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Article

Characterization of Positive Invariance of Quadratic Convex Sets for Discrete Systems Using Optimization Approaches

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Abstract: Positive invariant set is an important concept of dynamic systems. The purpose of this paper is to study the sufficient and necessary conditions that the set of ellipsoids and the Lorenz cone are positive invariant sets of discrete time systems. By means of nonlinear programming and induced norm, the problem of positive invariance is formulated as an optimization problem, and the equivalent dual form optimization is also presented using the dual optimization method. Our results provide more alternative methods for determining the positive invariance of quadratic form convex sets from the point of view of optimization and dual optimization. The effectiveness of this method is demonstrated by numerical examples.

Keywords: discrete-time dynamic systems; positive invariance; ellipsoid; Lorenz cone; optimization

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1. Introduction

Positive invariant set is an important concept of dynamic systems. As long as the initial state and the subsequent trajectory of the system are always in a certain positive invariant set, the state quantity of the system can be guaranteed to remain in the positive invariant set. Due to its good properties, positive invariant sets play an important role in the study of system stability analysis and feedback controller design [1–5]. Lyapunov stability theory provides theoretical support to the study the stability of dynamic systems. Bitsoris firstly proposed in [6] the sufficient and necessary conditions for the polyhedra set to be the positive invariant set of a linear discrete system. Ellipsoids and Lorenz cones have also been studied extensively as classical convex sets, but both of them have quadratic forms and the Lorenz cone itself has a second constraint, which makes the study of positive invariance of them more difficult than that of polyhedral sets. The Riccati equation approach is proposed in [7] to be the ellipsoidal set of a linear discrete time system. The maximum invariant ellipsoid for discrete time systems is obtained in [8,9] using linear matrix inequalities and bilinear matrix inequalities. A special class of Lorenz cones is constructed in [10] using the Dickin ellipsoid and some hyperplanes, and the invariant cone of the given system is studied. The ellipsoidal positive invariant set of the Lorenz system has been estimated in [11] for all positive values of the parameters of the Lorenz system, and the minimum volume value of the ellipsoid has been obtained. There are many ways to determine that a set is a positive invariant set of the dynamic system. The Lyapunov function is a classical method for studying the stability of systems, and a new method for solving the positive invariant set of Lorenz chaotic systems was obtained in [12] by constructing the Lyapunov function. The problem of finite-time stability of closed invariant sets of a class of nonlinear systems is discussed in [13] by using Lyapunov functions. [14] is an excellent review paper about the conditions for positive invariance, in an algebraic perspective the sufficient and necessary conditions for ellipsoidal sets to be positive invariant sets of linear systems. [15] investigated sufficient conditions for the existence of robust positive invariant sets for switching systems with average dwell time on the basis of a novel sequence-based technique. In [16], a sufficient and necessary condition for an ordered class of sets to

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be a positive invariant set for nonlinear discrete time systems is proposed from the point of view of the existence of monotone mappings. The condition for ellipsoidal invariance is proposed in [17,18] by the method of linear matrix inequality (LMI), and the optimal solution of the LMI problem is combined to determine whether the maximum ellipsoid is obtained. The S-procedure is an effective method to study the positive invariance conditions for quadratic convex sets, and sufficient and necessary conditions for ellipsoidal sets and Lorenz cones to be positive invariant sets of linear systems are given in [19] based on Lyapunov stability and S-procedure, but the invariance conditions for Lorenz cones are more complicated. The nonlinear programming techniques in [20,21] provide new alternative directions for the invariance condition. The problem of estimating the maximum robust invariant set for discrete time nonlinear regenerative systems in an optimal control framework is considered in [22]. The study of polyhedral sets positive invariance by optimization is investigated in [23,24]. The ellipsoid and Lorenz cone invariance condition in a nonconvex optimization form for continuous time systems is given in [25], and the existence problem of the solution is discussed in conjunction with KKT theorem.

Although there have been many methods to verify that a convex set is a positive invariant set of a dynamic system, most of them are polyhedral sets and for linear dynamic systems, there are few results for a quadratic convex set like the Lorenz cone which itself has a second constraint. The main contribution of this paper is the positive invariance condition of ellipsoids and Lorenz cones by virtue of optimization approach and dual optimization. For convex sets of complex quadratic forms like Lorenz cones three sufficient and necessary conditions to the positive invarianance of a discrete time system are proposed, which has not been studied before as far as we know. The proposed Lagrange dual and Wolfe dual optimization methods in [26,27] can simplify the feasible domain of the primal problem and make the verification process simpler. The method proposed in this paper establishes a connection among positive invariance conditions, optimization, dual optimization and induced norm, which provides an additional methods for the positive invariance of quadratic convex sets for nonlinear and linear discrete time dynamic systems.

The rest of this paper is orgnized as follows: Section 2 provides some preparatory knowledge and definitions. In Section 3, the sufficient and necessary conditions for ellipsoids positive invariance of nonlinear and linear discrete time systems are studied respectively. In Section 4, the positive invariance condition of Lorenz cone is studied. Illustrative numerical example is given in Section 5. Conclusions of this paper are summarized in Section 6.

Notations. $x_k, x_{k+1} \in R^n$ denotes the state vector, where $k \in \mathbb{N} \cup \{0\}$. The set of real numbers is given by R. R^n represents a column vector in dimension $n \times 1$. $R^{n\times n}$ represents a real square matrix of $n \times n$ dimension. $Q \succ 0$ means Q is a positive definite matrix.

2. Mathematical Preliminaries

2.1. Discrete-time dynamic systems

In this paper, we mainly consider discrete time dynamic systems, and the forms of linear and nonlinear discrete-time dynamic systems are given in the following:

$$x_{k+1} = Ax_k. \tag{1}$$

$$x_{k+1} = f_d(x_k). \tag{2}$$

where, A is a n by n dimensional matrix. $x_k, x_{k+1} \in R^n$, denotes the state vector, where $k \in \mathbb{N} \cup \{0\}$. $f_d(x_k)$ denotes a continuous differentiable function on $R^n \to R^n$.

Definition 1 (Positively invariant set)**.** The set D *is an positively invariant set of discrete time systems if and only if* $x_k \in D$ *implies* $x_{k+1} \in D$ *for all* $k \in N$.

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2.2. Convex sets

In this paper, we mainly study classical convex sets with quadratic forms, namely, ellipsoidal sets and Lorenz cones. The ellipsoidal set is defined as

$$S = \{x \in R^n \mid x^T Q x \leq 1\}, \tag{3}$$

where $Q \in R^{n \times n}$, and $Q \succ 0$. The ellipsoidal set is also often writen as an unit ball in the form of a quadratic norm, namely

$$S(Q, 1) = \{x \in R^n \mid \sqrt{x^T Q x} \leq 1\}, \tag{4}$$

that is

$$\|x\|_Q = \sqrt{x^T Q x}. \tag{5}$$

Any ellipsoid whose center is not at the origin can be transformed into an ellipsoid whose center is at the origin. Therefore, this paper discusses the invariance condition for the ellipsoidal set whose center is at the origin.
Lorenz cone is defined as

$$S_L = \{x \in R^n \mid x^T P x \leq 0, x^T P u_n \leq 0\}, \tag{6}$$

where $P \in R^{n \times n}$ is a symmetric nonsingular matrix with only one negative eigenvalue λ_n. u_n is the eigenvector corresponding to the negative eigenvalue λ_n.

2.3. Lagrange function

For optimization problem in the following form:

$$\begin{aligned} min \quad & f(x) \\ s.t \quad & g_i(x) \leq 0, \quad i = 1, 2, \ldots, m, \\ & y_j(x) = 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{7}$$

Define the Lagrange function as follows:

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{n} \mu_j y_j(x). \tag{8}$$

where λ_i, μ_j is called the Lagrange operator and required that $\lambda_i \geq 0, i = 1, 2, ..., m$.

2.4. Wolfe dual theory

Let $f(x)$ be a convex differentiable function with respect to $x \in R^n$, $g_i(x)$ is a differentiable convex function. Denote the gradient of the function $g(x)$ by $\nabla g(x)$, i.e. $\nabla g(x) = (\frac{\partial g(x)}{\partial x_1}, \frac{\partial g(x)}{\partial x_2}, \cdots, \frac{\partial g(x)}{\partial x_n})^T$. The primal problem is in the following form:

$$\begin{aligned} min \quad & f(x) \\ s.t \quad & g_i(x) \leq 0, \quad i = 1, 2, ..., m. \end{aligned} \tag{9}$$

The Wolfe dual form is:

$$\begin{aligned} max \quad & f(x) + \sum_{i=1}^{m} c_i g_i(x) \\ s.t \quad & \nabla f(x) = -\sum_{i=1}^{m} c_i \nabla g_i(x), \quad c_i \geq 0. \end{aligned} \tag{10}$$

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2.5. Slater condition

For the convex programming problem $min\{f(x) \mid g(x) \leq 0, h(x) = 0, x \in R^n\}$, if there exists a feasible point $\bar{x}$ such that $g(\bar{x}) < 0$, the programming problem is said to satisfy the Slater constraint qualification, which is also known as the Slater condition.

3. Invariance conditions for ellipsoids

In this section, we study the necessary and sufficient conditions for ellipsoids to be positive invariant sets of discrete time systems. By virtue of Lagrange duality and Wolfe duality, three equivalent optimization models for invariance conditions of nonlinear discrete systems are given. The case of $f_d(x_k) = Ax_k$ and the corresponding invariance conditions are discussed.

3.1. Formulation of positive invariance conditions

The invariance conditions for discrete time systems include linear and nonlinear systems are discussed respectively in this section.

Theorem 1. *For the ellipsoidal set (3) and nonlinear discrete time system (2), the sufficient and necessary condition for the ellipsoidal set to be a positive invariant set of the nonlinear discrete time system is that the optimal value of the following optimization problem is nonnegative.*

$$\begin{aligned} \min_x \quad & 1 - f_d^T(x) Q f_d(x) \\ s.t \quad & x^T Q x - 1 \leq 0. \end{aligned} \tag{11}$$

Proof. The set S is a positive invariant set of the system (2) if and only if $x_k \in S$, and $x_{k+1} \in S$. It is necessary to satisfy

$$x_k^T Q x_k - 1 \leq 0, \tag{12}$$

$$1 - f_d(x)^T Q f_d(x) \geq 0, \tag{13}$$

(12) holds implies that (13) also holds. It is necessary that all function values of (13) are greater than or equal to zero, i.e., the minimum value of the (13) is satisfied by being non-negative. Then an optimization is formulated as

$$\begin{aligned} \min_x \quad & 1 - f_d^T(x) Q f_d(x) \\ s.t \quad & x^T Q x - 1 \leq 0. \end{aligned} \tag{14}$$

□

When the system is a linear discrete time system, i.e., with $f_d(x) = Ax$, the invariance condition is presented by Theorem 2.

Theorem 2. *The ellipsoid set S is a positive invariant set of the linear discrete time system (1) if and only if the optimal value of the following optimization problem is nonnegative.*

$$\begin{aligned} \min_x \quad & 1 - x^T A^T Q A x \\ s.t \quad & x^T Q x - 1 \leq 0. \end{aligned} \tag{15}$$

Since the proof procedure of Theorem 2 is similar to that of Theorem 1, we do not repeat it.

For ellipsoidal set in quadratic norm form, we apply the induced norm to give sufficient and necessary condition for the ellipsoidal.

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Lemma 1. *A vector norm* $\| \bullet \|$ *on* C^n *is known, and for any square matrix* $M \in C^{n\times n}$*, let* $\| M \| = \sup\limits_{\|x\|=1} \| Mx \|$*, then* M *is said to be the induced norm of the vector norm* $\| x \|$*.*

Theorem 3. *The ellipsoid set* $S(Q,1)$ *is a positive invariant set of the linear discrete time system* $x_{k+1} = Ax_k$ *if and only if the optimal value of this optimization problem below is positive.*

$$\begin{aligned} \min_x \quad & -x^T(A^TQA - Q)x \\ s.t \quad & x^TQx - 1 = 0. \end{aligned} \tag{16}$$

Proof. From the definition of the induced norm in Lemma 1, a sufficient and necessary condition for the ellipsoid set $S(Q,1)$ to be a positive invariant set of a linear discrete time system is that (17) holds for all x.

$$\|A\|_Q = \sup_{\sqrt{x^TQx}=1} \sqrt{(Ax)^TQ(Ax)} < 1. \tag{17}$$

That is,

$$\begin{aligned} (Ax)^TQ(Ax) - 1 < 0 &\Rightarrow x^TA^TQAx - x^TQx < 0, \\ &\Rightarrow x^T(A^TQA - Q)x < 0, \\ &\Rightarrow A^TQA - Q \prec 0. \end{aligned} \tag{18}$$

Transformed (18) into an optimization problem with constraints as

$$\begin{aligned} \min_x \quad & -x^T(A^TQA - Q)x \\ s.t \quad & x^TQx - 1 = 0. \end{aligned} \tag{19}$$

□

From Theorems 1 and 2, we formulate the problem of positive invariance as the optimization problems. In practice, the dual method sometimes can simplify the primal problem, then we also present the Lagrangian dual and Wolfe dual forms of (11), (15) and (16). Note that since the constraint function in (16) is nonconvex, the Wolfe dual is invalid.

3.2. Lagrange dual

The Lagrange dual is a convex optimization problem regardless of whether the primal problem is convex or not, and the dual gives at least a lower bound on the optimal solution of the original problem. In this section, we will consider the Lagrange dual of the primal problem. Theorem 4 gives positive invariance conditions in the Lagrange dual form of the ellipsoidal set for the nonlinear dynamic systems.

Theorem 4. *Consider the nonlinear discrete time system* $x_{k+1} = f_d(x_k)$ *and the ellipsoidal set be* $S = \{x \in R^n \mid x^TQx \le 1\}$*, where* $Q \in R^{n\times n}$*, and* $Q \succ 0$*. Let* $1 - f_d^T(x)Qf_d(x)$ *be a continuous differentiable function with respect to* x*. The ellipsoid set* S *is a positive invariant set of the nonlinear discrete system* (2) *if and only if there exists* $\lambda \ge 0$*, such that the optimal value of the following optimization problem is non-negative.*

$$\max_{\lambda\ge 0}\min_x \quad 1 - f_d^T(x)Qf_d(x) + \lambda(x^TQx - 1) \tag{20}$$

Proof. Consider (11) as the primal problem and set it as $P(x)$. Let the Lagrange multiplier be λ, $\lambda \ge 0$, then the Lagrange function is

$$L(x,\lambda) = 1 - f_d^T(x)Qf_d(x) + \lambda(x^TQx - 1). \tag{21}$$

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Then we have

$$\max_{\lambda\geq 0} \quad 1 - f_d^T Q f_d(x) + \lambda(x^T Q x - 1) = \begin{cases} \infty, & otherwise \\ P(x), & x^T Q x \leq 1 \end{cases}$$

Then $\min_{x\in R^n} \max_{\lambda\geq 0} L(x,\lambda)$ is equivalent to the primal problem that satisfies the constraint, i.e.

$$\min_{x\in R^n} \max_{\lambda\geq 0} L(x,\lambda) = \min_{x\in R^n} \{1 - f_d^T Q f_d(x) | x^T Q x - 1 \leq 0\} \tag{22}$$

Then the Lagrange dual of the primal problem (11) is

$$\max_{\lambda\geq 0} \min_{x\in R^n} 1 - f_d^T Q f_d(x) + \lambda(x^T Q x - 1) \tag{23}$$

Since the Lagrange dual satisfies

$$\min_{x\in R^n} \max_{\lambda\geq 0} L(x,\lambda) \geq \max_{\lambda\geq 0} \min_{x\in R^n} L(x,\lambda). \tag{24}$$

As a result, the optimal value of the primal problem must be non-negative when the optimal value of the dual problem (20) is non-negative. □

Remark 1. *Lagrange duals are only useful for dual problems when the optimal value function of the inner optimization problem can be reduced to an analytic formula. Therefore, it is necessary to satisfy in Theorem 4 that* $1 - f_d^T(x) Q f_d(x)$ *is a continuous function differentiable with respect to* x.

Similarly, we give sufficient and necessary conditions for the ellipsoidal set to be a positive invariant set of the linear discrete system.

Theorem 5. *Let the linear discrete system be* $x_{k+1} = Ax_k$ *and the ellipsoidal set be* $S = \{x \in R^n \mid x^T Q x \leq 1\}$, *where* $Q \in R^{n\times n}$, *and* $Q \succ 0$. *Then the ellipsoidal set is a positive invariant set of the linear discrete system if and only if there exists* $\lambda \in [0,1]$ *such that the optimal value of the following optimization problem is non-negative.*

$$\max_{0\leq\lambda\leq 1} \min_{x} \quad 1 - x^T A^T Q A x + \lambda(x^T Q x - 1) \tag{25}$$

Proof. The process of proving Lagrange dual is similar to Theorem 3, here we focus on the range of values of Lagrange multipliers.
Let the Lagrange function of the primal problem be

$$L(x,\lambda) = 1 - x^T A^T Q A x + \lambda(x^T Q x - 1), \quad \lambda \geq 0. \tag{26}$$

When $x = 0$, the positive invariance condition should also satisfy $L(0,\lambda) \geq 0$, i.e.

$$1 - \lambda \geq 0,$$

that is

$$0 \leq \lambda \leq 1. \tag{27}$$

□

Theorem 6. *[19] The ellipsoidal set* S *is a positive invariant set of the linear discrete system* $x_{k+1} = Ax_k$ *if and only if there exists* $\lambda \in [0,1]$, *such that* $A^T Q A - \lambda Q \preceq 0$.

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Proof. From the proof of Theorem 5, it follows that if the ellipsoid set S is a positive invariant set of the linear discrete system (1), then there exists $\lambda \in [0,1]$ such that the optimal value of the optimization problem (25) is nonnegative. Then we have

$$1 - x^T A^T Q A x + \lambda(x^T Q x - 1) \geq 0$$

which means,

$$x^T A^T Q A x - \lambda x^T Q x \leq 1 - \lambda$$

that is,

$$x^T (A^T Q A - \lambda Q) x \leq 0$$

i.e.

$$A^T Q A - \lambda Q \preceq 0.$$

The proof is completed.
□

Remark 2. *Theorem 5 was once proved in [19] using S-procedure, and in this paper we use to give a novel proof from an optimization point of view, showing a direct connection between positive invariant sets and pairwise optimization.*

Theorem 7. *The linear discrete system is $x_{k+1} = Ax_k$, and when the ellipsoid is given by the quadratic norm of the form (4), then the ellipsoid is a positive invariant set of the linear discrete system (1) if and only if there exists $\lambda \leq 0$ such that the optimal value of the following optimization problem is positive.*

$$\max_{\lambda \leq 0} \min_{x} \quad -x^T (A^T Q A - Q) x + \lambda(x^T Q x - 1) \tag{28}$$

Remark 3. *The proof of Theorem 7 is similar to the proof in Theorem 4, and we omit the proof here.*

By comparing the primal optimization problem with the optimization problem after Lagrange dual, we can see that the feasible domain of the dual programming problem is simpler than that of the primal problem. Next, we discuss the case when the primal problem is convex optimization problem, at which time we apply the Wolfe dual approach to the primal optimization problem.

3.3. Wolfe dual forms

When the objective function is convex and the constraint is also convex, the primal optimization problem can be transformed into the form of Wolfe dual. We study the Wolfe dual form for the ellipsoid positive invariance set for the discrete time systems.

Remark 4. *Since the primal problem is a convex optimization problem and the Slater's condition is satisfied, the strong duality theorem holds. Therefore, the optimal value of the dual problem is theoretically equivalent to the primal problem. Then, we can use the positivity or negativity of the optimal value result to determine whether the ellipsoidal set is a positive invariant set of the discrete time systems or not.*

Theorem 8. *The ellipsoidal set $S = \{x \in R^n \mid x^T Q x \leq 1\}$, where $Q \in R^{n \times n}$, and $Q \succ 0$. The nonlinear discrete system is given by (2). Let $1 - f_d^T(x) Q f_d(x)$ be a convex function differentiable with respect to x and*

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$x^TQx - 1$ be a convex function, then the ellipsoidal set S is a positive invariant set of nonlinear discrete systems (2) if and only if there exists $\lambda \geq 0$, such that the optimal value of the following problem is nonnegative.

$$\begin{aligned} \max_{x\in R^n} \quad & 1 - f_d^T Q f_d(x) + \lambda(x^TQx - 1), \\ s.t \quad & \nabla(f_d^T Q f_d(x)) = \lambda \nabla(x^TQx). \end{aligned} \tag{29}$$

Proof. From the primal problem (11), both the objective function and the constraint are convex functions then Wolfe's dual theorem can be applied, i.e.

$$\begin{aligned} \max_{x\in R^n} \quad & 1 - f_d^T Q f_d(x) + \lambda(x^TQx - 1), \\ s.t \quad & \nabla(f_d^T Q f_d(x)) = \lambda \nabla(x^TQx). \end{aligned} \tag{30}$$

□

Theorem 9. *Let the linear discrete system be (1) and the ellipsoid set $S = \{x \in R^n \mid x^TQx \leq 1\}$, where $Q \in R^{n\times n}$, and $Q \succ 0$. Assume $1 - x^TA^TQAx$ be a convex function and $x^TQx - 1$ be a convex function, then the ellipsoidal set S is a positive invariant set of linear discrete systems (1) if and only if there exists $\lambda \in [0, 1]$, such that the optimal value of the following problem is nonnegative.*

$$\begin{aligned} \max_{x\in R^n} \quad & 1 - x^TA^TQAx + \lambda(x^TQx - 1), \\ s.t \quad & \lambda Q - A^TQA = 0. \end{aligned} \tag{31}$$

Proof. From the assumptions, it is clear that (15) is a convex optimization problem, so Wolfe's dual theory can be applied. In particular, when $x = 0$, there is $1 - \lambda \geq 0$, i.e., $0 \leq \lambda \leq 1$. The Wolfe dual of (15) is

$$\begin{aligned} \max_{x\in R^n} \quad & 1 - x^TA^TQAx + \lambda(x^TQx - 1), \\ s.t \quad & \lambda Q - A^TQA = 0. \end{aligned}$$

□

Remark 5. *By comparing Theorems 4 and 5 with Theorems 8 and 9, it can be seen that the Wolfe dual applies to the case where the primal problem is a convex optimization problem, while the Lagrange dual applies to the more general case.*

4. Positive invariance conditions for Lorenz cone

Lorenz cone is also one of the classical convex sets with quadratic form, but the study of its positive invariance is more complicated because Lorenz cone contains the second constraint itself. The positive invariance condition for Lorenz cone given in this paper is simpler. Similar to the ellipsoid, we give the following three equivalent sufficient and necessary conditions on the Lorenz cone positive invariance for nonlinear discrete time systems.

Theorem 10. *Lorenz cone (6) is a positive invariant set for a nonlinear discrete time system (2) if and only if the optimal value of the following optimization problem is nonnegative.*

$$\begin{aligned} \min_{x\in R^n} \quad & -f_d(x)^TPf_d(x), \\ s.t \quad & f_d(x)^TPu_n \leq 0 \\ & x^TPx \leq 0 \\ & x^TPu_n \leq 0. \end{aligned} \tag{32}$$

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If $f_d(x) = Ax$, the Lorenz cone S_L is a positive invariant set of linear discrete time systems, then the optimal value of the following optimization problem needs to be nonnegative.

$$\begin{aligned} \min_{x\in R^n} \quad & -x^T A^T PAx, \\ s.t \quad & x^T A^T Pu_n \leq 0 \\ & x^T Px \leq 0 \\ & x^T Pu_n \leq 0. \end{aligned} \tag{33}$$

Proof. First, if the Lorenz cone is a positive invariant set of the nonlinear discrete system (2) if and only if $x_k \in S_L$, and $x_{k+1} \in S_L$. i.e., it needs to satisfy

$$x_k^T Px_k \leq 0, x_k^T Pu_n \leq 0,$$

and

$$f(x_k)^T Pf(x_k) \leq 0, f(x_k)^T Pu_n \leq 0.$$

Translated into an optimization problem, i.e.

$$\begin{aligned} \min_{x\in R^n} \quad & -f_d(x)^T Pf_d(x) \\ s.t \quad & f_d(x)^T Pu_n \leq 0, \\ & x^T Px \leq 0, \\ & x^T Pu_n \leq 0. \end{aligned}$$

When the discrete time system is linear, i.e., when $f(x_k) = Ax$, the positive invariance condition is

$$\begin{aligned} \min_{x\in R^n} \quad & -x^T A^T PAx \\ s.t \quad & x^T A^T Pu_n \leq 0, \\ & x^T Px \leq 0, \\ & x^T Pu_n \leq 0. \end{aligned}$$

□

Next, the optimization problem in (32) and (33) is transformed into its equivalent Lagrange dual optimization form.

Theorem 11. *Consider the nonlinear discrete time systems be $x_{k+1} = f_d(x_k)$ and the Lorenz cone given by (6). Let $-f_d^T(x)Pf_d(x)$ be a continuous differentiable function with respect to x. Then the Lorenz cone S_L is a positive invariant set of the nonlinear discrete system (2) if and only if there exists $\lambda, \mu, \eta \geq 0$, such that the optimal value of the following optimization problem is non-negative.*

$$\max_{\lambda,\mu,\eta\geq 0} \min_{x\in R^n} \quad -f_d^T(x)Pf_d(x) + \lambda(f_d(x)^T Pu_n) + \mu(x^T Px) + \eta(x^T Pu_n) \tag{34}$$

If $f_d(x) = Ax$, the Lorenz cone S_L is a positive invariant set of linear discrete systems, then the optimal value of the following optimization problem needs to be nonnegative.

$$\max_{\lambda,\mu,\eta\geq 0} \min_{x\in R^n} \quad -x^T A^T PAx + \lambda(x^T A^T Pu_n) + \mu(x^T Px) + \eta(x^T Pu_n) \tag{35}$$

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Proof. Taking (32) as the primary problem and introducing the multiplier $\lambda, \mu, \eta \geq 0$, one can write its Lagrange function, i.e.

$$-f_d^T(x)Pf_d(x) + \lambda(f_d(x)^T Pu_n) + \mu(x^T Px) + \eta(x^T Pu_n).$$

Then, let the primary problem be $P(x)$, we have

$$\begin{aligned}&\max_{\lambda,\mu,\eta\geq 0} \quad -f_d^T(x)Pf_d(x) + \lambda(f_d(x)^T Pu_n) + \mu(x^T Px) + \eta(x^T Pu_n)\\ &= \begin{cases} \infty, & otherwise \\ P(x), & f_d(x)^T Pu_n \leq 0, x^T Px \leq 0, x^T Pu_n \leq 0. \end{cases}\end{aligned}$$

Therefore $\min_{x\in R^n} \max_{\lambda,\mu,\eta\geq 0} L(x, \lambda, \mu, \eta)$ is equivalent to (32), and the Lagrange dual of (32) is

$$\max_{\lambda,\mu,\eta\geq 0} \min_{x\in R^n} \quad -x^T A^T PAx + \lambda(x^T A^T Pu_n) + \mu(x^T Px) + \eta(x^T Pu_n)$$

Since the optimal value of the primal problem must be greater than or equal to the optimal value of its Lagrange dual problem, when the optimal value of the Lagrange dual problem is non-negative, the optimal value of the primal problem must be non-negative. When $f(x_k) = Ax$, the Lagrange dual problem of (33) is

$$\max_{\lambda,\mu,\eta\geq 0} \min_{x\in R^n} \quad -x^T A^T PAx + \lambda(x^T A^T Pu_n) + \mu(x^T Px) + \eta(x^T Pu_n).$$

□

When the optimization problem in (32) and (33) is convex optimization, it can be transformed into the equivalent Wolfe dual optimization.

Theorem 12. *The Lorenz cone $S_L = \{x \in R^n \mid x^T Px \leq 0, x^T Pu_n \leq 0\}$, where $P \in R^{n\times n}$. Let (32) be a convex optimization problem then the Lorenz cone S_L is a positive invariant set of nonlinear discrete systems (2) if and only if there exists $\lambda, \mu, \eta \geq 0$ such that the optimal value of the following problem is nonnegative.*

$$\begin{aligned}\max_{x\in R^n} \quad & -f_d^T Pf_d(x) + \lambda(f_d(x)^T Pu_n) + \mu(x^T Px) + \eta(x^T Pu_n),\\ s.t \quad & \nabla(f_d(x)^T Pf_d(x)) = \lambda\nabla(f_d(x)^T Pu_n) + \mu\nabla(x^T Px) + \eta\nabla(x^T Pu_n).\end{aligned} \tag{36}$$

If $f_d(x) = Ax$, when (33) is convex optimization, the Lorenz cone S_L is a positive invariant set of linear discrete systems, if and only if there exists $\lambda, \mu, \eta \geq 0$, such that the optimal value of the following optimization problem needs to be nonnegative.

$$\begin{aligned}\max_{x\in R^n} \quad & -x^T A^T PAx + \lambda(x^T A^T Pu_n) + \mu(x^T Px) + \eta(x^T Pu_n),\\ s.t \quad & 2(A^T PA - \mu P)x = \lambda A^T Pu_n + \eta Pu_n.\end{aligned} \tag{37}$$

The proof of Theorem 12 is similar to the proof of Theorem 8, and we omit it here.

5. Numerical Examples

In this section we present some numerical examples to verify the theorems presented in Section 3 and Section 4. Note that for convenience, we set the initial state of the system for each example to the origin, and the change of the initial state does not affect the final determination of whether it is a positively invariant set or not.

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Example 1. *The ellipsoid set is* $S = \{(x_1^{(k)}, x_2^{(k)}) \mid (x_1^{(k)})^2 + (x_2^{(k)})^2 \leq 1\}$, *the nonlinear discrete system be* $x_1^{(k+1)} = \frac{\sqrt{x_1^{(k)}+x_2^{(k)}}}{2}$, $x_2^{(k+1)} = \frac{\sqrt{x_1^{(k)}-3x_2^{(k)}}}{2}$.

In method 1, using (11) *of Theorem* 1 *to verify the positive invariance of this ellipsoid set. We transform the invariance problem into an optimization problem, i.e.*

$$\begin{aligned} \min_x \quad & 1 - f_d^T(x) Q f_d(x) \\ s.t \quad & x^T Q x \leq 1. \end{aligned}$$

First, we simplify the objective function.

$$\begin{aligned} & \min_x \quad 1 - \Big(\frac{\sqrt{x_1^{(k)} + x_2^{(k)}}}{2} \quad \frac{\sqrt{x_1^{(k)} - 3x_2^{(k)}}}{2}\Big) \\ = & \min_x \quad 1 - \Big[\frac{x_1^{(k)} + x_2^{(k)}}{4} + \frac{x_1^{(k)} - 3x_2^{(k)}}{4}\Big] \\ = & \min_x \quad 1 - \Big[\frac{x_1^{(k)} - x_2^{(k)}}{2}\Big] \\ = & \min_x \quad 1 - \frac{x_1^{(k)}}{2} + \frac{x_2^{(k)}}{2}. \end{aligned}$$

Next, we simplify the constraint.

$$\begin{aligned} & \begin{bmatrix} x_1^{(k)} & x_2^{(k)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \\ = \quad & (x_1^{(k)})^2 + (x_2^{(k)})^2 \leq 1. \end{aligned}$$

The simplified optimization problem is

$$\begin{aligned} \min_x \quad & 1 - \frac{x_1^{(k)}}{2} + \frac{x_2^{(k)}}{2} \\ s.t \quad & (x_1^{(k)})^2 + (x_2^{(k)})^2 \leq 1. \end{aligned}$$

Taking the initial state as $[0; 0]$, *the optimal value of this optimization problem is obtained by MATLAB as* $0.2929 > 0$, *then the ellipsoidal set S is a positive invariant set of the nonlinear discrete system.*

In method 2, apply (20) *in Theorem* 4 *to verify whether the ellipsoid set is a positive invariant set of this nonlinear discrete system. The optimization problem in method 1 is transformed into its Lagrange dual optimization form, i.e*

$$\begin{aligned} & \max_{\lambda \geq 0} \min_x \quad 1 - f_d^T(x) Q f_d(x) + \lambda(x^T Q x - 1) \\ \Rightarrow & \max_{\lambda \geq 0} \min_x \quad 1 - \frac{x_1^{(k)}}{2} + \frac{x_2^{(k)}}{2} + \lambda[(x_1^{(k)})^2 + (x_2^{(k)})^2 - 1] \end{aligned}$$

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First, let the inner optimization problem be $g(x_1^{(k)}, x_2^{(k)})$*, and we take the partial derivatives of* $x_1^{(k)}$*,* $x_2^{(k)}$ *in* $g(x_1^{(k)}, x_2^{(k)})$ *to be equal to zero, respectively, i.e*

$$\frac{\partial g}{x_1^{(k)}} = -\frac{1}{2} + 2\lambda x_1^{(k)} = 0 \Rightarrow x_1^{(k)} = \frac{1}{4\lambda},$$
$$\frac{\partial g}{x_2^{(k)}} = \frac{1}{2} + 2\lambda x_2^{(k)} = 0 \Rightarrow x_2^{(k)} = -\frac{1}{4\lambda}.$$

By substituting $x_1^{(k)}$ *and* $x_2^{(k)}$ *into the optimization function, we can get the function which is only related to* λ*. When* $\lambda = 0.5$*, the optimal value of the function is* $1 > 0$*. Therefore, the ellipsoid set is the positive invariant set of the nonlinear discrete system.*

In method 3, Since the optimization problem in Method 1 is convex optimization problem, we can apply (29) in Theorem 8 to solve this problem. That is, we need to find u satisfying $u > 0$ *such that the optimal value of the objective function is nonnegative.*

$$\max_{x\in R^n} \quad 1 - f_d^T Q f_d(x) + \lambda(x^T Q x - 1),$$
$$s.t \quad \nabla(f_d^T Q f_d(x)) = \lambda \nabla(x^T Q x).$$

In this example, we let $\lambda = 0.25$ *and substitute the objective function to obtain the value* $0.25 > 0$*. Therefore, the ellipsoidal set is the positive invariant set of this nonlinear discrete system.*

Example 2. *Let the linear discrete system* $x_{k+1} = Ax_k$*, where* $A = \begin{bmatrix} 0 & -3.2 \\ -0.1 & 0.4 \end{bmatrix}$ *and the set of ellipsoids is* $S = \{(x_1^{(k)})^2 + (x_2^{(k)})^2 \le 1$*. Next, we apply (15) in Theorem 2 to verify it. Substituting the data in this example into (15) yields an optimization problem.*

$$\min_x \quad 1 - \begin{bmatrix} x_1^{(k)} & x_2^{(k)} \end{bmatrix} \begin{bmatrix} 0 & -0.1 \\ -3.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3.2 \\ -0.1 & 0.4 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}$$
$$s.t \quad (x_1^{(k)})^2 + (x_2^{(k)})^2 - 1 \le 0.$$

First, the objective function is simplified.

$$\begin{aligned} & 1 - \begin{bmatrix} x_1^{(k)} & x_2^{(k)} \end{bmatrix} \begin{bmatrix} 0 & -0.1 \\ -3.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3.2 \\ -0.1 & 0.4 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \\ = & 1 - \begin{bmatrix} 0.01x_1^{(k)} - 0.04x_2^{(k)} & -0.04x_1^{(k)} + 10.4x_2^{(k)} \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \\ = & 1 - 0.01(x_1^{(k)})^2 + 0.08x_1^{(k)}x_2^{(k)} - 10.4(x_2^{(k)})^2. \end{aligned}$$

The constraint function is $(x_1^{(k)})^2 + (x_2^{(k)})^2 - 1 \le 0$*. The optimization function is finally reduced to*

$$\min_x \quad 1 - 0.01(x_1^{(k)})^2 + 0.08x_1^{(k)}x_2^{(k)} - 10.4(x_2^{(k)})^2,$$
$$s.t \quad (x_1^{(k)})^2 + (x_2^{(k)})^2 - 1 \le 0.$$

Taking the initial state as $[0; 0]$ *and the optimal value of the optimization function as* $-9.4002 < 0$*, then the ellipsoidal set is not a positive invariant set of this linear discrete system.*

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Example 3. *The linear discrete system is $x_{k+1} = Ax_k$, where $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$. And the ellipsoid set is $(x_1^{(k)})^2 + (x_2^{(k)})^2 \leq 1$.*

In method 1, using (15) in Theorem 2, the problem of verifying that the ellipsoid set is a positive invariant set of a linear discrete system is transformed into an optimization problem.

$$\begin{aligned} \min_x \quad & 1 - x^T A^T Q A x \\ s.t \quad & x^T Q x - 1 \leq 0. \end{aligned}$$

Where

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, the matrices A, Q are substituted into the objective function to obtain

$$\begin{aligned} \min_x \quad & \begin{bmatrix} x_1^{(k)} & x_2^{(k)} \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \\ = \quad & 1 - 0.25(x_1^{(k)})^2 - 0.25(x_2^{(k)})^2. \end{aligned}$$

Then the optimization function is

$$\begin{aligned} \min_x \quad & 1 - 0.25(x_1^{(k)})^2 - 0.25(x_2^{(k)})^2 \\ s.t \quad & (x_1^{(k)})^2 + (x_2^{(k)})^2 - 1 \leq 0. \end{aligned}$$

The initial state is set as $rand(2,1)$ in MATLAB, and the optimal value of the objective function is $0.7500000998413878 > 0$. Therefore, the ellipsoid set S is the positive invariant set of the linear discrete system.

In Method 2, the ellipsoid set is written in the form of (4), and the invariance condition of the ellipsoid set is transformed into an optimization problem by using (16) in Theorem 3.

$$\begin{aligned} \min_x \quad & -x^T(A^T Q A - Q)x \\ s.t \quad & x^T Q x - 1 = 0. \end{aligned}$$

Where

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the matrix A and Q are substituted and simplified to obtain the following optimization problem, i.e

$$\begin{aligned} \min_x \quad & -0.75(x_1^{(k)})^2 - 0.75(x_2^{(k)})^2 \\ s.t \quad & (x_1^{(k)})^2 + (x_2^{(k)})^2 - 1 = 0. \end{aligned}$$

The initial state is set as $rand(2,1)$ in MATLAB, and the optimal value of the objective function is $0.750000000000004 > 0$. Therefore, the ellipsoid set S is the positive invariant set of the linear discrete system.

In method 3, using (25) in Theorem 5 to determine whether the ellipsoid set is a positive invariant set of the linear discrete system.

$$\max_{0 \leq \lambda \leq 1} \min_x \quad 1 - x^T A^T Q A x + \lambda(x^T Q x - 1)$$

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That is

$$\max_{0\le\lambda\le1}\min_{x}\quad 1-0.25(x_1^{(k)})^2-0.25(x_2^{(k)})^2+\lambda((x_1^{(k)})^2+(x_2^{(k)})^2-1)$$

Let the inner optimization function be $g(x_1^{(k)},(x_2^{(k)})$*, and we take the partial derivatives with respect to* $x_1^{(k)},x_2^{(k)}$ *to be equal to zero respectively, i.e*

$$\frac{\partial g}{x_1^{(k)}}=2(\lambda-0.25)x_1^{(k)}=0,$$
$$\frac{\partial g}{x_2^{(k)}}=2(\lambda-0.25)x_2^{(k)}=0.$$

When $x_1=x_2=0$*, the optimal value is* $1>0$ *by substituting it into the objective function. When* $x_1,x_2\neq0$*,* $\lambda=0.25$ *can be obtained, and the optimal value is* $0.75>0$ *by substituting it into the objective function. Therefore, the set of ellipsoids is the positive invariant set of this linear discrete system.*

In method 4, apply (28) in Theorem 7 to determine the positive invariance of the ellipsoid set. That is, we need to find $\lambda<0$*, such that the optimal value of the optimization problem is positive. Substituting the data in this example into (28) and simplifying gives*

$$\max_{\lambda\le0}\min_{x}\quad 0.75(x_1^{(k)})^2+0.75(x_2^{(k)})^2+\lambda((x_1^{(k)})^2+(x_2^{(k)})^2)-1)$$

The optimal value is 0.75 *when* $\lambda=0.75$*. Therefore, this ellipsoidal set is the positive invariant set of this linear discrete system.*

In method 5, apply (29) in Theorem 8 to determine the positive invariance of the ellipsoid set. We need to verify whether we can find a λ *satisfying the conditions such that the optimal value of the optimization problem is nonnegative.*

$$\begin{aligned}\max_{x\in R^n}\quad & 1-x^TA^TQAx+\lambda(x^TQx-1),\\ s.t\quad & \lambda Q-A^TQA=0.\end{aligned}$$

Where

$$A=\begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix},Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

The constraint condition is satisfied when $\lambda=0.25$*, and the value of* $0.75>0$ *can be obtained by substituting* λ *into the objective function. Therefore, the ellipsoid set is the positive invariant set of the linear discrete system.*

Example 4. *The Lorenz cone is represented by* $S_L=\{(x_1^{(k)})^2-(x_2^{(k)})^2\le0,x_2^{(k)}\ge0\}$*, and the nonlinear discrete system is* $x_1^{(k+1)}=[-(x_1^{(k)})^2+2x_2^{(k)}-x_1^{(k)}]^{\frac{1}{2}},x_2^{(k+1)}=[(x_2^{(k)})^2-x_2^{(k)}-2x_1^{(k)}]^{\frac{1}{2}}$*. In method one, we apply (32) in Theorem 10 to determine whether the Lorenz cone is a positive invariant set of this nonlinear discrete system.*

$$\begin{aligned}\min_{x\in R^n}\quad & -f_d(x)^TPf_d(x),\\ s.t\quad & f_d(x)^TPu_n\le0\\ & x^TPx\le0\\ & x^TPu_n\le0.\end{aligned}$$

Where

$$P=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},u_n=\begin{bmatrix}0\\ 1\end{bmatrix}.$$

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Substituting P, u_n into the optimization problem and simplifying, we obtain

$$\begin{aligned}\min_{x\in R^n} \quad & (x_1^{(k)})^2 + (x_2^{(k)})^2 - x_1^{(k)} - 4x_2^{(k)},\\ s.t \quad & -\sqrt{(x_2^{(k)})^2 - x_2^{(k)} - 2x_1^{(k)}} \le 0\\ & (x_1^{(k)})^2 - (x_2^{(k)})^2 \le 0\\ & -x_2^{(k)} \le 0.\end{aligned}$$

The optimal value is 0*, so the Lorenz cone is the positive invariant set of the nonlinear discrete system.*

In method 2, apply (34) *in Theorem* 11*.*

Since the optimization problem in method 1 is convex optimization, Wolfe dual can be used to solve it. Substituting the data in this example into (34) *and simplifying, we get*

$$\begin{aligned}\max_{x\in R^n} \quad & (x_1^{(k)})^2 + (x_2^{(k)})^2 - x_1^{(k)} - 4x_2^{(k)} + \lambda\left(-\sqrt{(x_2^{(k)})^2 - x_2^{(k)} - 2x_1^{(k)}}\right)\\ & + \mu((x_1^{(k)})^2 - (x_2^{(k)})^2) + \eta(-x_2^{(k)}),\\ s.t \quad & \begin{bmatrix} 2x_1^{(k)} - 1 \\ 2x_2^{(k)} - 4\end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{\sqrt{(x_2^{(k)})^2 - x_2^{(k)} - 2x_1^{(k)}}} \\ -\frac{2x_2^{(k)} - 1}{2\sqrt{(x_2^{(k)})^2 - x_2^{(k)} - 2x_1^{(k)}}}\end{bmatrix} + \mu \begin{bmatrix} 2x_1^{(k)} \\ -2x_2^{(k)}\end{bmatrix} + \eta \begin{bmatrix} 0 \\ -1\end{bmatrix}.\end{aligned}$$

When $\lambda = 0, \mu = 0.75, \eta = 2.5$*, the value of the objective function is* $1.625 > 0$*. So the Lorenz cone is the positive invariant set of this nonlinear discrete system.*

Example 5. *Lorenz cone is* $S_L = \{(x_1^{(k)})^2 + (x_2^{(k)})^2 - (x_3^{(k)})^2 \le 0, (x_3^{(k)}) \ge 0\}$*, linear discrete time system is represented by* $x_{k+1} = Ax_k$*, where* $A = [0.5, 0, 0; 0, 0.5, 0; 0, 0, 1]$*.*

In method 1, apply (33) *in Theorem* 10 *is applied to verify, then the sufficient and necessary condition for the Lorenz cone to be a positive invariant set of this linear discrete-time system is that the optimal value of the following optimization problem is nonnegative, i.e*

$$\begin{aligned}\min_{x\in R^n} \quad & -x^T A^T P A x,\\ s.t \quad & x^T A^T P u_n \le 0\\ & x^T P x \le 0\\ & x^T P u_n \le 0.\end{aligned}$$

Where

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1\end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{bmatrix}, u_n = \begin{bmatrix} 0 \\ 0 \\ 1\end{bmatrix}$$

Substituting the values of A, P, u_n*, into the optimization framework yields the optimal value of* 0*, which is non-negative, then the Lorenz cone is a positive invariant set of this linear discrete system.*

In method 2, apply (37) *in Theorem* 12 *to determine whether the Lorenz cone is a positive invariant set of this linear discrete system. We need to find the parameter* $\lambda, \mu, \eta \ge 0$ *such that the optimal value of the optimization problem is non-negative.*

$$\max_{\lambda,\mu,\eta\ge 0} \min_{x\in R^n} \quad -x^T A^T P A x + \lambda(x^T A^T P u_n) + \mu(x^T P x) + \eta(x^T P u_n)$$

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For the inner optimization function, substituting A, P, u_n, and simplifying it yields

$$\begin{aligned}&\min_{x} \quad g(x_1^{(k)}, x_2^{(k)}, x_3^{(k)})\\ =&\min_{x} \quad (\mu - 0.25)(x_1^{(k)})^2 + (\mu - 0.25)(x_2^{(k)})^2 + (1-\mu)(x_3^{(k)})^2 - (\lambda+\eta)x_3^{(k)}\end{aligned}$$

Taking the partial derivatives of each of the variables in the function $g(x_1^{(k)}, x_2^{(k)}, x_3^{(k)})$ and making them equal to zero, i.e.

$$\begin{aligned}\frac{\partial g}{\partial x_1^{(k)}} &= 2(\mu - 0.25)x_1^{(k)} = 0,\\ \frac{\partial g}{\partial x_2^{(k)}} &= 2(\mu - 0.25)x_2^{(k)} = 0,\\ \frac{\partial g}{\partial x_1^{(k)}} &= 2(1-\mu)x_3^{(k)} - (\lambda+\eta) = 0.\end{aligned}$$

We get $\mu = 0.25$ and $x_3^{(k)} = \frac{2}{3}(\lambda+\eta)$. Substituting it into the inner optimization function, we obtain

$$\max_{\lambda\geq 0, \eta \geq 0} -\frac{1}{3}(\lambda+\eta)^2$$

Solution to the optimal value is zero, so the Lorenz cone is invariant set is linear discrete system.

When using Wolfe duality, it is necessary to determine whether the original problem is convex optimization. In this case, the objective function is not convex, so the Wolfe dual cannot be used.

6. Conclusions

In this paper, the sufficient and necessary conditions for determining the ellipsoidal set and Lorenz cone as positive invariant sets for discrete time dynamic systems are given by virtue of the optimization and the dual optimization method. The positive invariance condition of ellipsoidal set and Lorenz cone is formulated as an optimization problem. In particular, for ellipsoid sets, we propose an invariant condition derived from the induction norm and then formulate as an optimization problem. On this basis, a novel optimization model is proposed to determine that convex sets of quadratic form are positive invariant sets of discrete time systems. The equivalent invariance condition is obtained by using Lagrange duality and Wolfe duality. It is also interesting that the invariance conditions obtained in this paper relate algebraic problems to optimization problems and provide more alternative methods for the calculation of positive invariance sets of ellipsoids and Lorenz cones for nonlinear and linear dynamic systems.

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