2^T(\varepsilon) + P_1(\varepsilon)S_{w,3} P_3^T(\varepsilon) + \varepsilon P_2(\varepsilon)S_{w,5} P_3^T(\varepsilon) + \varepsilon P_3(\varepsilon)S_{w,6} P_3^T(\varepsilon), \\
G_2(P_1(\varepsilon), P_2(\varepsilon), P_3(\varepsilon), P_4(\varepsilon), P_5(\varepsilon), \varepsilon) &= P_2(\varepsilon)\bar{A}_5 + P_3(\varepsilon)\bar{A}_8 + \bar{A}_1^T P_2(\varepsilon) \\
&+ \bar{A}_4^T P_4^T(\varepsilon) + \varepsilon \bar{A}_7^T P_5^T(\varepsilon) - P_3(\varepsilon)P_5^T(\varepsilon) + P_1(\varepsilon)S_{w,1} P_2(\varepsilon) + \varepsilon P_2(\varepsilon)S_{w,2}^T P_2(\varepsilon) \\
&+ \varepsilon P_3(\varepsilon)S_{w,3}^T P_2(\varepsilon) + P_1(\varepsilon)S_{w,2} P_4(\varepsilon) + \varepsilon P_2(\varepsilon)S_{w,4} P_4(\varepsilon) + \varepsilon P_3(\varepsilon)S_{w,5}^T P_4(\varepsilon) \\
&+ \varepsilon P_1(\varepsilon)S_{w,3} P_5^T(\varepsilon) + \varepsilon^2 P_2(\varepsilon)S_{w,5} P_5^T(\varepsilon) + \varepsilon^2 P_3(\varepsilon)S_{w,6} P_5^T(\varepsilon), \\
G_3(P_1(\varepsilon), P_2(\varepsilon), P_3(\varepsilon), P_5(\varepsilon), P_6(\varepsilon), \varepsilon) &= \bar{A}_4^T P_5(\varepsilon) + \bar{A}_7^T P_6(\varepsilon) + P_2(\varepsilon)S_{w,2}^T P_3(\varepsilon) \\
&+ P_3(\varepsilon)S_{w,3}^T P_3(\varepsilon) + P_1(\varepsilon)S_{w,2} P_5(\varepsilon) + \varepsilon P_2(\varepsilon)S_{w,4} P_5(\varepsilon) + \varepsilon P_3(\varepsilon)S_{w,5}^T P_5(\varepsilon) \\
&+ P_1(\varepsilon)S_{w,3} P_6(\varepsilon) + \varepsilon P_2(\varepsilon)S_{w,5} P_6(\varepsilon) + \varepsilon P_3(\varepsilon)S_{w,6} P_6(\varepsilon), \\
G_4(P_2(\varepsilon), P_4(\varepsilon), P_5(\varepsilon), \varepsilon) &= P_2^T(\varepsilon)\bar{A}_2 + P_4(\varepsilon)\bar{A}_5 + \varepsilon P_5(\varepsilon)\bar{A}_8 + \bar{A}_2^T P_2(\varepsilon) + \bar{A}_5^T P_4(\varepsilon) \\
&+ \varepsilon \bar{A}_8^T P_5^T(\varepsilon) - \varepsilon P_5(\varepsilon)P_5^T(\varepsilon) + \varepsilon P_2^T(\varepsilon)S_{w,1} P_2(\varepsilon) + \varepsilon P_4(\varepsilon)S_{w,2}^T P_2(\varepsilon) + \varepsilon^2 P_5(\varepsilon)S_{w,3}^T P_2(\varepsilon) \\
&+ \varepsilon P_2^T(\varepsilon)S_{w,2} P_4(\varepsilon) + \varepsilon P_4(\varepsilon)S_{w,4} P_4(\varepsilon) + \varepsilon^2 P_5(\varepsilon)S_{w,5}^T P_4(\varepsilon) + \varepsilon^2 P_2^T(\varepsilon)S_{w,3} P_5^T(\varepsilon) \\
&+ \varepsilon^2 P_4(\varepsilon)S_{w,5} P_5^T(\varepsilon) + \varepsilon^3 P_5(\varepsilon)S_{w,6} P_5^T(\varepsilon), \\
G_5(P_2(\varepsilon), P_3(\varepsilon), P_4(\varepsilon), P_5(\varepsilon), P_6(\varepsilon), \varepsilon) &= P_5(\varepsilon)\bar{A}_9 + \bar{A}_5^T P_5(\varepsilon) + \bar{A}_8^T P_6(\varepsilon) - P_5(\varepsilon)P_6(\varepsilon) \\
&+ P_2^T(\varepsilon)S_{w,1} P_3(\varepsilon) + P_4(\varepsilon)S_{w,2}^T P_3(\varepsilon) + \varepsilon P_5(\varepsilon)S_{w,3}^T P_3(\varepsilon) + \varepsilon P_2^T(\varepsilon)S_{w,2} P_5(\varepsilon) \\
&+ \varepsilon P_4(\varepsilon)S_{w,4} P_5(\varepsilon) + \varepsilon^2 P_5(\varepsilon)S_{w,5}^T P_5(\varepsilon) + \varepsilon P_2^T(\varepsilon)S_{w,3} P_6(\varepsilon) + \varepsilon P_4(\varepsilon)S_{w,5} P_6(\varepsilon) \\
&+ \varepsilon^2 P_5(\varepsilon)S_{w,6} P_6(\varepsilon), \\
G_6(P_3(\varepsilon), P_5(\varepsilon), P_6(\varepsilon), \varepsilon) &= P_5^T(\varepsilon)S_{w,2}^T P_3(\varepsilon) + P_6(\varepsilon)S_{w,3}^T P_3(\varepsilon) + P_3^T(\varepsilon)S_{w,2} P_5(\varepsilon) \\
&+ \varepsilon P_5^T(\varepsilon)S_{w,4} P_5(\varepsilon) + \varepsilon P_6(\varepsilon)S_{w,5}^T P_5(\varepsilon) + P_3^T(\varepsilon)S_{w,3} P_6(\varepsilon) + \varepsilon P_5^T(\varepsilon)S_{w,5} P_6(\varepsilon) \\
&+ \varepsilon P_6(\varepsilon)S_{w,6} P_6(\varepsilon);
\end{aligned} \tag{52}$$

$$\Lambda(t) = \text{diag}(\lambda_1(t), \dots, \lambda_{m_1}(t)). \tag{53}$$

Remark 6. The representation (42) and the system (46)-(51) differ considerably from the results of [13,22–25]. These representation and system are essentially novel results in the asymptotic analysis of parameter-dependent matrix Riccati algebraic equations with singularities, arising in infinite horizon H_∞ control problems, differential games and optimal control problems.

4.2. Zero-Order Asymptotic Solution of the Set of the Equations (46)- (51)

We seek the zero-order solution $\{P_{10}, P_{20}, P_{30}, P_{40}, P_{50}, P_{60},\}$ of the set of the equations (46)- (51). The equations for this asymptotic solution are obtained by setting formally $\varepsilon = 0$ in (46)- (51) and replacing $P_l(\varepsilon)$ with P_{l0} , ($l = 1, 2, \dots, 6$). Thus, we have

$$P_{10}\bar{A}_1 + \bar{A}_1^T P_{10} - P_{20}P_{20}^T - P_{30}P_{30}^T + P_{10}S_{w,1}P_{10} + D_1 = 0, \tag{54}$$

$$P_{10}\bar{A}_2 - P_{20}P_{40} = 0, \tag{55}$$

$$P_{20}\bar{A}_6 + P_{30}\bar{A}_9 + \bar{A}_1^T P_{30} - P_{20}P_{50} - P_{30}P_{60} + P_{10}S_{w,1}P_{30} = 0, \tag{56}$$

$$-P_{40}^2 + \Lambda = 0, \tag{57}$$

$$P_{40}\bar{A}_6 + \bar{A}_2^T P_{30} - P_{40}P_{50} = 0, \quad (58)$$

$$P_{50}^T \bar{A}_6 + P_{60}\bar{A}_9 + \bar{A}_6^T P_{50} + \bar{A}_9^T P_{60} - P_{50}^T P_{50} - P_{60}^2 + P_{30}^T S_{w,1} P_{30} = 0. \quad (59)$$

The equation (57) yields the unique symmetric positive definite solution

$$P_{40} = \Lambda^{1/2} = \text{diag}(\lambda_1^{1/2}, \dots, \lambda_{m_1}^{1/2}). \quad (60)$$

Substituting (60) into (55) and (58) and resolving the resulting equations with respect to P_{20} and P_{50} , respectively, we obtain

$$P_{20} = P_{10}\bar{A}_2\Lambda^{-1/2}, \quad (61)$$

$$P_{50} = \bar{A}_6 + \Lambda^{-1/2}\bar{A}_2^T P_{30}, \quad (62)$$

where $\Lambda^{-1/2} = \text{diag}(\lambda_1^{-1/2}, \dots, \lambda_{m_1}^{-1/2})$.

Furthermore, substitution of (61) and (62) into the equation (56) yields after a routine algebra

$$P_{30}(\bar{A}_9 - P_{60}) + (\bar{A}_1^T - P_{10}\bar{A}_2\Lambda^{-1}\bar{A}_2^T + P_{10}S_{w,1})P_{30} = 0. \quad (63)$$

Considering the equation (63) as an equation with respect to the unknown matrix P_{30} , we can choose its solution as:

$$P_{30} = 0, \quad (64)$$

which, along with (62), yields

$$P_{50} = \bar{A}_6. \quad (65)$$

Now, using the equations (61),(64),(65), we can convert the equations (54) and (59) into the following equations for P_{10} and P_{60} , respectively:

$$P_{10}\bar{A}_1 + \bar{A}_1^T P_{10} - P_{1,0}(\bar{A}_2\Lambda^{-1}\bar{A}_2^T - S_{w,1})P_{10} + D_1 = 0, \quad (66)$$

$$P_{60}\bar{A}_9 + \bar{A}_9^T P_{60} - P_{60}^2 + \bar{A}_6^T \bar{A}_6 = 0. \quad (67)$$

In what follows, we assume:

A3. The equation (66) has a solution P_{10} such that:

- (a) $P_{10}^T = P_{10}$;
- (b) all roots ν_l , ($l = 1, \dots, n$) of the polynomial equation

$$\det(\nu I_n - \bar{A}_1 + (\bar{A}_2\Lambda^{-1}\bar{A}_2^T - S_{w,1})P_{10}) = 0 \quad (68)$$

lie strictly inside the left-hand half-plane;

- (c) all roots κ_l , ($l = 1, \dots, n$) of the polynomial equation

$$\det(\kappa I_n - \bar{A}_1 + \bar{A}_2\Lambda^{-1}\bar{A}_2^T P_{10}) = 0 \quad (69)$$

lie strictly inside the left-hand half-plane.

A4.

$$\text{rank}[\bar{A}_6^T, \bar{A}_9^T \bar{A}_6^T, \dots, (\bar{A}_9^T)^{m-m_1} \bar{A}_6^T] = m - m_1. \quad (70)$$

Remark 7. By virtue of the results of [26], the assumption A4 provides the existence of the symmetric solution P_{60} to the equation (67), such that all roots χ_l , ($l = 1, \dots, m - m_1$) of the polynomial equation

$$\det(\chi I_{m-m_1} - \bar{A}_9 + P_{60}) = 0 \quad (71)$$

lie strictly inside the left-hand half-plane.

Remark 8. By virtue of the results of [27] and Remark 7, the assumptions A3 and A4 yield the uniqueness of the solution (64) to the equation (63).

4.3. Reduced H_∞ Control Problem

Let us set $\varepsilon = 0$ in the H_∞ Control Problem, consisting of the singularly perturbed system (1),(10) and the non-cheap control cost functional (11). Re-denoting the notations x , y , $\hat{u}$, w and $\hat{J}$ in the resulting system and functional with x_r , y_r , $\hat{u}_r$, w_r and $\hat{J}_r$, respectively, we obtain

$$dx_r(t)/dt = A_1 x_r(t) + A_2 y_r(t) + F_1 w_r(t), \quad t \geq 0, \quad x_r(0) = 0, \quad (72)$$

$$\hat{u}_r(t) = 0, \quad t \geq 0, \quad (73)$$

$$\hat{J}_r = \int_0^{+\infty} [x_r^T(t) D_1 x_r(t) + y_r^T(t) D_2 y_r(t) + \hat{u}_r^T(t) \hat{u}_r(t) - \gamma^2 w_r^T(t) w_r(t)] dt. \quad (74)$$

Due to (73), the functional (74) becomes

$$\hat{J}_r = \int_0^{+\infty} [x_r^T(t) D_1 x_r(t) + y_r^T(t) D_2 y_r(t) - \gamma^2 w_r^T(t) w_r(t)] dt. \quad (75)$$

It is seen that, in the set (72),(75), the variable $y_r(t)$, $t \geq 0$ does not satisfy any equation. Hence, we can choose this variable to satisfy a desirable property of the system (72) and the functional (75). The latter means that $y_r(t)$ can be chosen as a control in these system and functional. This observation means that the functional (75) depends on y_r and w_r , i.e., $\hat{J}_r = \hat{J}_r(y_r, w_r)$, where $y_r(\cdot)$ is the control, while $w_r(\cdot)$ is the disturbance. Similarly to Definition 1, we can formulate the H_∞ control problem with the dynamics (72) and the cost functional (75).

Since the matrix D_2 is not invertible (see the equations (9),(41)), then the H_∞ control problem (72),(75) cannot be solved by direct application of the method based on the matrix Riccati algebraic equation, i.e., it is singular (see, e.g., [13]). However, if the assumption A2 is fulfilled, the H_∞ control problem (72),(75) becomes as:

$$dx_r(t)/dt = A_1 x_r(t) + \bar{A}_2 y_{r,1}(t) + F_1 w_r(t), \quad t \geq 0, \quad x_r(0) = 0, \quad (76)$$

$$\hat{J}_r(y_{r,1}, w_r) = \int_0^{+\infty} [x_r^T(t) D_1 x_r(t) + y_{r,1}^T(t) \Lambda y_{r,1}(t) - \gamma^2 w_r^T(t) w_r(t)] dt, \quad (77)$$

where $y_{r,1}(t) \in \mathbb{R}^{m_1}$ is the upper block of the control vector $y_r(t)$ and it is the control in the H_∞ problem (76)-(77). Since the matrix Λ is positive definite, the H_∞ problem (76)-(77) is regular, i.e., the method based on the matrix Riccati algebraic equation can be used for its solution. We call this H_∞ control problem the Reduced H_∞ Control Problem (RHCP). Thus, **the assumption A2 guarantees the RHCP to be regular.**

Consider the function

$$y_{r,1}^*(x_r) = -\Lambda^{-1} \bar{A}_2 P_{10} x_r, \quad (78)$$

where P_{10} is the solution of the problem (66) mentioned in the assumption A3 (items (a) and (c)).

Taking into account that $S_{w,1} = \gamma^2 F_1 F_1^T$, we obtain (similarly to Lemma 1) the following assertion.

Lemma 2. Let the assumption A3 (items (a) and (c)) be valid. Then:

- (i) P_{10} is a positive semi-definite matrix;
- (ii) the controller (78) solves the RHCP.

4.4. Justification of the Asymptotic Solution to the Equations (46)-(51)

Lemma 3. Let the assumptions A2-A4 be valid. Then, there exists a positive number ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0]$, the set of the equations (46)-(51) has the solution $P_l(\varepsilon) = P_l^*(\varepsilon)$, ($l = 1, \dots, 6$), satisfying the inequalities

$$\|P_l^*(\varepsilon) - P_{l0}\| \leq a\varepsilon, \quad l = 1, \dots, 6, \quad (79)$$

where $a > 0$ is some constant independent of ε .

Proof. Let us transform the variables in the equations (46)-(51) as follows:

$$P_l(\varepsilon) = P_{l0} + \Delta_l(\varepsilon), \quad l = 1, \dots, 6, \quad (80)$$

where $\Delta_l(\varepsilon)$, ($l = 1, \dots, 6$) are new unknown matrices.

Consider the block-form matrix

$$\Delta(\varepsilon) = \begin{pmatrix} \Delta_1(\varepsilon) & \varepsilon\Delta_2(\varepsilon) & \varepsilon\Delta_3(\varepsilon) \\ \varepsilon\Delta_2^T(\varepsilon) & \varepsilon\Delta_4(\varepsilon) & \varepsilon^2\Delta_5(\varepsilon) \\ \varepsilon\Delta_3^T(\varepsilon) & \varepsilon^2\Delta_5^T(\varepsilon) & \varepsilon^2\Delta_6(\varepsilon) \end{pmatrix}. \quad (81)$$

Substitution of (80) into the equations (46)-(51) and use of the equations (54)-(59) and the block forms of the matrices $S_u(\varepsilon)$, $P(\varepsilon)$, A , S_w (see the equations (14),(42), (43),(44)), yield after a routine matrix algebra the equation for $\Delta(\varepsilon)$

$$\Delta(\varepsilon)\Theta(\varepsilon) + \Theta^T(\varepsilon)\Delta(\varepsilon) - \Delta(\varepsilon)(S_u(\varepsilon) - S_w)\Delta(\varepsilon) + \Gamma(\varepsilon) = 0, \quad (82)$$

where

$$\Theta(\varepsilon) = A - (S_u(\varepsilon) - S_w)P_0(\varepsilon), \quad P_0(\varepsilon) = \begin{bmatrix} P_{10} & \varepsilon P_{20} & \varepsilon P_{30} \\ \varepsilon P_{20}^T & \varepsilon P_{40} & \varepsilon^2 P_{50} \\ \varepsilon P_{30}^T & \varepsilon^2 P_{50}^T & \varepsilon^2 P_{60} \end{bmatrix}; \quad (83)$$

the matrix $\Gamma(\varepsilon)$ has the form

$$\Gamma(\varepsilon) = P_0(\varepsilon)A + A^T P_0(\varepsilon) - P_0(\varepsilon)(S_u(\varepsilon) - S_w)P_0(\varepsilon) + D. \quad (84)$$

Taking into account the symmetry of the matrix $\Gamma(\varepsilon)$, let us represent it in the block form

$$\Gamma(\varepsilon) = \begin{bmatrix} \Gamma_1(\varepsilon) & \Gamma_2(\varepsilon) & \Gamma_3(\varepsilon) \\ \Gamma_2^T(\varepsilon) & \Gamma_4(\varepsilon) & \Gamma_5(\varepsilon) \\ \Gamma_3^T(\varepsilon) & \Gamma_5^T(\varepsilon) & \Gamma_6(\varepsilon) \end{bmatrix}, \quad (85)$$

where the blocks have the same dimensions as the corresponding blocks of the matrix $P_0(\varepsilon)$.

Substituting the expression for the matrix $P_0(\varepsilon)$ (see the equation (83)) into the equation (84) and using the equations (54)-(59),(64),(65), we obtain the estimates of the blocks in (85)

$$\begin{aligned} \|\Gamma_1(\varepsilon)\| &\leq a_1\varepsilon, \quad \|\Gamma_2(\varepsilon)\| \leq a_1\varepsilon, \quad \|\Gamma_3(\varepsilon)\| \leq a_1\varepsilon^2, \quad \varepsilon \in (0, \varepsilon_1], \\ \|\Gamma_4(\varepsilon)\| &\leq a_1\varepsilon, \quad \|\Gamma_5(\varepsilon)\| \leq a_1\varepsilon^2, \quad \|\Gamma_6(\varepsilon)\| \leq a_1\varepsilon^4, \quad \varepsilon \in (0, \varepsilon_1], \end{aligned} \quad (86)$$

where $\varepsilon_1 > 0$ is some sufficiently small number and $a_1 > 0$ is some number independent of ε .

Now, let us analyze the matrix $\Theta(\varepsilon)$. Taking into account the equations (60),(61),(64),(65), we can represent this matrix in the block form as:

$$\Theta(\varepsilon) = \begin{bmatrix} \Theta_1(\varepsilon) & \Theta_2(\varepsilon) & \Theta_3(\varepsilon) \\ \frac{1}{\varepsilon}\Theta_4(\varepsilon) & \frac{1}{\varepsilon}\Theta_5(\varepsilon) & \Theta_6(\varepsilon) \\ \Theta_7(\varepsilon) & \Theta_8(\varepsilon) & \Theta_9(\varepsilon) \end{bmatrix},$$

$$\begin{aligned} \Theta_1(\varepsilon) &= \bar{A}_1 + S_{w,1}P_{10} + O(\varepsilon), \quad \Theta_2(\varepsilon) = \bar{A}_2 + O(\varepsilon), \quad \Theta_3(\varepsilon) = O(\varepsilon^2), \\ \Theta_4(\varepsilon) &= -\Lambda^{-1/2}\bar{A}_2^T P_{10} + \varepsilon(\bar{A}_4 + S_{w,2}^T P_{10}) + O(\varepsilon^2), \quad \Theta_5(\varepsilon) = -\Lambda^{1/2} + \varepsilon\bar{A}_5 + O(\varepsilon^2), \\ \Theta_6(\varepsilon) &= O(\varepsilon^2), \quad \Theta_7(\varepsilon) = \bar{A}_7 + S_{w,3}^T P_{10} + O(\varepsilon), \quad \Theta_8(\varepsilon) = \bar{A}_8 - \bar{A}_6^T + O(\varepsilon), \\ \Theta_9(\varepsilon) &= \bar{A}_9 - P_{60} + O(\varepsilon^2), \end{aligned} \quad (87)$$

where $O(\varepsilon)$ and $O(\varepsilon^2)$ denote matrices of corresponding dimensions satisfying the inequalities

$$\|O(\varepsilon)\| \leq a_2\varepsilon, \quad \|O(\varepsilon^2)\| \leq a_2\varepsilon^2, \quad \varepsilon \in [0, \varepsilon_1], \quad (88)$$

$a_2 > 0$ is some number independent of ε .

Let us estimate the real parts of the eigenvalues of the matrix $\Theta(\varepsilon)$ for all sufficiently small $\varepsilon > 0$. For this purpose, first, we permute simultaneously the second and third block rows and columns of the matrix $\Theta(\varepsilon)$. This permutation yields the matrix

$$\tilde{\Theta}(\varepsilon) = \begin{bmatrix} \Theta_1(\varepsilon) & \Theta_3(\varepsilon) & \Theta_2(\varepsilon) \\ \Theta_7(\varepsilon) & \Theta_9(\varepsilon) & \Theta_8(\varepsilon) \\ \frac{1}{\varepsilon}\Theta_4(\varepsilon) & \Theta_6(\varepsilon) & \frac{1}{\varepsilon}\Theta_5(\varepsilon) \end{bmatrix}. \quad (89)$$

For any $\varepsilon > 0$, the matrices $\Theta(\varepsilon)$ and $\tilde{\Theta}(\varepsilon)$ have the same set of eigenvalues.

Presenting the matrix $\Theta_6(\varepsilon)$ as $\frac{1}{\varepsilon}\Theta_6(\varepsilon)$ and applying the results of the work [28] (Theorem 2.1) in the particular (undelayed case) to the matrix $\tilde{\Theta}(\varepsilon)$, we can conclude the following. If the matrices $\Theta_5(0)$ and

$$\Phi \triangleq \begin{bmatrix} \Theta_1(0) & \Theta_3(0) \\ \Theta_7(0) & \Theta_9(0) \end{bmatrix} - \begin{bmatrix} \Theta_2(0) \\ \Theta_8(0) \end{bmatrix} \Theta_5^{-1}(0) [\Theta_4(0), 0] \quad (90)$$

are Hurwitz matrices, then there exists a positive number $\varepsilon_2 \leq \varepsilon_1$ such that, for all $\varepsilon \in (0, \varepsilon_2]$, $n + m - m_1$ eigenvalues $\theta_{k_1}(\varepsilon)$ of the matrix $\tilde{\Theta}(\varepsilon)$ satisfy the inequality

$$\operatorname{Re}\theta_{k_1}(\varepsilon) < -\beta_1, \quad k_1 = 1, \dots, n + m - m_1, \quad (91)$$

while the other m_1 eigenvalues $\theta_{k_2}(\varepsilon)$ satisfy the inequality

$$\operatorname{Re}\theta_{k_2}(\varepsilon) < -\beta_2/\varepsilon, \quad k_2 = n + m - m_1 + 1, \dots, n + m, \quad (92)$$

where $\beta_1 > 0$ and $\beta_2 > 0$ are some numbers independent of ε .

Due to (87)-(88),

$$\Theta_5(0) = -\Lambda^{1/2}, \quad (93)$$

meaning, along with the equations (41),(82), that $\Theta_5(0)$ is a Hurwitz matrix.

Proceed to the matrix Φ . Using the equations (87),(90) and the inequalities (88), we obtain after a routine matrix algebra

$$\Phi = \begin{bmatrix} \bar{A}_1 - (\bar{A}_2\Lambda^{-1}\bar{A}_2^T - S_{w,1})P_{10} & 0 \\ \bar{A}_7 - ((\bar{A}_8 - \bar{A}_6^T)\Lambda^{-1}\bar{A}_2^T - S_{w,3}^T)P_{10} & \bar{A}_9 - P_{60} \end{bmatrix}. \quad (94)$$

Thus, Φ is a Hurwitz matrix if and only if the matrices $\Phi_1 \triangleq \bar{A}_1 - (\bar{A}_2 \Lambda^{-1} \bar{A}_2^T - S_{w,1}) P_{10}$ and $\Phi_2 \triangleq \bar{A}_9 - P_{60}$ are Hurwitz matrices. Due to the assumption A3 (item b), Φ_1 is a Hurwitz matrix. Due to Remark 7, Φ_2 is a Hurwitz matrix. Hence, Φ is a Hurwitz matrix. Therefore, the inequalities (91) and (92) are satisfied.

Based on the inequalities (91),(92) and using the results of [21], we can rewrite the equation (82) in the equivalent form

$$\Delta(\varepsilon) = \int_0^{+\infty} \Psi^T(\sigma, \varepsilon) \left(\Gamma(\varepsilon) - \Delta(\varepsilon) (S_u(\varepsilon) - S_w) \Delta(\varepsilon) \right) \Psi(\sigma, \varepsilon) d\sigma, \quad \varepsilon \in (0, \varepsilon_2], \quad (95)$$

where, for any given $\varepsilon \in (0, \varepsilon_2]$, the $(n+m) \times (n+m)$ -matrix-valued function $\Psi(\sigma, \varepsilon)$ is the unique solution of the problem

$$d\Psi(\sigma, \varepsilon)/d\sigma = \Theta(\varepsilon)\Psi(\sigma, \varepsilon), \quad \sigma \geq 0, \quad \Psi(0, \varepsilon) = I_{n+m}. \quad (96)$$

We are going to estimate the matrix-valued function $\Psi(\sigma, \varepsilon)$. For this purpose, let us partition it into blocks as:

$$\Psi(\sigma, \varepsilon) = \begin{bmatrix} \Psi_1(\sigma, \varepsilon) & \Psi_2(\sigma, \varepsilon) & \Psi_3(\sigma, \varepsilon) \\ \Psi_4(\sigma, \varepsilon) & \Psi_5(\sigma, \varepsilon) & \Psi_6(\sigma, \varepsilon) \\ \Psi_7(\sigma, \varepsilon) & \Psi_8(\sigma, \varepsilon) & \Psi_9(\sigma, \varepsilon) \end{bmatrix}, \quad (97)$$

where the blocks are of the same dimensions as the corresponding blocks in (42).

By virtue of the results of [28] (Theorem 2.3), we have the following estimates of these blocks:

$$\begin{aligned} \|\Psi_{r_1}(\sigma, \varepsilon)\| &\leq a_3 \exp(-\beta_1 \sigma), \quad r_1 = 1, 4, 7, 9, \\ \|\Psi_{r_2}(\sigma, \varepsilon)\| &\leq a_3 \varepsilon \exp(-\beta_1 \sigma), \quad r_2 = 2, 8, \\ \|\Psi_{r_3}(\sigma, \varepsilon)\| &\leq a_3 \varepsilon^2 \exp(-\beta_1 \sigma), \quad r_3 = 3, 6, \\ \|\Psi_5(\sigma, \varepsilon)\| &\leq a_3 \left(\varepsilon \exp(-\beta_1 \sigma) + \exp(-\beta_2 \sigma / \varepsilon) \right), \\ &\sigma \geq 0, \quad \varepsilon \in (0, \varepsilon_3], \end{aligned} \quad (98)$$

where $0 < \varepsilon_3 \leq \varepsilon_2$ is some sufficiently small number; a_3 is some positive number independent of ε .

Now, let us apply the method of successive approximations to the equation (95). For this purpose, we consider the sequence of the matrices $\{\Delta_r(\varepsilon)\}_{r=0}^{+\infty}$ given as:

$$\Delta_{r+1}(\varepsilon) = \int_0^{+\infty} \Psi^T(\sigma, \varepsilon) \left(\Gamma(\varepsilon) - \Delta_r(\varepsilon) (S_u(\varepsilon) - S_w) \Delta_r(\varepsilon) \right) \Psi(\sigma, \varepsilon) d\sigma, \quad (99)$$

where $(r = 0, 1, \dots)$, $\varepsilon \in (0, \varepsilon_3]$; $\Delta_0(\varepsilon) = 0$; the matrices $\Delta_r(\varepsilon)$ have the block form

$$\Delta_r(\varepsilon) = \begin{pmatrix} \Delta_{r,1}(\varepsilon) & \varepsilon \Delta_{r,2}(\varepsilon) & \varepsilon \Delta_{r,3}(\varepsilon) \\ \varepsilon \Delta_{r,2}^T(\varepsilon) & \varepsilon \Delta_{r,4}(\varepsilon) & \varepsilon^2 \Delta_{r,5}(\varepsilon) \\ \varepsilon \Delta_{r,3}^T(\varepsilon) & \varepsilon^2 \Delta_{r,5}^T(\varepsilon) & \varepsilon^2 \Delta_{r,6}(\varepsilon) \end{pmatrix}, \quad (100)$$

and the dimensions of the blocks in each of these matrices are the same as the dimensions of the corresponding blocks in (81).

Using the block form of the matrices $S_u(\varepsilon)$, S_w , $\Gamma(\varepsilon)$, $\Psi(\sigma, \varepsilon)$, $\Delta_r(\varepsilon)$ (see the equations (14),(44),(85),(97),(100)), as well as using the estimates (86) and (98), we obtain the existence of a positive number $\varepsilon_0 \leq \varepsilon_3$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the sequence $\{\Delta_r(\varepsilon)\}_{r=0}^{+\infty}$ converges in the linear space of $(n+m) \times (n+m)$ -matrices. Moreover, the following inequalities are fulfilled:

$$\|\Delta_{r,l}(\varepsilon)\| \leq a\varepsilon, \quad r = 1, 2, \dots, \quad l = 1, \dots, 6,$$

where $a > 0$ is some number independent of ε , r and l .

Thus, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\Delta^*(\varepsilon) \triangleq \lim_{r \rightarrow +\infty} \Delta_r(\varepsilon)$$

is a solution of the equation (95) and, therefore, of the equation (82). Moreover, this solution has the block form similar to (81) and satisfies the inequalities

$$\|\Delta_l^*(\varepsilon)\| \leq a\varepsilon, \quad l = 1, \dots, 6.$$

This observation, along with the equation (80), proves the lemma. $\square$

Consider the matrix

$$P^*(\varepsilon) \triangleq \begin{bmatrix} P_1^*(\varepsilon) & \varepsilon P_2^*(\varepsilon) & \varepsilon P_3^*(\varepsilon) \\ \varepsilon (P_2^*(\varepsilon))^T & \varepsilon P_4^*(\varepsilon) & \varepsilon^2 P_5^*(\varepsilon) \\ \varepsilon (P_3^*(\varepsilon))^T & \varepsilon^2 (P_5^*(\varepsilon))^T & \varepsilon^2 P_6^*(\varepsilon) \end{bmatrix}, \quad \varepsilon \in (0, \varepsilon_0], \quad (101)$$

where $\{P_1^*(\varepsilon), P_2^*(\varepsilon), P_3^*(\varepsilon), P_4^*(\varepsilon), P_5^*(\varepsilon), P_6^*(\varepsilon)\}$ is the solution of the set of the equations (46)- (51) mentioned in Lemma 3.

Corollary 1. *Let the assumptions A2-A4 be valid. Then, there exists a positive number $\varepsilon^* \leq \varepsilon_0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, the assumption A1 is fulfilled with $P^*(\varepsilon)$ given by (101). Therefore, for these ε , all the statements of Lemma 1 are valid.*

Proof. First of all, let us note the following. Since the set of the equations (46)- (51) is equivalent to the equation (13) and $\{P_1^*(\varepsilon), P_2^*(\varepsilon), P_3^*(\varepsilon), P_4^*(\varepsilon), P_5^*(\varepsilon), P_6^*(\varepsilon)\}$ is a solution of (46)- (51) for all $\varepsilon \in (0, \varepsilon_0]$, then the matrix $P^*(\varepsilon)$ is a solution of (13). Since the matrix $P^*(\varepsilon)$ is symmetric, then the item (a) of the assumption A1 is fulfilled for all $\varepsilon \in (0, \varepsilon_0]$.

Proceed to the proof of the fulfillment of the item (b).

Consider the matrix $\mathcal{A}(\varepsilon)$ given in the equation (37).

Using the block form of the matrices $S_u(\varepsilon)$, A and $P^*(\varepsilon)$ (see the equations (14),(43) and (101)), as well as Lemma 3, the assumption A2 and the equations (60),(61),(64),(65), we can represent the matrix $\mathcal{A}(\varepsilon)$ in the following block form

$$\mathcal{A}(\varepsilon) = \begin{bmatrix} \mathcal{A}_1(\varepsilon) & \mathcal{A}_2(\varepsilon) & \mathcal{A}_3(\varepsilon) \\ \frac{1}{\varepsilon} \mathcal{A}_4(\varepsilon) & \frac{1}{\varepsilon} \mathcal{A}_5(\varepsilon) & \mathcal{A}_6(\varepsilon) \\ \mathcal{A}_7(\varepsilon) & \mathcal{A}_8(\varepsilon) & \mathcal{A}_9(\varepsilon) \end{bmatrix},$$

$$\begin{aligned} \mathcal{A}_1(\varepsilon) &= \bar{A}_1, \quad \mathcal{A}_2(\varepsilon) = \bar{A}_2, \quad \mathcal{A}_3(\varepsilon) = 0, \quad \mathcal{A}_4(\varepsilon) = -\Lambda^{-1/2} \bar{A}_2^T P_{10} + \varepsilon \bar{A}_4 + O(\varepsilon^2), \\ \mathcal{A}_5(\varepsilon) &= -\Lambda^{1/2} + \varepsilon \bar{A}_5 + O(\varepsilon^2), \quad \mathcal{A}_6(\varepsilon) = O(\varepsilon), \quad \mathcal{A}_7(\varepsilon) = \bar{A}_7 + O(1), \\ \mathcal{A}_8(\varepsilon) &= \bar{A}_8 - \bar{A}_6^T + O(\varepsilon), \quad \mathcal{A}_9(\varepsilon) = \bar{A}_9 - P_{60} + O(\varepsilon), \end{aligned} \quad (102)$$

where $O(1)$ denotes a matrix of corresponding dimension satisfying the inequality

$$c_1 \leq \|O(1)\| \leq c_2, \quad (103)$$

$0 < c_1 < c_2$ are some numbers independent of $\varepsilon \in (0, \varepsilon_0]$.

Let us estimate the real parts of the eigenvalues of the matrix $\mathcal{A}(\varepsilon)$ for all sufficiently small $\varepsilon > 0$. This estimation is done quite similarly to such an estimation for the matrix $\Theta(\varepsilon)$ in the proof of Lemma

3. Namely, first, we permute simultaneously the second and third block rows and columns of the matrix $\mathcal{A}(\varepsilon)$ which yields the matrix

$$\tilde{\mathcal{A}}(\varepsilon) = \begin{bmatrix} \mathcal{A}_1(\varepsilon) & \mathcal{A}_3(\varepsilon) & \mathcal{A}_2(\varepsilon) \\ \mathcal{A}_7(\varepsilon) & \mathcal{A}_9(\varepsilon) & \mathcal{A}_8(\varepsilon) \\ \frac{1}{\varepsilon}\mathcal{A}_4(\varepsilon) & \mathcal{A}_6(\varepsilon) & \frac{1}{\varepsilon}\mathcal{A}_5(\varepsilon) \end{bmatrix}. \quad (104)$$

For any $\varepsilon > 0$, the matrices $\mathcal{A}(\varepsilon)$ and $\tilde{\mathcal{A}}(\varepsilon)$ have the same set of eigenvalues.

Presenting the matrix $\mathcal{A}_6(\varepsilon)$ as $\frac{1}{\varepsilon}\mathcal{A}_6(\varepsilon)$ and applying the results of the work [28] (Theorem 2.1) in the particular (undelayed case) to the matrix $\tilde{\mathcal{A}}(\varepsilon)$, we can conclude the following. If the matrices $\mathcal{A}_5(0)$ and

$$Y \triangleq \begin{bmatrix} \mathcal{A}_1(0) & \mathcal{A}_3(0) \\ \mathcal{A}_7(0) & \mathcal{A}_9(0) \end{bmatrix} - \begin{bmatrix} \mathcal{A}_2(0) \\ \mathcal{A}_8(0) \end{bmatrix} \mathcal{A}_5^{-1}(0) [\mathcal{A}_4(0), 0] \quad (105)$$

are Hurwitz matrices, then there exists a positive number $\varepsilon^* \leq \varepsilon_0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, $n + m - m_1$ eigenvalues $\eta_{k_1}(\varepsilon)$ of the matrix $\tilde{\mathcal{A}}(\varepsilon)$ satisfy the inequality

$$\operatorname{Re} \eta_{k_1}(\varepsilon) < -\omega_1, \quad k_1 = 1, \dots, n + m - m_1, \quad (106)$$

while the other m_1 eigenvalues $\eta_{k_2}(\varepsilon)$ satisfy the inequality

$$\operatorname{Re} \eta_{k_2}(\varepsilon) < -\omega_2/\varepsilon, \quad k_2 = n + m - m_1 + 1, \dots, n + m, \quad (107)$$

where $\omega_1 > 0$ and $\omega_2 > 0$ are some numbers independent of ε .

Due to (102) and (88),

$$\mathcal{A}_5(0) = -\Lambda^{1/2}, \quad (108)$$

meaning, along with the equations (41),(82), that $\mathcal{A}_5(0)$ is a Hurwitz matrix.

Proceed to the matrix Y . Using the equations (102),(105) and the inequalities (88),(103), we obtain after a routine matrix algebra

$$Y = \begin{bmatrix} \bar{A}_1 - \bar{A}_2 \Lambda^{-1} \bar{A}_2^T P_{10} & 0 \\ \bar{A}_7 - (\bar{A}_8 - \bar{A}_6^T) \Lambda^{-1} \bar{A}_2^T P_{10} & \bar{A}_9 - P_{60} \end{bmatrix}. \quad (109)$$

Thus, Y is a Hurwitz matrix if and only if the matrices $Y_1 \triangleq \bar{A}_1 - \bar{A}_2 \Lambda^{-1} \bar{A}_2^T P_{10}$ and $Y_2 = \Psi_2 = \bar{A}_9 - P_{60}$ are Hurwitz matrices. Due to the assumption A3 (item c), Y_1 is a Hurwitz matrix. Due to Remark 7, Y_2 is a Hurwitz matrix. Hence, Y is a Hurwitz matrix. Therefore, the inequalities (106) and (107) are satisfied, meaning that the item (b) of the assumption A1 is fulfilled for all $\varepsilon \in (0, \varepsilon^*]$. Since both items of the assumption A1 are fulfilled for all $\varepsilon \in (0, \varepsilon^*]$, then the statements of Lemma 1 are valid for these ε . Thus, the corollary is proven. $\square$

Remark 9. Corollary 1 presents the ε -free solvability conditions of the HCCP.

5. Simplified Controller for the HCCP

Consider the following controller for the HCCP:

$$u_{0,\varepsilon}(z) = -\frac{1}{\varepsilon^2} B^T P_0(\varepsilon) z, \quad \varepsilon > 0, \quad (110)$$

where the matrix B is given in (12) and the matrix $P_0(\varepsilon)$ is given in (83).

Comparing the equations (16) and (110) with each other, one can conclude that the controller $u_{0,\varepsilon}(z)$ is obtained from the controller $u_\varepsilon^*(z)$ by replacing in the latter $P^*(\varepsilon)$ with $P_0(\varepsilon)$.

Substituting $u(t) = u_{0,\varepsilon}(z)$ into the system (1)-(2) and into the cost functional (6), and using the equations (7),(12) and (14), we obtain after some rearrangement the following system and cost functional:

$$dz(t)/dt = \mathcal{A}_0(\varepsilon)z(t) + Fw(t), \quad t \geq 0, \quad z(0) = 0, \quad (111)$$

$$J_0(w) \triangleq J(u_{0,\varepsilon}, w) = \int_0^{+\infty} [z^T(t) \mathcal{D}_0(\varepsilon)z(t) - \gamma^2 w^T(t)w(t)] dt, \quad (112)$$

where

$$\mathcal{A}_0(\varepsilon) = A - S_u(\varepsilon)P_0(\varepsilon), \quad \mathcal{D}_0(\varepsilon) = D + P_0(\varepsilon)S_u(\varepsilon)P_0(\varepsilon). \quad (113)$$

Remark 10. For a given $\varepsilon > 0$, the controller $u_{0,\varepsilon}(z)$ solves the HCCP if and only if

$$J_0(w) \leq 0 \quad \forall w(\cdot) \in L_2[0, +\infty; \mathbb{R}^q] \quad (114)$$

along trajectories of the system (111).

Theorem 1. Let the assumptions A2-A4 be valid. Then, there exists a positive number $\bar{\varepsilon}_0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}_0]$, the controller $u_{0,\varepsilon}(z)$ solves the HCCP.

Proof of the theorem is presented in Appendix A.

For the differential system (111), we consider the output equation

$$\zeta_{0,\varepsilon}(t) = (\mathcal{D}_0(\varepsilon))^{1/2} z(t), \quad t \geq 0, \quad \varepsilon \in (0, \bar{\varepsilon}_0], \quad (115)$$

where $(\mathcal{D}_0(\varepsilon))^{1/2}$ is the unique symmetric positive semi-definite square root of the symmetric positive semi-definite matrix $\mathcal{D}_0(\varepsilon)$.

Similarly to Remark 4, we have the following remark.

Remark 11. By virtue of the results of [3] and Theorem 1 and its proof, the assumptions A2-A4 guarantee that, for all $\varepsilon \in (0, \bar{\varepsilon}_0]$, the H_∞ norm $\gamma_{H,0}(\varepsilon) > 0$ of the system (111),(115) satisfies the inequality

$$\gamma_{H,0}(\varepsilon) < \gamma. \quad (116)$$

Moreover, for all $w(\cdot) \in L_2[0, +\infty; \mathbb{R}^q]$, all $\varepsilon \in (0, \bar{\varepsilon}_0]$ and any number $\bar{\gamma} \in (\gamma_{H,0}(\varepsilon), \gamma]$, the following inequality is valid:

$$\int_0^{+\infty} \left[\left(z_{0,\varepsilon}(t; w(\cdot)) \right)^T \mathcal{D}_0(\varepsilon) z_{0,\varepsilon}(t; w(\cdot)) - \bar{\gamma}^2 w^T(t)w(t) \right] dt \leq 0, \quad (117)$$

where $z_{0,\varepsilon}(t; w(\cdot))$ is the solution of the system (111).

6. Illustrative Example

Consider a particular case of the HCCP with the following data:

$$\begin{aligned} n = 1, \quad m = 2, \quad q = 2, \quad m_1 = 1, \quad \gamma = 0.5, \\ A_1 = -2, \quad A_2 = [1 \ 0], \quad A_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \\ B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_1 = 1, \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \quad (118)$$

yielding

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$S_u(\varepsilon) = \frac{1}{\varepsilon^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_w = 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
(119)

Let $P^*(\varepsilon)$ be the solution of the matrix Riccati algebraic equation (13) for the matrix-valued coefficients given in (119).

In this example, $P^*(\varepsilon)$ was calculated numerically using the MATLAB function `icare(A, [], D, [], [], [], -(S_u - S_w))`. Numerical simulation shows that the equation (13) with the data (119) admits the unique symmetric stabilizing solution for $0 < \varepsilon < \hat{\varepsilon} = 0.5745$.

In Figures 1 and 2, the real parts of the eigenvalues $\mu_i(\varepsilon)$, ($i = 1, 2, 3$) of the closed-loop matrix $\mathcal{A}(\varepsilon)$ (see the equation (37)) are depicted for $\varepsilon \in (0, \hat{\varepsilon})$.

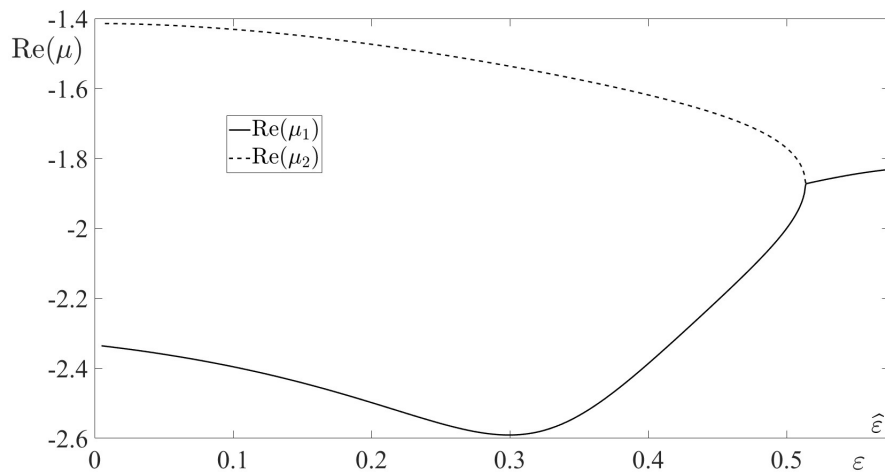


Figure 1. Real parts of $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$.

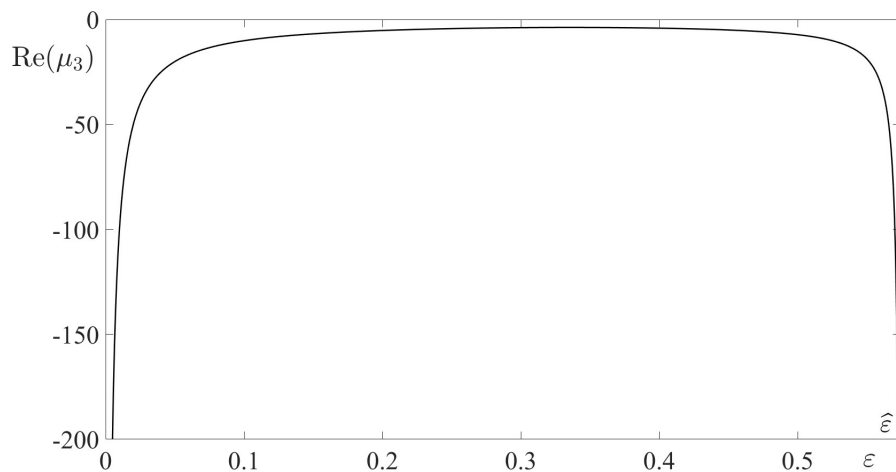


Figure 2. Real part of $\mu_3(\varepsilon)$.

It is seen that $\text{Re}(\mu_i(\varepsilon)) < 0$, ($i = 1, 2, 3$) for all $\varepsilon \in (0, \hat{\varepsilon})$, i.e., the MATLAB function `icare` returns the symmetric stabilizing solution of (13),(119) meaning that the assumption A1 is satisfied for all $\varepsilon \in (0, \hat{\varepsilon})$.

Figure 3 presents the H_∞ norm of the system (35),(38) in this example as a function of ε . It is seen that $\gamma_H(\varepsilon) < \gamma$ for all $\varepsilon \in (0, \hat{\varepsilon})$, which illustrates Remark 4. The H_∞ norm was calculated by implementing the MATLAB code

```
A_cl=A-Su*P;
D_cl=D+P*Su*P;
C_cl=sqrtm(D_cl);
system=ss(A_cl,F,C_cl,0);
hinfh=hinfnorm(system);
```

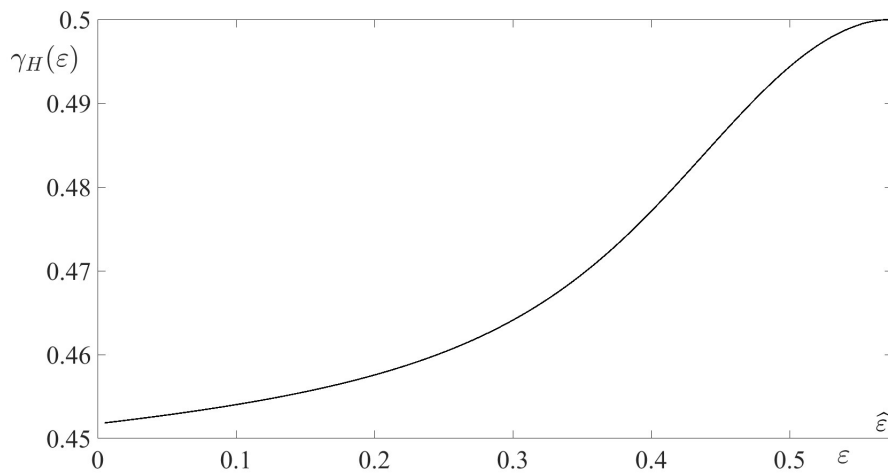


Figure 3. H_∞ -norm of the system (35),(38).

One can observe that, in agreement with the results of [3],

$$\gamma_H(\hat{\varepsilon}) = 0.5 = \gamma. \quad (120)$$

Using the equations (12),(14),(43)-(45),(118), we obtain that in this example the system (46)-(51) becomes as:

$$\begin{aligned} -4P_1(\varepsilon) - P_2^2(\varepsilon) - P_3^2(\varepsilon) + 4P_1^2(\varepsilon) + 1 + 4\varepsilon^2 P_2^2(\varepsilon) &= 0, \\ P_1(\varepsilon) - P_2(\varepsilon)P_4(\varepsilon) - 2\varepsilon P_2(\varepsilon) - \varepsilon P_3(\varepsilon)P_5(\varepsilon) \\ + 4\varepsilon P_1(\varepsilon)P_2(\varepsilon) + 4\varepsilon^2 P_2(\varepsilon)P_4(\varepsilon) &= 0, \\ P_2(\varepsilon) - 3P_3(\varepsilon) - P_2(\varepsilon)P_5(\varepsilon) - P_3(\varepsilon)P_6(\varepsilon) \\ + 4P_1(\varepsilon)P_3(\varepsilon) + 4\varepsilon^2 P_2(\varepsilon)P_5(\varepsilon) &= 0, \\ -P_4^2(\varepsilon) + 1 + 2\varepsilon P_2(\varepsilon) - \varepsilon^2 P_5^2(\varepsilon) + 4\varepsilon^2 P_2^2(\varepsilon) + 4\varepsilon^2 P_4^2(\varepsilon) &= 0, \\ P_4(\varepsilon) + P_3(\varepsilon) - P_4(\varepsilon)P_5(\varepsilon) - \varepsilon P_5(\varepsilon) - \varepsilon P_5(\varepsilon)P_6(\varepsilon) \\ + 4\varepsilon P_2(\varepsilon)P_3(\varepsilon) + 4\varepsilon^2 P_4(\varepsilon)P_5(\varepsilon) &= 0, \\ 2P_5(\varepsilon) - 2P_6(\varepsilon) - P_5^2(\varepsilon) - P_6^2(\varepsilon) + 4P_3^2(\varepsilon) + 4\varepsilon^2 P_5^2(\varepsilon) &= 0, \end{aligned} \quad (121)$$

where $P_1(\varepsilon)$, $P_2(\varepsilon)$, $P_3(\varepsilon)$, $P_4(\varepsilon)$, $P_5(\varepsilon)$ and $P_6(\varepsilon)$ are scalar unknowns.

Setting $\varepsilon = 0$ in the system (121) yields the particular case of the system (54)-(59):

$$\begin{aligned} -4P_{10} - P_{20}^2 - P_{30}^2 + 4P_{10}^2 + 1 &= 0, \\ P_{10} - P_{20}P_{40} &= 0, \\ P_{20} - 3P_{30} - P_{20}P_{50} - P_{30}P_{60} + 4P_{10}P_{30} &= 0, \\ -P_{40}^2 + 1 &= 0, \\ P_{40} + P_{30} - P_{40}P_{50} &= 0, \\ 2P_{50} - 2P_{60} - P_{50}^2 - P_{60}^2 + 4P_{30}^2 &= 0. \end{aligned} \quad (122)$$

Using the results of the Section 4.2 and the data (118) of the example, we obtain the following components of the solution to the system (122):

$$P_{30} = 0, \quad P_{40} = 1, \quad P_{50} = 1. \quad (123)$$

The components P_{10} and P_{60} satisfy the equations

$$3P_{10}^2 - 4P_{10} + 1 = 0 \quad (124)$$

and

$$P_{60}^2 + 2P_{60} - 1 = 0, \quad (125)$$

respectively.

The component P_{20} satisfies the equation

$$P_{20} = P_{10}. \quad (126)$$

Let us start with the equation (124). This quadratic equation has two solutions $P_{10,1} = 1/3$ and $P_{10,2} = 1$. It is directly verified that $P_{10} = P_{10,1}$ satisfies the assumption A3, while $P_{10} = P_{10,2}$ does not satisfy this assumption. Thus,

$$P_{10} = \frac{1}{3}, \quad (127)$$

which, along with (126), yields

$$P_{20} = \frac{1}{3}, \quad (128)$$

Proceed to the equation (125). Before solving this equation, let us check the fulfillment of the assumption A4.

Due to the data (118) of the example, $\bar{A}_6 = 1$, $\bar{A}_9 = -1$ and $m - m_1 = 1$. Substituting these values into the equation (70), we immediately obtain the fulfillment of the assumption A4. Therefore, the equation (125) has the solution mentioned in Remark 7. This solution is

$$P_{60} = \sqrt{2} - 1. \quad (129)$$

The absolute values of the errors $\Delta_l(\varepsilon)$, defined in (80), for $l = 1, 2, 3$, and for $l = 4, 5, 6$ are shown in Figures 4 and 5, respectively. For the sake of clear demonstration, the errors are depicted for $\varepsilon \in (0, 0.5)$. It is seen that these errors are of the order ε , which illustrates Lemma 3.

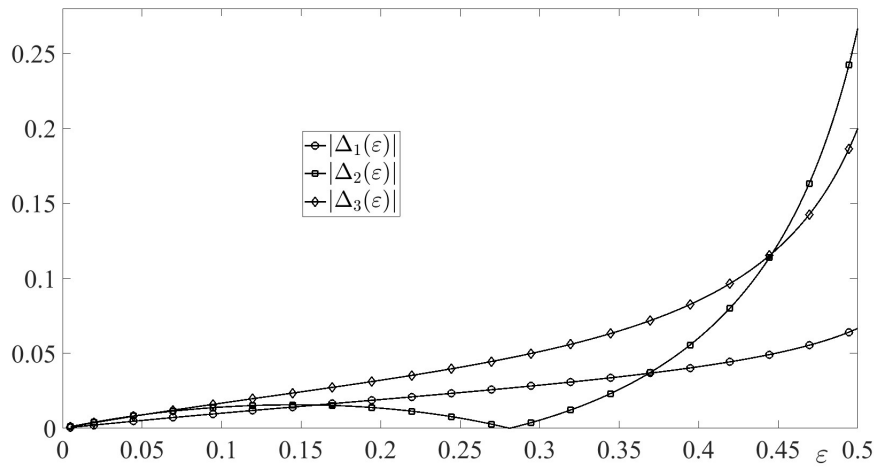


Figure 4. Absolute errors of P_{l0} , $l = 1, 2, 3$.

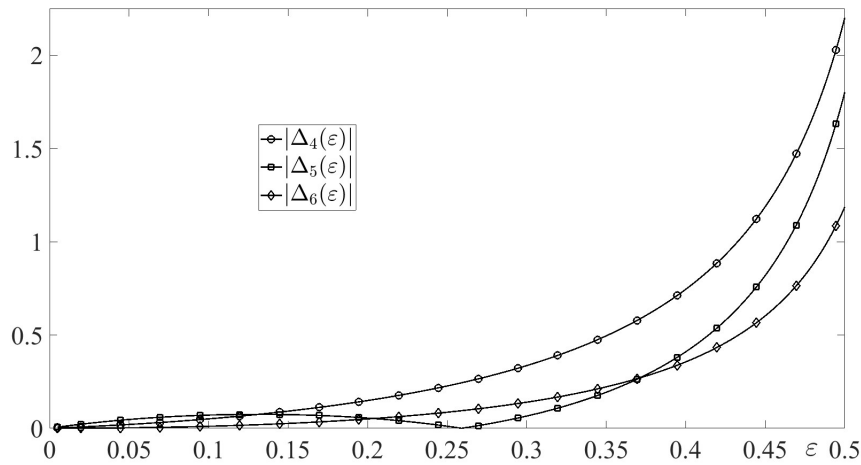


Figure 5. Absolute errors of P_{l0} , $l = 4, 5, 6$.

Using the equations (83) and (123),(127)-(129), we have in this example

$$P_0(\varepsilon) = \begin{bmatrix} \frac{1}{3} & \frac{\varepsilon}{3} & 0 \\ \frac{\varepsilon}{3} & \varepsilon & \varepsilon^2 \\ 0 & \varepsilon^2 & \varepsilon^2(\sqrt{2}-1) \end{bmatrix}. \quad (130)$$

Furthermore, using the equations (7),(110),(118),(130), we obtain the simplified controller in this example

$$u_{0,\varepsilon}(z) = \begin{bmatrix} -\frac{x}{3\varepsilon} - \frac{y_1}{\varepsilon} - y_2 \\ -y_1 - (\sqrt{2}-1)y_2 \end{bmatrix}, \quad (131)$$

where $z = \text{col}(x, y_1, y_2)$; x, y_1, y_2 are scalar state variables; x is a slow state variable, while $y = \text{col}(y_1, y_2)$ is a fast state variable.

Moreover, in this example the matrix-valued coefficient $\mathcal{A}_0(\varepsilon)$ of the closed-loop system (111) is

$$\mathcal{A}_0(\varepsilon) = \begin{bmatrix} -2 & 1 & 0 \\ -\frac{1}{3\varepsilon} & -\frac{1}{\varepsilon} & 0 \\ 0 & -1 & -\sqrt{2} \end{bmatrix}, \quad (132)$$

and (111) becomes

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x(t) \\ y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} -2 & 1 & 0 \\ -\frac{1}{3\varepsilon} & -\frac{1}{\varepsilon} & 0 \\ 0 & -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \quad t \geq 0, \\ x(0) &= 0, \quad y_1(0) = 0, \quad y_2(0) = 0, \end{aligned} \quad (133)$$

where $w(t) = \text{col}(w_1(t), w_2(t)) \in L_2[0, +\infty; \mathbb{R}^2]$.

The matrix $\mathcal{D}_0(\varepsilon)$, appearing in the cost functional (112), becomes

$$\mathcal{D}_0(\varepsilon) = \begin{bmatrix} \frac{10}{9} & \frac{1}{3} & \frac{\varepsilon}{3} \\ \frac{1}{3} & 2 + \varepsilon^2 & \varepsilon + \varepsilon^2(\sqrt{2} - 1) \\ \frac{\varepsilon}{3} & \varepsilon + \varepsilon^2(\sqrt{2} - 1) & \varepsilon^2 + \varepsilon^2(\sqrt{2} - 1)^2 \end{bmatrix}. \quad (134)$$

By a routine algebra, it is shown that the eigenvalues of the matrix (132) are

$$\begin{aligned} \mu_{i,0}(\varepsilon) &= \frac{-2\varepsilon - 1 \pm \sqrt{4\varepsilon^2 - 16\varepsilon/3 + 1}}{2\varepsilon}, \quad i = 1, 2, \\ \mu_{3,0}(\varepsilon) &\equiv -\sqrt{2}, \end{aligned} \quad (135)$$

and these eigenvalues satisfy the inequalities $\text{Re}(\mu_{i,0}(\varepsilon)) < 0$ for all $\varepsilon > 0$. Thus, the simplified controller (131) stabilizes the system (1)-(2) with the data (118) for all $\varepsilon > 0$.

In this example, the Riccati algebraic equation (A1) was solved numerically using the MATLAB function `icare`. Numerical simulation showed that this equation admits the unique symmetric stabilizing solution $\mathcal{P}(\varepsilon)$ for $0 < \varepsilon < \tilde{\varepsilon} = 0.3625$.

Thus, the simplified controller (131) solves the HCCP for $\varepsilon \in (0, \tilde{\varepsilon})$.

The absolute values of the errors $\Delta_{0,l}(\varepsilon) = |\mathcal{P}_l(\varepsilon) - P_{l0}(\varepsilon)|$ for $l = 1, 2, 3$, and for 4, 5, 6 are shown in Figures 6 and 7, respectively, for $\varepsilon \in (0, \tilde{\varepsilon})$. It is seen that these errors are of the order ε , which illustrates Lemma A2.

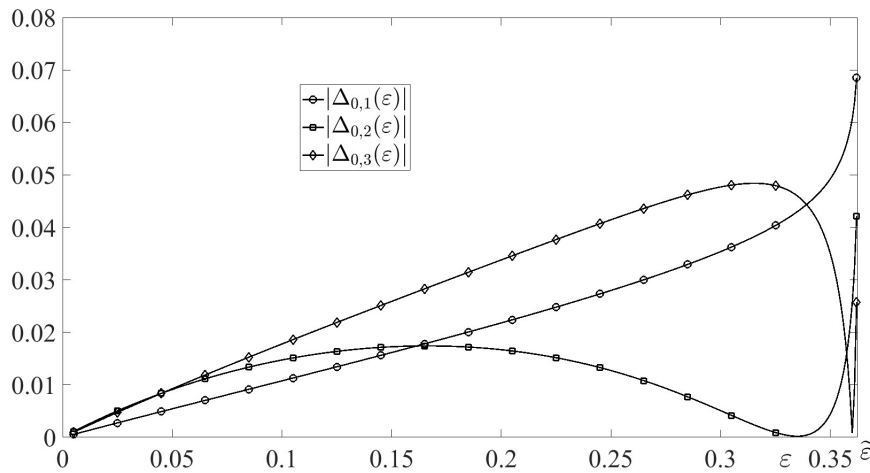


Figure 6. Values of $|\Delta_{0,l}(\varepsilon)|$, $l = 1, 2, 3$.

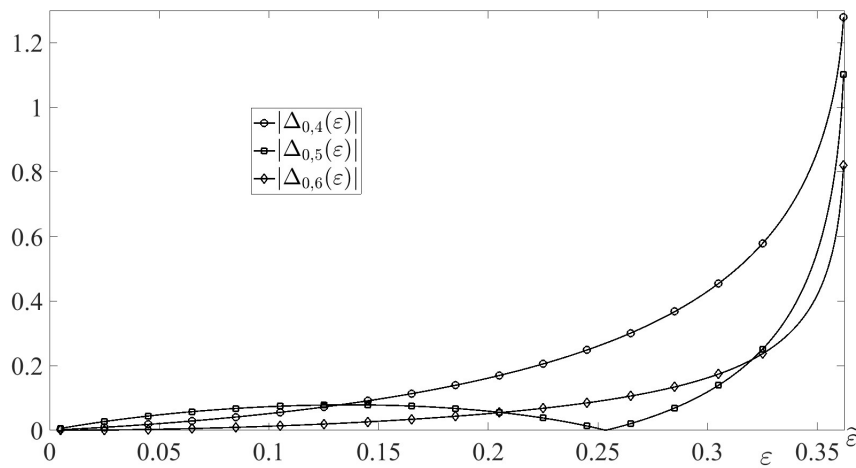


Figure 7. Values of $|\Delta_{0,l}(\varepsilon)|$, $l = 4, 5, 6$.

In Figure 8, the H_∞ norm $\gamma_{H,0}(\varepsilon)$ of the differential system (133) with the output (115) and the matrix $\mathcal{D}_0(\varepsilon)$, given by (134), is depicted for $\varepsilon \in (0, \tilde{\varepsilon})$. It is seen that, for these values of ε , $0 < \gamma_{H,0}(\varepsilon) < \gamma = 0.5$. The latter means that the simplified controller (131) solves the HCCP (1)-(2),(6) with the data (118) not only for all $\varepsilon \in (0, \tilde{\varepsilon})$ and $\gamma = 0.5$, but also for all $\varepsilon \in (0, \tilde{\varepsilon})$ and any performance level $\bar{\gamma} \in (\gamma_{H,0}(\varepsilon), \gamma)$ instead of γ in the cost functional.

In addition, let us note that, similarly to (120), $\gamma_{H,0}(\tilde{\varepsilon}) = 0.5 = \gamma$.

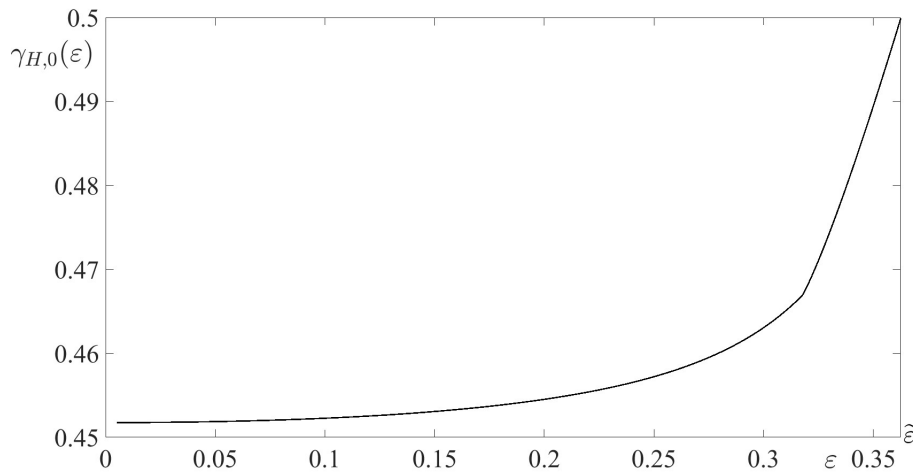


Figure 8. H_∞ -norm of system (111),(115).

7. Conclusions

In this paper, an infinite-horizon H_∞ control problem for a linear differential equation and a quadratic cost functional was studied in the case where the control cost is much smaller than the state cost. This smallness is due to the presence of the small multiplier ε^2 ($\varepsilon > 0$ is a small parameter) in the control cost. Because of this feature, the considered H_∞ problem is a cheap control problem. One more feature of this H_∞ problem is the following. The problem's dynamics has both, slow and fast, state variables, and the current quadratic cost of the fast state variable in the cost functional is a positive semi-definite (but non-zero) quadratic form. By the solvability conditions, the solution of the considered H_∞ control problem was reduced to the solution of the game-theoretic matrix Riccati algebraic equation perturbed by the small parameter $\varepsilon > 0$. This equation represents a new type of ε -dependent matrix Riccati algebraic equations arising in the control theory. The zero-order asymptotic solution to the aforementioned matrix Riccati algebraic equation was derived. This derivation requires

the developing the considerably novel approach. Based on the asymptotic solution to the matrix Riccati algebraic equation, the ε -free conditions, providing the existence of the solution to the considered H_∞ cheap control problem for all sufficiently small $\varepsilon > 0$, were established. The simplified controller, solving this problem for all sufficiently small $\varepsilon > 0$, was designed. Using the theoretical results, the academic example was solved. Numerical simulation in this example shows that a very accurate approximation of the solution to the H_∞ cheap control problem is obtained even for not too small values of $\varepsilon > 0$.

Appendix A. Proof of Theorem 1

Proof of Theorem 1 is based on two auxiliary lemmas.

Appendix A.1. Auxiliary Lemmas

Lemma A1. *Let the assumptions A2-A4 be valid. Then, there exists a positive number $\bar{\varepsilon}_1$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}_1]$, the real parts of all eigenvalues of the matrix $\mathcal{A}_0(\varepsilon)$ are smaller than some negative number independent of ε .*

Proof. The lemma is proven just similarly to the proof of Corollary 1 (the property of the matrix $\mathcal{A}(\varepsilon)$). $\square$

Consider the following matrix Riccati algebraic equation with respect to the $(n + m) \times (n + m)$ -matrix $\mathcal{P}$:

$$\mathcal{P}\mathcal{A}_0(\varepsilon) + \mathcal{A}_0^T(\varepsilon)\mathcal{P} + \mathcal{P}S_w\mathcal{P} + \mathcal{D}_0(\varepsilon) = 0. \quad (\text{A1})$$

Lemma A2. *Let the assumptions A2-A4 be valid. Then, there exists a positive number $\bar{\varepsilon}_2$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}_2]$, the equation (A1) has a solution $\mathcal{P} = \mathcal{P}(\varepsilon)$ of the block form*

$$\mathcal{P}(\varepsilon) = \begin{bmatrix} \mathcal{P}_1(\varepsilon) & \varepsilon\mathcal{P}_2(\varepsilon) & \varepsilon\mathcal{P}_3(\varepsilon) \\ \varepsilon\mathcal{P}_2^T(\varepsilon) & \varepsilon\mathcal{P}_4(\varepsilon) & \varepsilon^2\mathcal{P}_5(\varepsilon) \\ \varepsilon\mathcal{P}_3^T(\varepsilon) & \varepsilon^2\mathcal{P}_5^T(\varepsilon) & \varepsilon^2\mathcal{P}_6(\varepsilon) \end{bmatrix},$$

where the matrices $\mathcal{P}_1(\varepsilon)$, $\mathcal{P}_4(\varepsilon)$ and $\mathcal{P}_6(\varepsilon)$ are of the dimensions $n \times n$, $m_1 \times m_1$ and $(m - m_1) \times (m - m_1)$, respectively, and these matrices are symmetric.

Moreover, the matrices $\mathcal{P}_l(\varepsilon)$, $(l = 1, \dots, 6)$ satisfy the inequalities

$$\|\mathcal{P}_l(\varepsilon) - P_{l0}\| \leq \bar{a}\varepsilon, \quad l = 1, \dots, 6,$$

where the matrices P_{l0} , $(l = 1, \dots, 6)$ are obtained in Section 4.2; $\bar{a} > 0$ is some constant independent of ε .

Proof. The lemma is proven quite similarly to Lemma 3. $\square$

Appendix A.2. Main Part of the Proof

Consider the Lyapunov-like function

$$\mathcal{V}[z] = z^T \mathcal{P}(\varepsilon)z, \quad z \in \mathbb{R}^{n+m}, \quad \varepsilon \in (0, \bar{\varepsilon}_0], \quad (\text{A2})$$

where $\mathcal{P}(\varepsilon)$ is the solution of the equation (A1), mentioned in Lemma A2; $\bar{\varepsilon}_0 = \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\}$.

Let

$$\mathcal{V}_0(t; w(\cdot)) \triangleq \mathcal{V}[z_0(t; w(\cdot))], \quad (\text{A3})$$

where, for any given $w(\cdot) \in L_2[0, +\infty; \mathbb{R}^q]$, $z_0(t; w(\cdot))$ is the unique solution of the initial-value problem (111). This solution is locally absolutely continuous in the entire interval $[0, +\infty)$.

Differentiating $\mathcal{V}_0(t; w(\cdot))$ with respect to $t \in [0, +\infty)$, we obtain the following expression:

$$\frac{d\mathcal{V}_0(t; w(\cdot))}{dt} = 2 \left(\frac{dz_0(t; w(\cdot))}{dt} \right)^T \mathcal{P}(\varepsilon) z_0(t; w(\cdot)), \quad t \geq 0. \quad (\text{A4})$$

Using the equations (111),(A1), we can convert (A4) after a routine algebra to the following expression:

$$\begin{aligned} \frac{d\mathcal{V}_0(t; w(\cdot))}{dt} = & - \left(z_0(t; w(\cdot)) \right)^T (\mathcal{D}_0(\varepsilon) + \mathcal{P}(\varepsilon) S_w \mathcal{P}(\varepsilon)) z_0(t; w(\cdot)) \\ & + 2w^T(t) F^T \mathcal{P}(\varepsilon) z_0(t; w(\cdot)), \quad t \geq 0. \end{aligned} \quad (\text{A5})$$

Finally, using the notation

$$w_0(t) \triangleq \gamma^{-2} F^T \mathcal{P}(\varepsilon) z_0(t; w(\cdot)),$$

the expression (A5) can be rewritten as:

$$\begin{aligned} \frac{d\mathcal{V}_0(t; w(\cdot))}{dt} = & - \left(z_0(t; w(\cdot)) \right)^T \mathcal{D}_0(\varepsilon) z_0(t; w(\cdot)) + \gamma^2 w^T(t) w(t) \\ & - \gamma^2 (w(t) - w_0(t))^T (w(t) - w_0(t)), \quad t \geq 0. \end{aligned} \quad (\text{A6})$$

Let $z_0 \in \mathbb{R}^{n+m}$ be any given vector and t_0 be any non-negative number. Let $\bar{z}_0(t)$, $t \geq t_0$ be the solution of the differential equation in (111), generated by the disturbance $w(t) \equiv 0$ and the initial condition $z(t_0) = z_0$. Due to Lemma A1,

$$\lim_{t \rightarrow +\infty} \bar{z}_0(t) = 0 \quad \forall \varepsilon \in (0, \bar{\varepsilon}_0]. \quad (\text{A7})$$

Let

$$\bar{\mathcal{V}}_0(t) \triangleq \mathcal{V}[\bar{z}_0(t)], \quad t \geq t_0. \quad (\text{A8})$$

Then, quite similarly to the expression (A6), we obtain

$$\frac{d\bar{\mathcal{V}}_0(t)}{dt} = -\bar{z}_0^T(t) \mathcal{D}_0(\varepsilon) \bar{z}_0(t) - \gamma^2 \bar{w}_0^T(t) \bar{w}_0(t), \quad t \geq t_0, \quad (\text{A9})$$

where

$$\bar{w}_0(t) \triangleq \gamma^{-2} F^T \mathcal{P}(\varepsilon) \bar{z}_0(t), \quad t \geq t_0.$$

Since $\mathcal{D}_0(\varepsilon)$ is a positive semi-definite matrix for all $\varepsilon \in (0, \bar{\varepsilon}_0]$, the expression (A9) yields the inequality

$$-\frac{d\bar{\mathcal{V}}_0(t)}{dt} \geq 0, \quad t \geq t_0, \quad \varepsilon \in (0, \bar{\varepsilon}_0].$$

Integrating this inequality from $t = t_0$ to $+\infty$ and using the equations (A2),(A7),(A8) and that $\bar{z}_0(t_0) = z_0$, we obtain

$$z_0^T \mathcal{P}(\varepsilon) z_0 \geq 0, \quad \varepsilon \in (0, \bar{\varepsilon}_0]. \quad (\text{A10})$$

In this inequality z_0 is any $(n+m)$ -dimensional vector. This observation directly yields the positive semi-definiteness of the matrix $\mathcal{P}(\varepsilon)$ for all $\varepsilon \in (0, \bar{\varepsilon}_0]$.

Based on the inequality (A10), we continue to treat the expression (A6).

Since $\gamma^2(w(t) - w_0(t))^T(w(t) - w_0(t)) \geq 0$, this expression implies the inequality

$$\left(z_0(t; w(\cdot))\right)^T \mathcal{D}_0(\varepsilon) z_0(t; w(\cdot)) - \gamma^2 w^T(t) w(t) \leq -\frac{d\mathcal{V}_0(t; w(\cdot))}{dt}, \quad t \geq 0. \quad (\text{A11})$$

Integrating this inequality from $t = 0$ to any $t \geq 0$, using the equation (A2) and taking into account that $z_0(0; w(\cdot)) = 0$, we obtain

$$\begin{aligned} \int_0^t \left[\left(z_0(\sigma; w(\cdot))\right)^T \mathcal{D}_0(\varepsilon) z_0(\sigma; w(\cdot)) - \gamma^2 w^T(\sigma) w(\sigma) \right] d\sigma \\ \leq -\left(z_0(t; w(\cdot))\right)^T \mathcal{P}(\varepsilon) z_0(t; w(\cdot)), \quad t \geq 0. \end{aligned} \quad (\text{A12})$$

Since $\mathcal{P}(\varepsilon)$ is a positive semi-definite matrix, the inequality (A12) yields

$$\int_0^t \left[\left(z_0(\sigma; w(\cdot))\right)^T \mathcal{D}_0(\varepsilon) z_0(\sigma; w(\cdot)) - \gamma^2 w^T(\sigma) w(\sigma) \right] d\sigma \leq 0, \quad t \geq 0, \quad (\text{A13})$$

which, along with the inclusion $w(t) \in L_2[0, +\infty; \mathbb{R}^q]$, implies

$$\int_0^t \left(z_0(\sigma; w(\cdot))\right)^T \mathcal{D}_0(\varepsilon) z_0(\sigma; w(\cdot)) d\sigma \leq \gamma^2 \|w(\cdot)\|_{L_2(0, +\infty)}^2, \quad t \geq 0. \quad (\text{A14})$$

Since the matrix $\mathcal{D}_0(\varepsilon)$ is symmetric and positive semi-definite for all $\varepsilon \in (0, \bar{\varepsilon}_0]$, the integral in the left-hand side of the inequality (A14) is a non-decreasing function of $t \geq 0$ and this function is bounded from above. Therefore, this integral has a finite limit for $t \rightarrow +\infty$. This observation, along with the equation (112) and the inequality (A13), directly yields the inequality (114). This conclusion, along with Remark 10, completes the proof of the theorem.

References

1. Doyle, J.C.; Glover, K.; Khargonekar, P.P.; Francis, B. State-space solution to standard H_2 and H_∞ control problem. *IEEE Trans. Autom. Control* **1989**, *34*, 831–847.
2. Doyle, J.; Francis, B.; Tannenbaum, A. *Feedback Control Theory*; Macmillan Publishing Co.: London, UK, 1990.
3. Başar, T.; Bernhard, P. *H_∞ -Optimal Control and Related Minimax Design Problems: A Dynamic Games Approach*; Birkhauser: Boston, MA, USA, 1991.
4. Petersen, I.; Ugrinovski, V.; Savkin, A.V. *Robust Control Design Using H_∞ Methods*; Springer: London, UK, 2000.
5. Gershon, E.; Shaked, U.; Yaesh, I. *H_∞ -Control and Estimation of State-multiplicative Linear Systems*; Lecture Notes in Control and Information Science Series, Volume 318, Springer: London, UK, 2005.
6. Martynyuk, A.A.; Martynyuk-Chernienko, Yu.A. *Uncertain Dynamical Systems: Stability and Motion Control*; CRC Press: Florida, USA, 2011.
7. Glizer, V.Y.; Turetsky, V. *Robust Controllability of Linear Systems*; Nova Science Publishers: New-York, NY, USA, 2012.
8. Chang, X.H. *Robust Output Feedback H_∞ Control and Filtering for Uncertain Linear Systems*; Springer: Berlin, Germany, 2014.
9. Fridman, E.; Poznyak, A.; Rodriguez, F.J.B. *Robust Output LQ Optimal Control via Integral Sliding Modes*; Birkhauser: Basel, Switzerland, 2014.
10. Petersen, I.R.; Tempo, R. Robust control of uncertain systems: classical results and recent developments. *Autom. J. IFAC* **2014**, *50*, 1315–1335.
11. Yedavalli, R.K. *Robust Control of Uncertain Dynamic Systems: A Linear State Space Approach*; Springer: New York, NY, USA, 2014.
12. Kersten, J.; Rauh, A.; Aschemann, H. Analyzing uncertain dynamical systems after state-space transformations into cooperative form: verification of control and fault diagnosis. *Axioms* **2021**, *10*, 88.
13. Glizer, V.Y.; Kelis, O. *Singular Linear-Quadratic Zero-Sum Differential Games and H_∞ Control Problems: Regularization Approach*; Birkhauser: Basel, Switzerland, 2022.

14. Aggarwal, V.; Agarwal, M. Control of uncertain systems. In *Springer Handbook of Automation*; Nof, S.Y. Ed.; Springer Handbooks, Springer, Cham. 2023.
15. Petersen, I.R. Linear-quadratic differential games with cheap control. *Syst. Control Lett.* **1986**, *8*, 181–188.
16. Toussaint, G.J.; Başar, T. Achieving nonvanishing stability regions with high-gain cheap control using H^∞ techniques: the second-order case. *Systems Control Lett.* **2001**, *44*, 79–89.
17. Glizer, V.Y. H_∞ cheap control for a class of linear systems with state delays. *J. Nonlinear Convex Anal.* **2009**, *10*, 235–259.
18. Glizer, V.Y. Solution of a singular H_∞ control problem for linear systems with state delays. Proceedings of the 2013 European Control Conference, pp. 2843–2848, 2013 (published on-line at IEEE Xplore Digital Library <http://ieeexplore.ieee.org/>).
19. Glizer, V.Y.; Kelis, O. Singular infinite horizon zero-sum linear-quadratic differential game: saddle-point equilibrium sequence. *Numer. Algebra Control Optim.* **2017**, *7*, 1–20.
20. Vasil'eva, A.B.; Butuzov, V.F.; Kalachev, L.V. *The Boundary Function Method for Singular Perturbation Problems*; SIAM Books: Philadelphia, PA, USA, 1995.
21. Gajic, Z.; Qureshi, M.T.J. *Lyapunov Matrix Equation in System Stability and Control*; Dover Publications: Mineola, NY, USA, 2008.
22. Pan, Z.; Başar, T. H_∞ -optimal control for singularly perturbed systems. Part I: Perfect state measurements. *Autom. J. IFAC* **1993**, *29*, 401–423.
23. Pan, Z.; Başar, T. H_∞ -optimal control for singularly perturbed systems. Part II: Imperfect state measurements. *IEEE Trans. Autom. Control* **1994**, *39*, 280–299.
24. Gajic, Z.; Lim, M.T. *Optimal Control of Singularly Perturbed Linear Systems and Applications. High Accuracy Techniques*; Marsel Dekker: New York, NY, 2001.
25. O'Reilly, J. Partial cheap control of the time-invariant regulator. *Internat. J. Control* **1983**, *37*, 909–927.
26. Kalman, R.E. Contributions to the theory of optimal control. *Bol. Soc. Mat Mexicana* **1960**, *5*, 102–119.
27. Daletskii, Ju.L.; Krein, M.G. *Stability of Solutions Of Differential Equations In Banach Space*; Translations of Mathematical Monographs, Volume 43, American Mathematical Society: Providence, RI, USA, 1974.
28. Glizer, V.Y. Blockwise estimate of the fundamental matrix of linear singularly perturbed differential systems with small delay and its application to uniform asymptotic solution. *J. Math. Anal. Appl.* **2003**, *278*, 409–433.

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