Article

Not peer-reviewed version

The Retention of Information in the Presence of Increasing Entropy Using Lie Algebras defines Fibonacci Type Sequences

Joseph Johnson *

Posted Date: 8 May 2025

doi: 10.20944/preprints202505.0500.v1

Keywords: Fibonacci numbers; Golden Ratio; Lie algebra; Lie group; Entropy; Information; Living Systems; General Linear Group in n dimensions

Article

The Retention of Information in the Presence of Increasing Entropy Using Lie Algebras Defines Fibonacci Type Sequences

Joseph E. Johnson

Physics Department, University of South Carolina, April 28, 2025; jjohnson@sc.edu

Abstract: In the General Linear Lie algebra of continuous linear transformations in n dimensions, we show that unequal Abelian scaling transformations on the components of a vector can stabilize the system information in the presence of Markov component transformations on the vector which alone would lead to increasing entropy. The more interesting results follow from seeking Diophantine (integer) solutions with the result that the system can be stabilized with constant information for each of a set of entropy rates k = 1,2,3,…. . The first of these, the simplest, where k=1, results in the Fibonacci sequence, with information determined by the golden mean, and Fibonacci interpolating functions. Other interesting results are that a new set of higher order generalized Fibonacci sequences, functions, golden means, and geometric patterns that emerge for k=2, 3,…. Specifically, we will define the k^{th} order Golden Mean as $\Phi_k = k/2 + \sqrt{(k/2)^2+1}$ for k =1, 2, 3, … One can easily observe that one can form a right triangle with sides of 1 and k/2 and that this will give a hypotenuse of $\sqrt{(k/2)^2+1}$. Thus, the sum of the k/2 side plus the hypotenuse of these triangles so proportioned will give geometrically the exact value of the Golden Means for any value of k relative to the third side with a value of unity. The sequential powers of the matrix $(k^2+1,k,k,1)$ for any integer value of k, provide a generalized Fibonacci sequence. Also using the general equation $\Phi_k = k/2 + \sqrt{(k/2)^2+1}$ for k =1,2,3, one can easily prove that $\Phi_k = k + 1/\Phi_k$ which is a generalization of the familiar equation $\Phi = 1 + 1/\Phi$. We suggest that one could look for these new ratios and patterns in nature with the possibility that all of these systems are connected with the retention of information in the presence of increasing entropy. Thus, we show that two components of the General Linear Lie algebra (GL(n, R)), acting simultaneously with certain parameters, can stabilize the information content of a vector over time.

Keywords: Fibonacci numbers; Golden Ratio; Lie algebra; Lie group; Entropy; Information; Living Systems; General Linear Group in n dimensions

1. Introduction

The identification of symmetries in nature, as well as their violation, is of the highest importance. The theories of quantum mechanics rest upon the Heisenberg Lie algebra for the four momentum and space-time, and in special relativity on the Lorentz Lie algebra which contains both the rotation and Lorentz transformation symmetries jointly called the Poincare Lie algebra whose representations define the fundamental particles along with the inversions of space, time, and particle antiparticle discrete groups. These have been combined into an extended Poincare Lie algebra by the author [1]. The vast array of fundamental particles and their interactions is now based upon the SU(3)xSU(2)xU(1) "Standard Model" which has had phenomenal success [2] leaving only the theory of General Relativity otherwise framed in nonlinear differential equations. The author has proposed an extended form of Lie algebra to include Einstein's general relativity equations for gravitation [3].

The author has also separately studied the General Linear Lie algebra[4], GL(n, R), that describes the Lie algebra that generates all continuous nonsingular matrix transformations in n dimensions over the real numbers. He proved that they are composed of exactly two subalgebras: (1) an Abelian Lie algebra that consists of matrices with a "1" on the n different positions on the diagonal which

 doi:10.20944/preprints202505.0500.v1

generate continuous exponential expansions and contractions of the associated n axes. (2) A second Lie algebra that consists of a "1" in the (i, j) position along with a "-1" in the (j, j) position that generates transformations that move one over the plane perpendicular to the vector (1, 1, ….1) in an n-dimensional space. The actions of the associated group transfer a fraction of the vector component at one position, and adds it to another component thus conserved the sum of the components of the vector. If one only takes the positive linear combinations of these Lie generators, one obtains a Lie monoid (a group without an inverse) whose transformations exactly give the Markov Monid (MM) transformations on the vector. These Markov transformations describe increasing entropy thus lowering the "information content" in the vector while the diagonal transformations can increase the information content in a vector with their expansions and contractions of components. The MM transformations describe diffusion of one liquid in another and the dispersal of order in a system of increasing entropy.

In this paper we describe how the matrix generators of Fibonacci sequences are a combination of both these MM transformations by increasing information with diagonal transformations A(n) while the MM(n^2-n) transformations attempt to lower the system information. The Fibonacci sequence thus is able to maintain constant information (for a period of time) using A(n) to counter increasing entropy MM. It is well known that closed systems never increase their order, and, except for cyclic systems, they continuously increase their entropy, which is a metric of the total system disorder. This is true both in thermodynamics and in information systems, as information is stored as special physical states of matter and energy. Thus on the surface it is perplexing that living things, as subsystems of larger closed systems, can become increasingly organized, and that groups of living things such as human societies can dramatically increase their order as subsystems of a larger closed system. However this does not violate the increase of entropy of the system as a whole. We understand that this is accomplished by utilizing two sources of relative order for such organizing systems where, for example, energy flows from a more ordered domain to states of lower order such as from a hot reservoir (e.g. the sun) to a colder one (e.g. the earth or outer space). In some sense we could visualize the self organizing subsystem to be "feeding" on the source of order available so as to maintain and increase its own order. It is natural and customary to imagine that this entire process of retaining and even increasing order in the presence of increasing entropy to be extremely complex (in the technical sense) certainly highly nonlinear. But we will show that within the framework of an extremely simple and fully linear system that while entropy is increasing; it is possible to utilize hot and cold type reservoirs (more precisely an information source and sink) for the subsystem to retain order (or information) as entropy for the combined system increases. To accomplish this we need to carefully define order, information, entropy, and the mathematical structure of the system to be studied. Our work will rest upon previous work by the author concerning entropy and diffusion using continuous (Lie) groups with a decomposition of the general linear group in n dimensions. Beginning with the works of Einstein on random motion of molecules, and Markov on diffusion using Markov transformations over a century ago, diffusion with the associated increases of entropy has been described by Markov transformations and the associated diffusion equations in the continuous limit. It is the very essence of a diffusion process that these transformations are irreversible and thus have no inverse transformation. Thus, our approach using continuous group theory seems at first paradoxical as all groups have an inverse for each transformation but becomes apparent upon closer analysis in the following.

The General Linear Group in n Dimensions:

We will explore the general linear group in n dimensions, GL(n,R) over the real numbers which we have previously shown [4] can be decomposed into a Markov type Lie group (MT) and an Abelian scaling group (A). First consider the Abelian scaling group, which is generated by the Lie Algebra matrix representation that consists of the n elements L^{ii} that have value of '1' on a single diagonal position (ii) and a '0' in every other position. Thus the group consists of elements like A(a) = exp ($a_{ii}L^{ii}$) which is a matrix with all off-diagonal elements having values of '0', and diagonal values of $e^{a_{ii}}$ .and thus these transformations simply scale each axis, i, by these multipliers.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 May 2025 doi:10.20944/preprints202505.0500.v1

One recalls that a Markov transformation is a linear transformation on a vector space that preserves the sum of the elements of the vector and when acting upon a vector with non-negative components, it transforms it into a new vector all with non-negative components. The MT (Lie) group (the non-Abelian part of the general linear group) is the group of transformations in n dimensions that preserves the sum of the elements of the vector upon which they act without regard to whether these elements are positive or negative. The MT Lie group was found to contain all valid Markov transformations when a particular basis is used for the generating Lie Algebra. The MT Lie algebra basis consists of all linear combinations of elements that have a value of '1' in the ij position, and a '-1' in the jj position, with all other elements being '0'. Thus, each of these elements has columns that each sum to '0', and which close as a Lie algebra with n(n-1) basis elements. By exponentiation they generate all linear transformations that preserve the sum of components of a vector upon which it acts. As the sum of components is invariant (as opposed to the sum of squares for the motion on a sphere), then we can say that this sum represents a conserved entity (such as probability, money, or substance) which is being redistributed by the transformation. But as many of these transformations generate new vectors that have some negative components, these are unacceptable as Markov transformations.

Continuous Markov Transformations – The Markov Monoid (MM)

It can then be shown that with the basis elements just defined, that if one takes only non-negative linear combinations ($L(a) = \exp(a_{ij}L^{ij})$ where $a_{ij} >= 0$, then one gets all continuous Markov transformations in n dimensions that are continuously connected to the identity. This process removes the (unacceptable) inverse transformations and thus makes this Lie Group into a Lie Monoid (which we call the Markov Monoid (MM)) which is a group without inverses. The values a_{ij} give the exponential rate of transfer, redistribution, and diffusion of the conserved substance from component j to component i. It is easy to show that the MT transformations are linear transformations that move one over the hyperplane perpendicular to the vector (1,1,1,…1) in n dimensions and that the MM transformations are those which constrain the transformations to the positive hyperquadrant of the n-dimensional space. These transformations are very intuitive and if one chooses only those elements that are just below or just above the diagonal for a_{ij} then one gets a random walk in one dimension over a lattice of positions represented by the components x_i as probabilities or amount of substance. Reframed as a Hilbert space with continuous positions replacing the discrete values of 'i' then one obtains a continuous diffusion model in one dimension.

Either of these models takes one from a highly organized state of perfect information and minimum entropy into one of maximum entropy at the final equilibrium state if a_{ij} is multiplied by a continuous parameter t. In our work here we will assume that all states are equally likely and thus equilibrium is the uniform distribution over all states. One can use the Shannon definition of entropy as $S = \Sigma_i \ x_i \log_2 x_i$ or the Renyi second order entropy defined by $S = \log_2 \{n \ \Sigma_i (x_i^{2})\}$. We will use the later of these which in fact differs only slightly in value from the Shannon formula. For a two component system this becomes $S = \log_2 \{2 \ \Sigma_i (x_1^2 + x_2^2) \}$. Thus, when the substance or probability is in one or the other state (i.e. $x_1 = 1$, $x_2 = 0$ or conversely) then one obtains $S = \log_2 \{2 (1 + 0)\} = 1$ (i.e. one bit of information). Then after an infinite time with maximum diffusion equally in both directions: $M(t) = \exp t (L^{12} + L^{21}) = (½)*(1,1; 1,1)$ as t approaches infinity and thus $S = \log_2 \{2 \ \Sigma_i ((1/2)^2 + (1/2^2) \} = 0$ (zero bits of information) no matter what the original state was. Thus we can conclude that the Markov transformations will distribute the conserved entity (such as probability) equally to all available states in an irreversible way that perfectly mimics the diffusion of dye in a liquid, dust in a room, and our general view of entropy being maximized irreversibly. The inverse transformations in MT are not available as they lead to unphysical states of negative probability.

The Abelian Group Combined with the Markov Monoid

However, although the Abelian group does not conserve the total substance or probability as the MM transformations do, it still does not lead to negative states but rather these transformations can expand or contract any axis independently with an exponential multiplier $A(a) = \exp(a)$ where a can be a positive or negative real number making the quantity larger or smaller in that direction. It is easier to think of the conserved entity as money rather than probability, and then the Abelian group would be like having a bank infuse or remove the money supply for any component at a rate proportional to its values. One visual model in two dimensions would be to have a tank with water divided into two halves separated by a membrane and with one half clear and the other half with dye. If the dye is on the left this could represent a bit value of '1' and if the dye is on the right it would represent a '0'. Larger numbers of tanks would represent the physical storage of any number of bits of information. If a small hole is made in the membrane than diffusion will occur so that no matter what the original state the red dye is on, eventually the two sides will be evenly colored and all information will be lost as represented by the MM transformation above for a two component system representing the amount of dye in each side. The information of the system can here be measured by the Renyi' entropy. Now the action of the Abelian group can allow the amount of water to grow exponentially in one side while decreasing exponentially in the other side, simultaneously with the action of the MM transformation. This can be done at a rate that will exactly cancel the increase of entropy and thus keep the information of the system constant as determined by the ratio of color in the two sides (thus effectively renormalizing the length of the vector continuously to make it have a value of unity in the computation of information).

Thus the linear transformations that we wish to study are combinations of the scaling transformations acting simultaneously with the MM transformations. $A(s) = \exp\{\rho(1,0; 0,-1) + \gamma(1,0; 0,1)\} = (e^{+\rho+\gamma}, 0 ; 0, e^{-\rho+\gamma})$. Likewise, if we assume that each component is equally likely to have the red dye (the tank is divided into two equal parts with no preference) then if follows that the MM transformation is given simultaneously by $M(s) = \exp\{\sigma(-1, +1; +1, -1)\}$. Acting simultaneously, we thus wish to study the three-parameter transformation $F(s) = \exp(t\{\rho+\gamma-\sigma, \sigma; \sigma, -\rho+\gamma-\sigma\})$. As overall growth will not affect the information level (since both components are multiplied by the same value and the ratio is constant) then we will later remove the $\gamma-\sigma$ part thus leaving only two free parameters.

Retention of Information in the Presence of Entropy:

We have simplified the four parameter general linear transformation by making the equilibrium state one with equal values of x and y and thus removing any asymmetry of the Markov transformation. Secondly, since information is only dependent upon the ratio of x and y, then we can remove the overall growth transformation represented by the same exponential factor for both components and thus leaving only the asymmetric Abelian transformation that is useful in increasing the information. We now explicitly understand the two components of this transformation where the Markov component proceeds at a relative rate of 's' to transform the two components of a vector into a new vector with ever increasing nearness to the state of maximum entropy where both components are equal and thus where we could not identify whether the 'bit' of information so represented, was a '1' or a '0'. But simultaneous with the increase of entropy, the Abelian transformation unequally acts upon the two components reducing one by a proportional factor and increasing the other.

The most general (A + MM) infinitesimal transformation is given by:

$$G(\varepsilon) = \{ 1+\varepsilon(\gamma+\alpha-\sigma), \varepsilon(\sigma+\nu) ; \varepsilon(\sigma-\nu) , 1+\varepsilon(\gamma-\alpha-\sigma) \}$$

where σ is the multiplier of the symmetric part of the MM transformation and ν is the multiplier of the asymmetric part of the MM transformation. The γ gives an overall growth as one of the two Abelian generators, while α gives the other complementary unequal growth for the Abelian group. When acting upon a vector (x,y) and treating t as infinitesimal, we get two coupled first order differential equations:

$$dx/dt = (\gamma+\alpha-\sigma)\, x + (\sigma+\nu)y$$

and

$$dy/dt = (\sigma-\nu)x \quad + (\gamma-\alpha-\sigma)y$$

which can be combined into the single second order equation

$$d^2x/dt^2 + 2(\sigma-\gamma)\, dx/dt + (\gamma^2 - 2\gamma\sigma - \alpha^2 + \nu^2)x = 0$$

where

Assuming a solution $x(t) = A\, e^{\rho t}$ then one finds

$$\rho = (\sigma-\gamma) \pm \beta \text{ where } \quad \beta^2 = \alpha^2 + \sigma^2 - \nu^2$$

One can also expand the general linear transformation:

$G(t) = \exp t\, \{\gamma-\sigma+\alpha\, ,\, \sigma+\nu\, ;\, \sigma-\nu\, ,\, \gamma-\sigma-\alpha\, \}$ and, with some work, collect terms to get:

$$G(t) = e^{(\sigma-\gamma)t}\{\, ch(\beta t) + (\alpha/\beta)sh(\beta t)\, ,\, ((\sigma+\nu)/\beta)sh(\beta t)\, ;\, ((\sigma-\nu)/\beta)sh(\beta t)\, ,\, ch(\beta t) - (\alpha/\beta)sh(\beta t)\}$$

One can easily prove that with the removal of the overall exponential factor , σ-γ, gives a G′(t)), which has unit determinant at all times $|G'(t)| = 1$. We now wish to make two simplifications: (1) We again restrict the system to have a symmetric equilibrium and thus $\nu = 0$ and thus $\beta^2 = \alpha^2 + \sigma^2$. (2) Also since the overall exponential factor does not affect the information content which is determined by the ratio of the two components, we can remove $e^{(\sigma-\gamma)t}$ thus leaving the solution as:

$$G(t) = \{\, ch(\beta t) + (\alpha/\beta)sh(\beta t)\, ,\, (\sigma/\beta)sh(\beta t)\, ;\, (\sigma/\beta)sh(\beta t)\, ,\, ch(\beta t) - (\alpha/\beta)sh(\beta t)\}$$

Also in the limit of very large t, the ch and sh terms become equal and can be factored out with the normalization of the sum as $x'+y'=1$ thus leaving the information as

$I = \log_2 \{2\, \{\, x'^2 + y'^2\}$ where

$x' = \{(1+\alpha/\beta)x + (\sigma/\beta)y\}/N$ and $y' = \{(\sigma/\beta)x + (1- \alpha/\beta)y\}/N$

where $N = x'+y'$ renormalizes the sum to unity.

If one begins with a state of perfect information, x=1, y=0 then the information as t approaches infinity becomes, using these last equations,

$$I = \log_2 \{2\, [(1+\alpha/\beta)^2 + (\sigma/\beta)^2]/\, [1 + \alpha/\beta + \sigma/\beta]^2\} = \log_2 \{4(\alpha+\beta)\beta/(\alpha+\beta+\sigma)^2\}$$

Thus if we only have Markov diffusion via σ with no Abelian unequal growth (i.e. $\alpha = 0$ and $\beta=\sigma$) one gets $I = \log_2 (1) = 0$ and thus no information and maximum entropy at an infinite time in the future. Thus, in the case of a pure Markov transformation with σ nonzero but with no Abelian transformation (and thus $\alpha = 0$ and $\beta=\sigma$) then $x' = y'$ and the limiting case has no information, taken for example from an initial state of perfect information (1,0).

But when α is nonzero, there is thus always some information as x′ and y′ are not. By adjusting the value of α, then we can maintain the information level as we desire, but at the price that the system is taking substance (energy, money, …) from some source at an exponential rate which cannot be maintained indefinitely.

Restriction to Integer Solutions (Diophantine Type Equation):

Quantization to discrete values is not just a property of quantum mechanics and the atomic and nuclear domains. Quantization occurs in almost all branches of science and for diverse reasons. Quantization specifically occurs in life forms [5]: A tree or bush does not grow one huge trunk with a single leave of ever-increasing size growing in a continuous fashion. In fact discreteness is at the core of social and biological systems that thrive on diversity and numbers rather than a single massive entity that grows continuously. This requires a quantization into integer units of leaves, stems, seeds, pods, and component organs.

For those reasons we now leave the continuous case and ask what integer solutions there are to the system where entropy is countered by uneven growth. Thus, we set $G(t) = (n\, ,\, k\, ;\, k,\, m)$ where n, k, m are all integers for some value of t. Since we have removed the overall exponential growth, it is easy to prove that the remaining G(t) has unit determinant for all values of t as the system evolves and this restricts us to $nm-k^2 = 1$. Beginning with the simplest possible solutions, we must have some minimal value of k which we can take as k = 1, 2, 3, … One series of integer solutions for these values of k are where $n = k^2 + 1$ and m = 1 as this gives a unit determinant for every value of k in this series Det $G(t) = (k^2 + 1)(1) - k^2 = 1$). One notes that with each increasing value of k, that one is introducing larger and larger rates of entropy increase over time and thus the required values of α must increase accordingly. Solutions for $n = k^2 + 1$ and m = 1 and k = 1,2,3,… give:

With G(t) = { ch(βt) + (α/β)sh(βt) , ((σ/β)sh(βt) ; ((σ)/β)sh(βt) , ch(βt) - (α/β)sh(βt)} = (n,k; k,m) one obtains:

$2\,ch(\beta t) = n+m = k^2+2$ or $ch(\beta t) = (k^2+2)/2$

$(2\alpha/\beta)\,sh(\beta t) = k^2$ or $sh(\beta t) = (\beta k^2)/(2\alpha)$

$(\sigma/\beta)\,sh(\beta t) = k$ or $sh(\beta t) = \beta k/\sigma$ and thus $2\alpha = \sigma k$

Where $\beta^2 = \alpha^2 + \sigma^2 = \alpha^2 + (2\alpha/k)^2$

The parameter s is not determined explicitly as it is adjustable with the time parameter. One can select units of time which correspond to σ = 1 rate of diffusion and let that rate scale the time parameter. Our results do not depend upon an explicit value of s. The equilibrium ratio of at t = ∞ is thus determined beginning with a vector of compete information (1,0) and transformed to (x′,y′) at t = ∞ to give

$x' = ch(\beta t) + (\alpha/\beta)sh(\beta t)$ and $y' = (\sigma/\beta)sh(\beta t)$

thus $x'/y' = \beta/\sigma + \alpha/\sigma = k/2 + \sqrt{(k/2)^2+1}$ for k = 1, 2, 3,….

With information $I = \log_2\{2(x'^2+y'^2)/(x'+y')^2\}$

Fibonacci and related sequences as solutions (k=1):

The Fibonacci sequence and the limiting ratio of adjacent values, called the Golden Mean, has a vast history and literature since Fibonacci's work in the 13th century and culminating in the Fibonacci Quarterly [6] which is devoted exclusively to related investigations. In addition to the elegance of the associated mathematics, the most intriguing aspect is the presence of such numerical sequences in an extremely diverse array of living things.

We will now explore the limiting value of the ratio of $x/y = k/2 + \sqrt{(k/2)^2+1}$ for k = 1, 2, 3,…(which we define as Φ_k). which shows that the information level is stabilized at these fixed values as t approaches infinity beginning with a state of perfect information (1,0). Thus the simplest result of combating diffusion with an uneven growth is with k=1 where we get x′/y′, the ratio of the limited values, to be:

$\Phi_1 = ½ + \sqrt{(1/2)^2+1} = (1+\sqrt{5})/2 = 1.618034$ which is often called the Golden Ratio or Golden Mean and which is the limiting ratio of adjacent Fibonacci numbers.

By taking the matrix $G = (k^2+1, k; k, 1)$ for k=1 one obtains the Fibonacci sequence in higher powers of this matrix as: (2,1;1,1), (5,3;3,2) (13,8;8,5) etc. or 1,1,2,3,5, 8, 13, 21… where a value $n = n_{-1} + n_{-2}$ (i.e. the sum of the last two values).

Let us now explore higher order Fibonacci type sequences with k= 2 to obtain the next order Golden Ratio of $k/2 + \sqrt{(k/2)^2+1}$ with k=2 or

$\Phi_2 = 1 + \sqrt{2} = 2.4142$.

The k=2 associated sequence is given by the powers of (5,2;2,1) which give the sequence 0,1, 2, 5, 12, 29, 70, 169,… where $n = 2n_{-1} + n_{-2}$ (i.e. twice the last value plus the value before that).

Now for the next level with k=3, we obtain the next order Golden Ratio of

$\Phi_3 = k/2 + \sqrt{(k/2)^2+1} = (3+\sqrt{13})/2 = 3.3027756$.

The k= 3 associated sequence is given by the powers of (10,3;3,1) which give the sequence 0, 1, 3, 10, 33, 109, … where $n = 3n_{-1} + n_{-2}$

Recalling that we defined the kth order Golden Mean as $\Phi_k = k/2 + \sqrt{(k/2)^2+1}$ for k =1,2,3, .. one can easily observe that one can form a right triangle with sides of 1 and k/2 and that this will give a hypotenuse of $\sqrt{(k/2)^2+1}$. Thus the sum of the k/2 side plus the hypotenuse of these triangles so proportioned will give geometrically the exact value of the Golden Means for any value of k relative to the third side with a value of unity. That type of construction for the Fibonacci Golden Mean is well known and is the foundation of multiple geometrical constructions which can now be easily extended to these other Fibonacci-like sequences and ratios.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 May 2025 doi:10.20944/preprints202505.0500.v1

General Discussion of the Associated Generating Functions, and Differential Equations for the Fibonacci Sequences of Different Orders

The general form of interpolating functions for Fibonacci sequences of different order and the associated differential equations have already been shown to be

$G(\varepsilon) = \exp \{ 1+\varepsilon(\gamma+\alpha-\sigma), \varepsilon(\sigma+\nu) ; \varepsilon(\sigma-\nu) , 1+\varepsilon(\gamma-\alpha-\sigma) \}$

and when acting upon a vector (x,y) and treating ε as infinitesimal value of t, we can get two coupled first order differential equations:

$dx/dt = (\gamma+\alpha-\sigma) x + (\sigma+\nu)y$ and $dy/dt = (\sigma-\nu)x + (\gamma-\alpha-\sigma)y$.

which can be combined into the single second order equation $d^2x/dt^2 + 2(\sigma-\gamma) dx/dt + (\gamma^2 -2\gamma\sigma -\alpha^2 + \nu^2)x = 0$. Assuming a solution

$x(t) = A e^{\rho t}$ then one finds $\rho = (\sigma-\gamma) \pm \beta$ where $\beta^2 = \alpha^2 + \sigma^2 - \nu^2$

The general solution to this expansion is given by

$G(t) = \exp t \{\gamma+\alpha , \sigma+\nu ; \sigma-\nu , \gamma-\alpha \}$ or

$G(t) = e^{(\sigma-\gamma)t} \{ ch(\beta t) + (\alpha/\beta)sh(\beta t) , ((\sigma+\nu)/\beta)sh(\beta t) ; ((\sigma-\nu)/\beta)sh(\beta t) , ch(\beta t) - (\alpha/\beta)sh(\beta t)\}$

When the parameters are set to the discrete values as $(n,k; k,m) = (k^2+1,k,k,1)$ for any integer value of k, then it follows that as the time evolves, the continuous functions pass through each of the new discrete quantized set of integer values when time itself is an integer multiple of the first value. This follows because that first value of the time gives the first matrix so multiples of that time value gives products of the original matrix. It also follows that the functions that are elements of G(t) are interpolating functions of all of the multiple Fibonacci sequences although it should be noted that these functions pass through alternate values of each of the Fibonacci sequences.

Prediction of a Possible Occurrence of $\Phi_k = k/2 + \sqrt{((k/2)^2+1)}$ for k =1,2,3, in Nature

It occurs to one that these higher order Fibonacci type sequences of numbers as well as generalized Golden Mean values might also be observed in nature just as we find many instances of Fibonacci numbers and the first Golden Mean $\Phi_1 = 1.618$. We can conjecture that this might occur for the reasons like those that initiated this investigation, namely that some systems that experience increasing entropy in nature could stabilize their 'information content' or 'internal order' by utilizing unequal growth from sources and sinks utilizing the Abelian scaling transformations that have been derived. It is probably more unusual to find such higher order generalized Fibonacci type sequences and the next few Golden Means but it could be an important prediction. That is because this prediction of something that could be observed in nature is based almost exclusively on mathematical reasoning and devoid of experimental data. It simply rests on the fact that entropy increases in nature and that living things must be able to sustain their information content and internal structure over time in order to 'survive' not unlike standing waves on a string give $f_n = nf_1$. In fact this is the most fundamental aspect of life: that it is capable of maintaining order. One of the very interesting properties of the Golden Mean Φ_1, is that $\Phi_1 = 1 + 1/ \Phi_1$. Now using the general equation $\Phi_k = k/2 + \sqrt{((k/2)^2+1)}$ for k =1,2,3, one can easily prove that $\Phi_k = k + 1/ \Phi_k$.

Another Generator and Fibonacci Interpolating Functions:

The generator that we have been using, L = (1/2, 1; 1, -1/2) had ν = 0 (no asymmetry in the entropy equilibrium) and γ = 1 (thus γ+σ=0 and no overall exponential growth or decay of the system). We now wish to look at this same matrix but with an additional overall growth factor of γ = 2 giving L = (1,1;1,0). This new L is an interesting Lie algebra generator because when acting as a discrete transformation on (# of old rabbit pairs, # new rabbit pairs), it gives the original Fibonacci numerical sequence in both components when beginning with (0,1) i.e. one new pair of rabbit pairs. The sequential discrete action is (0.1), (1,0), (1,1), (2,1), (3,2), (5,3), …. etc. obeying the original rule that a new pair must age into an older pair for one cycle and that every old pair creates a new pair every cycle. When taken as the generator of a continuous Lie group transformation it gives the interesting set of Fibonacci interpolating functions which are exponential expansion terms multiplied by Fibonacci numbers:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 May 2025 doi:10.20944/preprints202505.0500.v1

$G(t) = e^{tL} = (f''(t) , f'(t); f'(t), f(t))$ where

$f(t) = 1 + 0t + 1\, t^2/2! + 1\, t^3/3! + 2\, t^4/4! + 3\, t^5/5! + 5\, t^6/6! \ldots$

and where f′ and f″ are the first and second derivatives of this function which interpolates the Fibonacci numbers.

By returning to the original form of G(t) in terms of the ch and sh functions one can write this in that previous form. One notes that this Lie generator, L(1) = (1,1;1,0) has the property that $L(1)^2$ = (2,1;1,1) and thus is the 'square root' of the defining Diophantine equation.

Generators of the Generalized Fibonacci Numbers.

One recalls that we found the generalized Diophantine sequential matrices to be of the form R = (k^2+1, k; k, 1) and one can easily show that L(k) = (k,1;1;0) is such that $L(k)^2$=R. This suggests that this Lie generator will provide the expanded forms of the Fibonacci generalized functions. One notes that this L(k) for k=1 gives the previous traditional Fibonacci sequence where each number is the sum of the two previous values. This also follows from the fact that this L(k) gives a sequence where each generalized Fibonacci number is equal to k * the previous value plus the second previous value (ie F(n) = k*F(n-1) + F(n-2)). We can thus generalize our previous result as

$G(t) = e^{tL(k)} = (f''_k(t) , f'_k(t); f'_k(t), f_k(t)$ where now

$f_k(t) = 1 + 0t + 1\, t^2/2! + k\, t^3/3! + (k^2+1)t^4/4! + (k^3+2k)t^5/5! + (k^4+3k^2+1)t^6/6! \ldots$ and where again f″ and f′ represent the second and first derivative of f.

Linear Transformations that Include the Source and Sink of Order:

So far we have only looked at a two component system with internal diffusion (x_1, x_2) and we have treated the 'source' and 'sink' of energy, money, some entity or order as external to this system. Here we close the system by adding a source component, y_1, (the sun, a bank, or some source of 'order') and also a 'sink' component, y_2 for that same entity. Thus now we have a closed system that conserves the entity with transformations on the four dimensional vector (y_1, y_2, x_1, x_2). The four-by-four infinitesimal transformation that does this is

$$L = (1, 0, -\alpha, 0 ; 0, 1, 0, +\alpha ; 0, 0, (1+\alpha-\sigma), +\sigma; 0, 0, +\sigma, (1-\alpha-\sigma)).$$

One can verify that the sum of each column is equal to one and thus this is a Markov type transformation. The transformation on (x_1, x_2) is the same as before but recall that we 'infused' a fraction of x_1, α, into x_1 and removed a fraction α of x_2 from x_2. While the fractions were the same, they were fractions of two different numbers and thus the quantities were not equal. In the new matrix transformation, the fraction α of x_2 that was removed from x_2 is now transferred into (the sink of the order) y_2 thus conserving that quantity. That action is still in keeping with a true Markov transformation because a positive fraction of one component (x_2) is transferred to another component (y_2). However, the transformation is infusing x_1 at a positive rate, (+α), proportional to x_1 itself which, when exponentiated leads to exponential growth of x_1. That infused substance here comes from y_1 and thus y_1 loses substance at a rate proportional to x_1. If x_1 is larger than y_1 then this will result in y_1 becoming negative and thus this transformation is outside of the Markov Monoid and is in the part of the Markov type transformations which preserve the sum of components but not their positive definiteness. So this part of the transformation must rely on the fact that the 'source of entity', y_1, must be so large that during the time of action of the transformation, that y_1 will not be used up and go negative due to the exponential growth of x_1. This is not a problem if the resource y1 is many orders of magnitude larger than the subsystem (x1, x_2). An analogy would be the sun as a source of energy, y_1, and the earth (or empty space) as the sink y_2. Thus also at some point, the subsystem that is maintaining the information level (x_1, x_2) must transfer its information content to a much smaller system (z_1, z_2) while maintaining the ratio of the two components with $z_1/z_2 = x_1/x_2$. Thus, the original system must cease to exist at some point and must reproduce, i.e. pass its 'information content' to a new small system that can then exponentially grow.

 doi:10.20944/preprints202505.0500.v1

Conclusions:

Within the framework of the continuous general linear (Lie) group acting just on a two dimensional space we have studied how the Abelian group of diagonal scaling transformations, acting unequally upon the components, can counter the loss of information and thus the natural increase in entropy that obtains from different levels (k) of continuous Markov transformations: G(t) = $e^{t(\gamma+\alpha,\ \sigma+\nu;\ \sigma-\nu,\ \gamma-\alpha)}$. We concentrated on cases where the entropy equilibrium was equal in both components and thus ν=0. We sought to what extent information would be retained if by different actions of the scaling transformation via the parameter α and we found that any non-zero value altered the equilibrium and retained some information. The interesting result followed when we asked what integer (Diophantine) type solutions might exist and we found an infinite sequence based upon an integer k = 1,2,3,… where k=1 reproduced the Fibonacci sequence and other results such as the associated interpolating functions, Golden Ratio, and geometric constructions. The natural conjecture is that perhaps the presence of the observation of the Fibonacci sequence in nature arises from an attempt to counter increasing disorder and loss of information by utilizing an unequal source and sink of entity such as energy. We of course know that this happens with all living things but is this a possible source of the Fibonacci sequence in nature. Based upon this Lie algebra decomposition of the general linear transformations, we were able to consider higher order levels of entropy indicated by k = 1, 2, 3,… and ask what Diophantine (integer) solutions issue. We found solutions for each value of k and the associated generalized Fibonacci sequence, generating and interpolating functions, generalized golden means, and geometric constructions. These generalizations give a much deeper insight into the Fibonacci numbers and the related mathematics.

The most interesting conjecture from this work is that if it is true that the Fibonacci sequence is a methodology of retaining order in living systems, then one might be able to observe these higher order generalized sequences, their geometric constructs in terms of elemental triangular ratios, and the generalized values of the golden means. If these could be observed it would be very remarkable that this rather pure mathematical model based upon the concept of entropy and information retention, and without any data input from nature, could predict something observable in the physical world.

It is obvious that although this investigation has only looked at a two component system, similar to a single bit of information as stored in a binary form, such elementary systems could be collected to represent complexity at any level as with molecular structures and vast sequences of coded bits of magnetic bits of information. This two node network is easily generalized and very complex systems can be constructed from such elemental systems allowing them to maintain any possible level of information and thus order and structure. The author has been able to show that every possible network [7] among nodes (as represented by non-negative non-diagonal values of a square connection matrix C_{ij}) is isomorphic to a Markov monoid. Even more general networks allow for the Abelian diagonal transformations as discussed here. Thus this simple exchange between the x and y coordinates could be thought of as an exchange between any given two nodes of a more complex network and thus order could be maintained in a network of any complexity by appropriately 'feeding substance' to nodes at specific rates to counter entropy of the network diffusion. The organizational structure of such networks could represent interesting problems in complex systems that are more solvable due to the underlying coupled linear systems here represented. Thus this investigation is relevant to other investigations in network theory. That is even more valid since social networks form and expand in order to increase the survival of the system. We are currently studying information retention, in the presence of entropy, in larger networks.

References:

1. Joseph E. Johnson, Position Operators and Proper Time in Relativistic Quantum Mechanics; Phys. Rev. Vol 181 1755-1764 (1969)
2. Mathew D. Schwartz, Quantum Field Theory and the Standard Model; Cambridge University Press. (2014)

 doi:10.20944/preprints202505.0500.v1

3. Joseph E. Johnson, General Relativity Framed as a Generalized Lie Algebra; Preprint July 17 2024. ASG.sc.edu
4. Joseph E. Johnson, Markov-type Lie Groups in GL(n,R); Journal of Math. Phys. 26 (2), 252, 1985
5. Stephene Douady and Yves Couder, Growth Patterns in Physical Sciences and Biology; Edited by Juan M. Garcia-Ruiz et al. Plenum Press (1993)
6. The Fibonacci Quarterly and Web page: http://www.fq.math.ca/
7. Joseph E. Johnson; Networks, Markov Lie Monoids, and Generalized Entropy; Computer Networks Security – Third International Workshop on Mathematical Methods, Models, and Architectures for Computer Network Security, Proceedings, 129-135, St. Petersburg, Russia, (September 2005),